Journal of Applied Analysis and Computation
Volume 10, Number 3, June 2020, 986–1004
Website:http://jaac-online.com/
DOI:10.11948/20190179

A NEW ITERATIVE METHOD FOR SOLVING SPLIT FEASIBILITY PROBLEM

Chanchal Garodia[1] and Izhar Uddin[1,†]

Abstract In this paper, we construct a new iterative algorithm and show that the newly introduced iterative algorithm converges faster than a number of existing iterative algorithms for contractive-like mappings. We present a numerical example followed by graphs to validate our claim. We prove strong and weak convergence results for approximating fixed points of generalized α-nonexpansive mappings. Again we reconfirm our results by an example and table. Further, we utilize our proposed algorithm to solve split feasibility problem.

Keywords Generalized α-nonexpansive mappings, fixed point, contractive-like mappings, iteration process, strong and weak convergence, split feasibility problem.

MSC(2010) 47H09, 47H10, 54H25.

1. Introduction

Fixed point theory is one of the fast growing topic of nonlinear functional analysis. It has many applications in finding out the solutions of ODE, PDE, variational inequalities and zero of monotone operators. Banach Contraction Principle is one of the prime result of fixed point theory. The early findings in fixed point theory revolve around generalization of Banach Contraction Principle. The whole mathematics community had to wait for the first fixed point theorem for nonexpansive mapping for 43 years. Let K be a nonempty closed convex subset of a uniformly convex Banach space E. Then, a mapping $T : K \to K$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$. A point $x \in K$ is said to be a fixed point of T if $Tx = x$. We will denote the set of fixed points of T by $F(T)$. T is called quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - p\| \leq \|x - p\|$ for all $x \in K$. It is well known that every nonexpansive mapping with a fixed point is quasi-nonexpansive mapping. One can observe that the well known Banach Contraction Principle is no longer true for nonexpansive mappings i.e. a nonexpansive mapping need not admit a fixed point on complete metric space. Also, Picard iteration need not be convergent for a nonexpansive map in a complete metric space. This led to the beginning of a new era of fixed point theory for nonexpansive mappings by using geometric properties. In 1965, Browder [7], Göhde [11] and Kirk [15] gave three basic existence results in respect of nonexpansive mappings.

† the corresponding author.
Email address:c.garodia85@gmail.com(C. Garodia), izharuddin1@jmi.ac.in(I. Uddin)
[1] Department of mathematics, Jamia Millia Islamia, New Delhi-110025, India

Following this, many mathematicians have introduced various generalizations and extensions of nonexpansive mappings. In 2008, Suzuki [22] introduced a new generalization of nonexpansive mappings and called the defining condition as Condition (C) which is also referred as Suzuki generalized nonexpansive mappings. A mapping $T : K \to K$ defined on a nonempty subset K of a Banach space E is said to satisfy the Condition (C) if

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\|$$

for all x and $y \in K$. Suzuki proved that the mappings satisfying the Condition (C) is weaker than nonexpansive and also obtained few results regarding the existence of fixed points for such mappings.

Later, in 2011, Aoyama and Kohsaka [3] introduced another generalization of nonexpansive mappings, namely, α-nonexpansive mappins and obtained few convergence results. A mapping $T : K \to K$ is said to be α-nonexpansive if there exists an $\alpha \in [0, 1)$ such that for all $x, y \in K$,

$$\|Tx - Ty\|^2 \leq \alpha\|Tx - y\|^2 + \alpha\|x - Ty\|^2 + (1 - 2\alpha)\|x - y\|^2.$$

It is obvious that every nonexpansive mapping is 0-nonexpansive and every α-nonexpansive mapping with a fixed point is quasi-nonexpansive. It is worth mentioning that nonexpansive mappings are always continuous but mappings satisfying the Condition (C) or α-nonexpansive mappings need not be continuous in general.

In 2017, Pant and Shukla [18] introduced the class of generalized α-nonexpansive mappings. A mapping $T : K \to K$ is said to be generalized α-nonexpansive if there exists an $\alpha \in [0, 1)$ such that

$$\frac{1}{2}\|x-Tx\| \leq \|x - y\| \Rightarrow \|Tx-Ty\| \leq \alpha\|Tx-y\|+\alpha\|Ty-x\|+(1-2\alpha)\|x-y\|$$

for all x and $y \in K$. They established some existence and convergence theorems for the newly introduced class of mappings. Clearly, every mapping satisfying the Condition (C) is a generalized α-nonexpansive mapping.

In the last few years many iterative processes have been obtained in different domains to approximate the fixed points of various classes of mappings. To name a few, we have Mann iteration [16], Ishikawa iteration [14], Noor iteration [17], Agarwal et al. iteration [2], Abbas and Nazir iteration [1], Thakur et al. iterations [24,25], M^* iteration [26], M iteration [27], K iteration [12] and K^* iteration [28]. Very recently, Piri et al. [19] introduced a new iteration process as follows:

$$\begin{cases} a_1 \in K, \\ c_n = T((1 - \beta_n)a_n + \beta_n Ta_n), \\ b_n = Tc_n, \\ a_{n+1} = (1 - \alpha_n)Tc_n + \alpha_n Tb_n, \end{cases} \tag{1.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. Authors proved that their iteration process (1.1) is having a better rate of convergence than a number of existing iteration processes. Further, they used their iteration process to obtain few convergence results involving generalized α-nonexpansive mappings.

Motivated and inspired by the research going on this direction, we introduce a new iteration process to approximate fixed points of generalized α-nonexpansive mappings. We define our process as follows:

$$\begin{cases} x_1 \in K, \\ z_n = T((1-\alpha_n)x_n + \alpha_n T x_n), \\ y_n = T((1-\beta_n)Tx_n + \beta_n T z_n), \\ x_{n+1} = T y_n, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$.

The aim of this paper is to prove that the newly defined iteration process (1.2) converges faster than iteration process (1.1) for contractive-like mappings. Also, we prove some weak and strong convergence results involving the iteration process (1.2) for generalized α-nonexpansive mappings. Further, we provide a numerical example to show that our process (1.2) converges faster than a number of existing iteration processes. In the last section, we discuss about the solution of split feasibility problem using our newly introduced iterative algorithm.

2. Preliminaries

For making our paper self contained, we collect some basic definitions and needed results.

Definition 2.1. A Banach space E is said to be uniformly convex if for each $\epsilon \in (0, 2]$ there is a $\delta > 0$ such that for $x, y \in E$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| > \epsilon$, we have

$$\left\| \frac{x+y}{2} \right\| < 1 - \delta.$$

Definition 2.2. A Banach space E is said to satisfy the Opial's condition if for any sequence $\{x_n\}$ in E which converges weakly to $x \in E$ i.e. $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in E$ with $y \neq x$.

Examples of Banach spaces satisfying this condition are Hilbert spaces and all l^p spaces $(1 < p < \infty)$. On the other hand, $L^p[0, 2\pi]$ with $1 < p \neq 2$ fail to satisfy Opial's condition.

A mapping $T : K \to E$ is demiclosed at $y \in E$ if for each sequence $\{x_n\}$ in K and each $x \in E$, $x_n \rightharpoonup x$ and $Tx_n \to y$ imply that $x \in K$ and $Tx = y$.

Let K be a nonempty closed convex subset of a Banach E, and let $\{x_n\}$ be a bounded sequence in E. For $x \in E$ write:

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius of $\{x_n\}$ relative to K is given by

$$r(\{x_n\}) = \inf\{r(x, x_n) : x \in K\}$$

and the asymptotic center $A(K,\{x_n\})$ of $\{x_n\}$ is defined as:

$$A(K,\{x_n\}) = \{x \in K : r(x,\{x_n\}) = r(K,\{x_n\})\}.$$

It is known that, in a uniformly convex Banach space, $A(K,\{x_n\})$ consists of exactly one point.

The following definitions about the rate of convergence were given by Berinde [6].

Definition 2.3. Let $\{a_n\}$ and $\{b_n\}$ be two real sequences converging to a and b respectively. Then, $\{a_n\}$ converges faster then $\{b_n\}$ if $\lim_{n\to\infty} \frac{\|a_n-a\|}{\|b_n-b\|} = 0$.

Definition 2.4. Let $\{u_n\}$ and $\{v_n\}$ be two fixed point iteration processes converging to the same fixed point p. If $\{a_n\}$ and $\{b_n\}$ are two sequences of positive numbers converging to zero such that $\|u_n - p\| \le a_n$ and $\|v_n - p\| \le b_n$ for all $n \ge 1$, then we say that $\{u_n\}$ converges faster than $\{v_n\}$ to p if $\{a_n\}$ converges faster then $\{b_n\}$.

The following lemma due to Schu [20] is very useful in our subsequent discussion.

Lemma 2.1. *Let E be a uniformly convex Banach space and $\{t_n\}$ be any sequence such that $0 < p \le t_n \le q < 1$ for some $p, q \in \mathbb{R}$ and for all $n \ge 1$. Let $\{x_n\}$ and $\{y_n\}$ be any two sequences of E such that $\limsup_{n\to\infty} \|x_n\| \le r$, $\limsup_{n\to\infty} \|y_n\| \le r$ and $\limsup_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r$ for some $r \ge 0$. Then, $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Now, we recall some important results involving generalized α-nonexpansive mappings.

Lemma 2.2 ([18]). *Let K be a nonempty subset of a Banach space E and $T : K \to K$ a generalized α-nonexpansive mapping. Then,*

(i) $F(T)$ is closed. Moreover, if E is strictly convex and K is convex, then $F(T)$ is convex.

(ii) If $F(T) \ne \emptyset$, then T is quasi-nonexpansive.

(iii)

$$\|x - Ty\| \le \frac{3+\alpha}{1-\alpha}\|x - Tx\| + \|x - y\|$$

for all x and $y \in K$.

Lemma 2.3 ([23]). *Let T be a generalized α-nonexpansive mapping defined on a nonempty closed subset K of a Banach space E with the Opial property. If a sequence $\{x_n\}$ converges weakly to z and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$, then $I - T$ is demiclosed at zero.*

In 1972, Zamifirescu [31] coined the idea of so-called Zamfirescu mappings which serves as an important generalization for contractions. A generalized version of the famous Banach contraction principle [4] was given using this kind of mappings in [31]. Later on, in 2004, Berinde [5] gave a more general class of mappings known as quasi-contractive mappings. Following this, Imoru and Olantiwo [13] gave the following definition:

Definition 2.5. A mapping $T : K \to K$ is known as contractive-like mapping if there exists a strictly increasing and continuous function $\varphi : [0,\infty) \to [0,\infty)$ with

$\varphi(0) = 0$ and a constant $\delta \in [0, 1)$ such that for all $x, y \in K$, we have

$$\|Tx - Ty\| \leq \delta\|x - y\| + \varphi(\|x - Tx\|).$$

Clearly, the class of contractive-like mappings is wider than the class of quasi-contractive mappings.

3. Rate of Convergence

In this section, first we show that our algorithm (1.2) converges faster than the algorithm (1.1) for contractive-like mappings.

Theorem 3.1. *Let T be a contractive-like mapping defined on a nonempty closed convex subset K of a Banach space E with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence defined by* (1.2)*, then $\{x_n\}$ converges faster than the iterative algorithm* (1.1).

Proof. From (1.2), for any $p \in F(T)$, we have

$$\begin{aligned}
\|z_n - p\| &= \|T((1 - \alpha_n)x_n + \alpha_n Tx_n) - p\| \\
&\leq \delta\|(1 - \alpha_n)x_n + \alpha_n Tx_n - p\| \\
&\leq \delta((1 - \alpha_n)\|x_n - p\| + \alpha_n\|Tx_n - p\|) \\
&\leq \delta((1 - \alpha_n)\|x_n - p\| + \alpha_n\delta\|x_n - p\|) \\
&= \delta(1 - (1 - \delta)\alpha_n)\|x_n - p\| \\
&\leq \delta\|x_n - p\|,
\end{aligned}$$

$$\begin{aligned}
\|y_n - p\| &= \|T((1 - \beta_n)Tx_n + \beta_n Tz_n) - p\| \\
&\leq \delta\|(1 - \beta_n)Tx_n + \beta_n Tz_n - p\| \\
&\leq \delta((1 - \beta_n)\|Tx_n - p\| + \beta_n\|Tz_n - p\|) \\
&\leq \delta((1 - \beta_n)\delta\|x_n - p\| + \beta_n\delta\|z_n - p\|) \\
&= \delta^2((1 - \beta_n)\|x_n - p\| + \beta_n\|z_n - p\|) \\
&\leq \delta^2((1 - \beta_n)\|x_n - p\| + \beta_n\delta\|x_n - p\|) \\
&\leq \delta^2(1 - (1 - \delta)\beta_n)\|x_n - p\| \\
&\leq \delta^2\|x_n - p\|.
\end{aligned}$$

and

$$\begin{aligned}
\|x_{n+1} - p\| &= \|Ty_n - p\| \\
&\leq \delta\|y_n - p\| \\
&\leq \delta^3\|x_n - p\| \\
&\ \ \vdots \\
&\leq \delta^{3n}\|x_1 - p\|.
\end{aligned}$$

Now, from (1.1) we get

$$\begin{aligned}\|c_n - p\| &= \|T((1-\beta_n)a_n + \beta_n T a_n) - p\| \\ &\leq \delta\|(1-\beta_n)a_n + \beta_n T a_n - p\| \\ &\leq \delta((1-\beta_n)\|a_n - p\| + \beta_n\|Ta_n - p\|) \\ &\leq \delta((1-\beta_n)\|a_n - p\| + \beta_n\delta\|a_n - p\|) \\ &= \delta(1-(1-\delta)\beta_n)\|a_n - p\| \\ &\leq \delta\|a_n - p\|,\end{aligned}$$

$$\begin{aligned}\|b_n - p\| &= \|Tc_n - p\| \\ &\leq \delta\|c_n - p\| \\ &\leq \delta^2\|a_n - p\|\end{aligned}$$

and

$$\begin{aligned}\|a_{n+1} - p\| &= \|(1-\alpha_n)Tc_n + \alpha_n T b_n - p\| \\ &\leq (1-\alpha_n)\|Tc_n - p\| + \alpha_n\|Tb_n - p\|) \\ &\leq \delta((1-\alpha_n)\|c_n - p\| + \alpha_n\|b_n - p\|) \\ &\leq \delta((1-\alpha_n)\|c_n - p\| + \alpha_n\delta\|c_n - p\|) \\ &= \delta(1-(1-\delta)\alpha_n)\|c_n - p\| \\ &\leq \delta\|c_n - p\| \\ &\leq \delta^2\|a_n - p\| \\ &\vdots \\ &\leq \delta^{2n}\|a_1 - p\|.\end{aligned}$$

Let $b_n = \delta^{3n}\|x_1 - p\|$ and $a_n = \delta^{2n}\|a_1 - p\|$, then

$$\begin{aligned}\tfrac{b_n}{a_n} &= \tfrac{\delta^{3n}\|x_1-p\|}{\delta^{2n}\|a_1-p\|} \\ &\to 0 \qquad \text{as} \qquad n\to\infty.\end{aligned}$$

Hence, $\{x_n\}$ converges faster than $\{a_n\}$. □

Now, we present a example of a contractive-like mapping which is not a contraction.

Example 3.1. Let $E = \mathbb{R}$ and $K = [0, 8]$. Let $T : K \to K$ be a mapping defined as

$$Tx = \begin{cases} \dfrac{x}{6}, & x \in [0,4) \\ \dfrac{x}{12}, & x \in [4,8]. \end{cases}$$

Proof. Clearly $x = 0$ is the fixed point of T. First, we prove that T is a contractive-like mapping but not a contraction. Since T is not continuous at $x = 4 \in [0, 8]$, so T is not a contraction. We show that T is a contractive-like mapping. For this, define $\varphi : [0, \infty) \to [0, \infty)$ as $\varphi(x) = \frac{x}{10}$. Then, φ is a strictly increasing as well as continuous function. Also, $\varphi(0) = 0$.

We need to show that

$$\|Tx - Ty\| \leq \delta\|x - y\| + \varphi(\|x - Tx\|) \tag{A}$$

for all $x, y \in [0, 8]$ and δ is a constant in $[0, 1)$.

Before going ahead, let us note the following. When $x \in [0, 4)$, then

$$\|x - Tx\| = \left\|x - \frac{x}{6}\right\| = \frac{5x}{6}$$

and

$$\varphi(\frac{5x}{6}) = \frac{x}{12}. \tag{3.1}$$

Similarly, when $x \in [4, 8]$, then

$$\|x - Tx\| = \left\|x - \frac{x}{12}\right\| = \frac{11x}{12}$$

and

$$\varphi(\frac{11x}{12}) = \frac{11x}{120}. \tag{3.2}$$

Consider the following cases:

Case A: Let $x, y \in [0, 4)$, then using (3.1) we get

$$\begin{aligned}
\|Tx - Ty\| &= \|\frac{x}{6} - \frac{y}{6}\| \\
&\leq \frac{1}{6}\|x - y\| \\
&\leq \frac{1}{6}\|x - y\| + \frac{x}{12} \\
&= \frac{1}{6}\|x - y\| + \varphi(\frac{5x}{6}) \\
&= \frac{1}{6}\|x - y\| + \varphi(\|x - Tx\|).
\end{aligned}$$

So (A) is satisfied with $\delta = \frac{1}{6}$.

Case B: Let $x \in [0, 4)$ and $y \in [4, 8]$ then using (3.1) we get

$$\begin{aligned}
\|Tx - Ty\| &= \|\frac{x}{6} - \frac{y}{12}\| \\
&= \|\frac{x}{12} + \frac{x}{12} - \frac{y}{12}\| \\
&\leq \frac{1}{12}\|x - y\| + \left\|\frac{x}{12}\right\| \\
&\leq \frac{1}{6}\|x - y\| + \varphi(\frac{5x}{6}) \\
&= \frac{1}{6}\|x - y\| + \varphi(\|x - Tx\|).
\end{aligned}$$

So (A) is satisfied with $\delta = \frac{1}{6}$.
Case C: Let $x \in [4,8]$ and $y \in [0,4)$ then using (3.2) we get

$$\begin{aligned}
\|Tx - Ty\| &= \|\frac{x}{12} - \frac{y}{6}\| \\
&= \|\frac{x}{6} - \frac{x}{12} - \frac{y}{6}\| \\
&\leq \frac{1}{6}\|x-y\| + \left\|\frac{x}{12}\right\| \\
&\leq \frac{1}{6}\|x-y\| + \left\|\frac{11x}{120}\right\| \\
&= \frac{1}{6}\|x-y\| + \varphi(\|x - Tx\|).
\end{aligned}$$

So (A) is satisfied with $\delta = \frac{1}{6}$.
Case D: Let $x, y \in [4,8]$ then using (3.2) we get

$$\begin{aligned}
\|Tx - Ty\| &= \|\frac{x}{12} - \frac{y}{12}\| \\
&\leq \frac{1}{12}\|x-y\| + \left\|\frac{11x}{120}\right\| \\
&\leq \frac{1}{6}\|x-y\| + \left\|\frac{11x}{120}\right\| \\
&= \frac{1}{6}\|x-y\| + \varphi(\|x - Tx\|).
\end{aligned}$$

So (A) is satisfied with $\delta = \frac{1}{6}$.
Consequently, (A) is satisfied for $\delta = \frac{1}{6}$ and $\varphi(x) = \frac{x}{10}$ in all the possible cases. Thus, T is a contractive-like mapping. □

Now, using T, we show that our iteration process (1.2) has a better rate of convergence. Set $\alpha_n = \beta_n = \gamma_n = \frac{n+5}{n+7}$ for each $n \in \mathbb{N}$. Then, we get the following tables and graphs with the initial value 7.5.

Table 1.

Step	Agarwal	Abbas	Thakur	M^*	K^*	New
1	7.5	7.5	7.5	7.5	7.5	7.5
2	0.449219	0.311686	0.266113	0.0602214	0.0244141	0.00569661
3	0.0371268	0.0151702	0.0141267	0.000448035	0.0000839571	7.06361×10^{-6}
4	0.00288764	0.000713562	0.000680173	3.04221×10^{-6}	2.59127×10^{-7}	7.9938×10^{-9}
5	0.000212794	0.0000325538	0.0000300014	1.90313×10^{-8}	7.2872×10^{-10}	8.33452×10^{-12}
6	0.0000149416	1.44502×10^{-6}	1.22241×10^{-6}	1.10543×10^{-10}	1.8899×10^{-12}	8.06847×10^{-15}
7	1.00446×10^{-6}	6.25812×10^{-8}	4.6333×10^{-8}	6.00095×10^{-13}	4.5646×10^{-15}	7.30012×10^{-18}
8	6.49138×10^{-8}	2.65063×10^{-9}	1.64349×10^{-9}	3.06171×10^{-15}	1.03506×10^{-17}	6.20758×10^{-21}

Table 2.

Step	Noor	Thakur New	K	M	Piri et al.	New
1	7.5	7.5	7.5	7.5	7.5	7.5
2	2.146	0.100911	0.0124783	0.0651042	0.0244141	0.00569661
3	0.551395	0.00139001	0.0000286472	0.000636306	0.0000839571	7.06361×10^{-6}
4	0.12825	0.0000180187	6.18921×10^{-8}	5.89172×10^{-6}	2.59127×10^{-7}	7.9938×10^{-9}
5	0.0272568	2.21304×10^{-7}	1.26692×10^{-10}	5.20733×10^{-8}	7.2872×10^{-10}	8.33452×10^{-12}
6	0.00533441	2.58985×10^{-9}	2.47106×10^{-13}	4.4198×10^{-10}	1.8899×10^{-12}	8.06847×10^{-15}
7	0.000967698	2.90173×10^{-11}	4.6144×10^{-16}	3.62021×10^{-12}	4.5646×10^{-15}	7.30012×10^{-18}
8	0.000163634	3.12545×10^{-13}	8.2836×10^{-19}	2.87318×10^{-14}	1.03506×10^{-17}	6.20758×10^{-21}

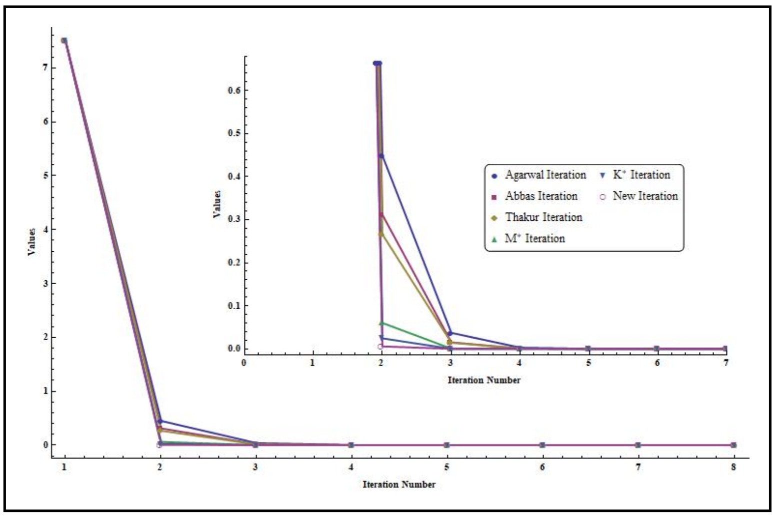

Figure 1. Graph corresponding to Table 1.

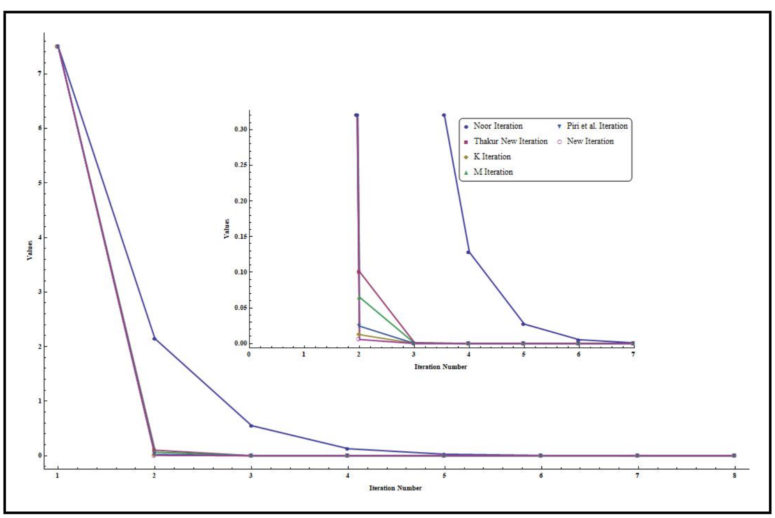

Figure 2. Graph corresponding to Table 2.

Clearly, our algorithm (1.2) converges at a faster rate for contractive-like mappings.

4. Convergence Results

First, we prove few lemmas which will be useful in obtaining convergence results.

Lemma 4.1. *Let T be a generalized α-nonexpansive mapping defined on a nonempty closed convex subset K of a Banach space E with $F(T) \neq \emptyset$. Let $\{x_n\}$ be the iterative sequence defined by the iteration process* (1.2). *Then, $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$.*

Proof. Let $p \in F(T)$. By Lemma 2.2(ii), T is quasi-nonexpansive, so we have

$$\begin{aligned} \|z_n - p\| &= \|T((1-\alpha_n)x_n + \alpha_n Tx_n) - p\| \\ &\leq \|(1-\alpha_n)x_n + \alpha_n Tx_n - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|Tx_n - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|x_n - p\| \\ &= \|x_n - p\| \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} \|y_n - p\| &= \|T((1-\beta_n)Tx_n + \beta_n Tz_n) - p\| \\ &\leq \|(1-\beta_n)Tx_n + \beta_n Tz_n - p\| \\ &\leq (1-\beta_n)\|Tx_n - p\| + \beta_n\|Tz_n - p\| \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n\|z_n - p\| \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n\|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{4.2}$$

Using (4.1) and (4.2), we get

$$\begin{aligned} \|x_{n+1} - p\| &= \|Ty_n - p\| \\ &\leq \|y_n - p\| \\ &\leq \|x_n - p\|. \end{aligned}$$

Thus, $\{\|x_n - p\|\}$ is bounded and non-increasing sequence of reals and hence $\lim_{n\to\infty} \|x_n - p\|$ exists. □

Lemma 4.2. *Let T be a generalized α-nonexpansive mapping defined on a nonempty closed convex subset K of a uniformly convex Banach space E. Let $\{x_n\}$ be the iterative sequence defined by the iteration process* (1.2). *Then, $F(T) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$.*

Proof. Suppose $F(T) \neq \emptyset$ and let $p \in F(T)$. Then, by Lemma 4.1, $\lim_{n\to\infty} \|x_n - p\|$ exists. Let

$$\lim_{n\to\infty} \|x_n - p\| = c. \tag{4.3}$$

From inequalities (4.1) and (4.2), we have

$$\limsup_{n\to\infty} \|y_n - p\| \le c \tag{4.4}$$

and

$$\limsup_{n\to\infty} \|z_n - p\| \le c. \tag{4.5}$$

Now,

$$c = \lim_{n\to\infty} \|x_{n+1} - p\| = \lim_{n\to\infty} \|Ty_n - p\|,$$

and

$$\|Ty_n - p\| \le \|y_n - p\|.$$

So,

$$c \le \liminf_{n\to\infty} \|y_n - p\|$$

which along with inequality (4.4) implies

$$\lim_{n\to\infty} \|y_n - p\| = c. \tag{4.6}$$

Now, consider

$$\begin{aligned} \|y_n - p\| &= \|T((1-\beta_n)Tx_n + \beta_n Tz_n) - p\| \\ &\le \|(1-\beta_n)Tx_n + \beta_n Tz_n - p\| \\ &\le (1-\beta_n)\|Tx_n - p\| + \beta_n\|Tz_n - p\| \\ &\le (1-\beta_n)\|x_n - p\| + \beta_n\|z_n - p\| \\ &= \|x_n - p\| + \beta_n(\|z_n - p\| - \|x_n - p\|). \end{aligned} \tag{4.7}$$

It follows that

$$\|y_n - p\| - \|x_n - p\| \le \beta_n(\|z_n - p\| - \|x_n - p\|).$$

Now, since $\{\beta_n\} \in (0,1)$, using (4.7) we have

$$\|y_n - p\| - \|x_n - p\| \le \frac{\|y_n - p\| - \|x_n - p\|}{\beta_n} \le \|z_n - p\| - \|x_n - p\|$$

which gives $\|y_n - p\| \le \|z_n - p\|$ and using (4.6) we get

$$c \le \liminf_{n\to\infty} \|z_n - p\|. \tag{4.8}$$

Owing to (4.5) and (4.8), we have

$$\lim_{n\to\infty} \|z_n - p\| = c. \tag{4.9}$$

Also, using the fact that T is quasi-nonexpansive we have $\|Tx_n - p\| \le \|x_n - p\|$, which gives

$$\limsup_{n\to\infty} \|Tx_n - p\| \le c. \tag{4.10}$$

From (4.1), we have

$$\|z_n - p\| \leq \|(1-\alpha_n)x_n + \alpha_n T x_n - p\| \leq \|x_n - p\|$$

which on using (4.3) and (4.9) gives

$$\lim_{n\to\infty} \|(1-\alpha_n)x_n + \alpha_n T x_n - p\| = c. \tag{4.11}$$

Using (4.3), (4.10), (4.11) and Lemma 2.1, we conclude that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. Conversely, suppose that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Let $p \in A(K, \{x_n\})$, we have

$$\begin{aligned} r(Tp, \{x_n\}) &= \limsup_{n\to\infty} \|x_n - Tp\| \\ &\leq \left(\tfrac{3+\alpha}{1-\alpha}\right) \limsup_{n\to\infty} \|Tx_n - x_n\| + \limsup_{n\to\infty} \|x_n - p\| \\ &= \limsup_{n\to\infty} \|x_n - p\| \\ &= r(p, \{x_n\}). \end{aligned}$$

This implies that $Tp \in A(K, \{x_n\})$. Since E is uniformly convex, $A(K, \{x_n\})$ is singleton, therefore we get $Tp = p$. □

Theorem 4.1. *Let T be a generalized α-nonexpansive mapping defined on a nonempty closed convex subset K of a Banach space E which satisfies the Opial's condition with $F(T) \neq \emptyset$. If $\{x_n\}$ is the iterative sequence defined by the iteration process* (1.2)*, then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. Let $p \in F(T)$. Then, from Lemma 4.1 $\lim_{n\to\infty} \|x_n - p\|$ exists. In order to show the weak convergence of the iteration process (1.2) to a fixed point of T, we will prove that $\{x_n\}$ has a unique weak subsequential limit in $F(T)$. For this, let $\{x_{n_j}\}$ and $\{x_{n_k}\}$ be two subsequences of $\{x_n\}$ which converges weakly to u and v respectively. By Lemma 4.1, we have $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$ and using the Lemma 2.3, we have $I - T$ is demiclosed at zero. So $u, v \in F(T)$.

Next, we show the uniqueness. Since $u, v \in F(T)$, so $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exists. Let $u \neq v$. Then, by Opial's condition, we obtain

$$\begin{aligned} \lim_{n\to\infty} \|x_n - u\| &= \lim_{j\to\infty} \|x_{n_j} - u\| \\ &< \lim_{j\to\infty} \|x_{n_j} - v\| \\ &= \lim_{n\to\infty} \|x_n - v\| \\ &= \lim_{k\to\infty} \|x_{n_k} - v\| \\ &< \lim_{k\to\infty} \|x_{n_k} - u\| \\ &= \lim_{n\to\infty} \|x_n - u\| \end{aligned}$$

which is a contradiction, so $u = v$. Thus, $\{x_n\}$ converges weakly to a fixed point of T. □

Now, we establish some strong convergence results.

Theorem 4.2. *Let K be a nonempty closed convex subset of a uniformly convex Banach space E and $T : K \to K$ be a generalized α-nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is defined by the iteration process* (1.2)*, then $\{x_n\}$ converges strongly to a point of $F(T)$ if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.*

Proof. If the sequence $\{x_n\}$ converges to a point $p \in F(T)$, then it is obvious that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.

For the converse part, assume that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. From Lemma 4.1, we have $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$, which gives

$$\|x_{n+1} - p\| \leq \|x_n - p\| \text{ for any } p \in F(T)$$

which yields

$$d(x_{n+1}, F(T) \leq d(x_n, F(T). \tag{4.12}$$

Thus, $\{d(x_n, F(T)\}$ forms a non-increasing sequence which is bounded below by zero as well, so we get that $\lim_{n\to\infty} d(x_n, F(T)$ exists. As, $\liminf_{n\to\infty} d(x_n, F(T) = 0$ so $\lim_{n\to\infty} d(x_n, F(T) = 0$.

Now, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a sequence $\{u_j\}$ in $F(T)$ such that $\|x_{n_j} - u_j\| \leq \frac{1}{2^j}$ for all $j \in \mathbb{N}$. From the proof of Lemma 4.1, we have

$$\|x_{n_{j+1}} - u_j\| \leq \|x_{n_j} - u_j\| \leq \frac{1}{2^j}$$

. Using triangle inequality, we get

$$\begin{aligned} \|u_{j+1} - u_j\| \leq & \ \|u_{j+1} - x_{n_{j+1}}\| + \|x_{n_{j+1}} - u_j\| \\ \leq & \ \tfrac{1}{2^{j+1}} + \tfrac{1}{2^j} \\ \leq & \ \tfrac{1}{2^{j-1}} \\ \to & \ 0 \text{ as } j \to \infty. \end{aligned}$$

So, $\{u_j\}$ is a Cauchy sequence in $F(T)$. By Lemma 2.2 $F(T)$ is closed, so $\{u_j\}$ converges to some $u \in F(T)$.

Again, owing to triangle inequality, we have

$$\|x_{n_j} - u\| \leq \|x_{n_j} - u_j\| + \|u_j - u\|.$$

Letting $j \to \infty$, we have $\{x_{n_j}\}$ converges strongly to $u \in F(T)$.

Since $\lim_{n\to\infty} \|x_n - u\|$ exists by Lemma 4.1, therefore $\{x_n\}$ converges strongly to $u \in F(T)$. □

A mapping $T : K \to K$ is said to satisfy the Condition (A) ([21]) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in K$, where $d(x, F(T)) = inf\{\|x - p\| : p \in F(T)\}$.

Now, we present a strong convergence result using the Condition (A).

Theorem 4.3. *Let K be a nonempty closed convex subset of a uniformly convex Banach space E and $T : K \to K$ be a generalized α-nonexpansive mapping with $F(T) \neq \emptyset$. If T satisfies the Condition (A) and $\{x_n\}$ is defined by the iteration process* (1.2)*, then $\{x_n\}$ converges strongly to a point of $F(T)$.*

Proof. From 4.12, $\lim_{n\to\infty} d(x_n, F(T))$ exists.

Also, by Lemma 4.2 we have $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

It follows from the Condition (A) that

$$\lim_{n\to\infty} f(d(x_n, F(T))) \leq \lim_{n\to\infty} \|x_n - Tx_n\| = 0,$$

so that $\lim_{n\to\infty} f(d(x_n, F(T))) = 0$.

Since f is a non decreasing function satisfying $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$, therefore $\lim_{n\to\infty} d(x_n, F(T)) = 0$.

By Theorem 4.2., the sequence $\{x_n\}$ converges strongly to a point of $F(T)$. □

Now, first we will construct an example of a generalized α-nonexpansive mapping which is neither a Suzuki generalized nonexpansive mapping nor a nonexpansive mapping. Then, using this example, we will show that our iteration scheme (1.2) has a better speed of convergence than number of existing iteration schemes.

Example 4.1. Let $E = \mathbb{R}$ with the usual norm and $K = [0, \infty)$. Let $T : K \to K$ be a mapping defined as

$$T(x) = \begin{cases} 0, & x \in [0, \frac{3}{2}) \\ \frac{5x}{13}, & x \in [\frac{3}{2}, \infty). \end{cases}$$

for all $x \in K$.

Proof. Clearly $x = 0$ is the fixed point of T. Then,

(i) Since T is not continuous at $x = \frac{3}{2}$, so T is not a nonexpansive map.

(ii) Let $x = 1$ and $y = \frac{3}{2}$, then

$$\frac{1}{2}\|x - Tx\| = \frac{1}{2} \leq \frac{1}{2} = \|x - y\|.$$

But

$$\|Tx - Ty\| = \frac{5y}{13} = \frac{15}{26} > \frac{1}{2} = \|x - y\|.$$

So, T is not a Suzuki generalized nonexpansive mapping.

(iii) Now, we prove that T is a generalized α-nonexpansive mapping. For this, let $\alpha = \frac{1}{3}$ and consider the following cases:

Case (A). When $x \in [\frac{3}{2}, \infty)$ and $y \in [0, \frac{3}{2})$ then,

$$\|Tx - Ty\| = |Tx - Ty| = \frac{5x}{13}.$$

Now,

$$\begin{aligned} \alpha\|Tx - y\| + \alpha\|Ty - x\| + (1 - 2\alpha)\|x - y\| &= \frac{1}{3}|Tx - y| + \frac{1}{3}|Ty - x| + \frac{1}{3}|x - y| \\ &= \frac{1}{3}|\frac{5x}{13} - y| + \frac{1}{3}|x| + \frac{1}{3}|x - y| \\ &\geq \frac{1}{3}|\frac{5x}{13} - y| + \frac{1}{3}|x - y| \\ &\geq \frac{6x}{13} \end{aligned}$$

$$> \frac{5x}{13} = \|Tx - Ty\|.$$

Case (B). When $x, y \in [\frac{3}{2}, \infty)$ then,

$$\|Tx - Ty\| = \frac{5}{13}\|x - y\| = \frac{5}{13}|x - y|.$$

Now,

$$\begin{aligned}
\alpha\|Tx - y\| + \alpha\|Ty - x\| + (1 - 2\alpha)\|x - y\| &= \frac{1}{3}|Tx - y| + \frac{1}{3}|Ty - x| + \frac{1}{3}|x - y| \\
&= \frac{1}{3}|\frac{5x}{13} - y| + \frac{1}{3}|x - \frac{5y}{13}| + \frac{1}{3}|x - y| \\
&\geq \frac{1}{3}|\frac{18x}{13} - \frac{18y}{13}| + \frac{1}{3}|x - y| \\
&= \frac{6}{13}|x - y| + \frac{1}{3}|x - y| \\
&> \frac{5}{13}|x - y| \\
&= \|Tx - Ty\|.
\end{aligned}$$

Case (C). When $x, y \in [0, \frac{3}{2})$ then,

$$\|Tx - Ty\| = 0.$$

So,

$$\begin{aligned}
\alpha\|Tx - y\| + \alpha\|Ty - x\| + (1 - 2\alpha)\|x - y\| &= \frac{1}{3}|Tx - y| + \frac{1}{3}|Ty - x| + \frac{1}{3}|x - y| \\
&\geq \|Tx - Ty\|.
\end{aligned}$$

Therefore, T is a generalized α-nonexpansive mapping with $\alpha = \frac{1}{3}$. □

Let $\alpha_n = \beta_n = \gamma_n = \frac{n}{n+10}$ for all $n \in \mathbb{N}$ and $x_1 = 70000.5$, then we get the following tables of iteration values and graphs.

Table 3.

Step	Agarwal	Abbas	Thakur	M^*	K^*	New
1	70000.5	70000.5	70000.5	70000.5	70000.5	70000.5
2	26786.3	11676.5	26657.1	9755.54	9228.9	3752.13
3	10126.3	2086.53	9920.18	1285.63	1099.54	190.181
4	3767.1	388.543	3583.13	160.776	119.736	9.14749
5	1376.1	74.1824	1251.84	19.1422	12.0315	0.0000
6	493.08	14.3753	422.386	2.17636	1.12452	0.0000
7	173.234	2.80895	137.583	0.0000	0.0000	0.0000
8	59.6767	0.332132	43.2732	0.0000	0.0000	0.0000

Table 4.

Step	Noor	Thakur New	K	M	Piri et al.	New
1	70000.5	70000.5	70000.5	70000.5	70000.5	70000.5
2	65942.7	10302.4	3962.48	9775.8	9228.9	3752.13
3	58718.0	1497.98	221.594	1297.8	1099.54	190.181
4	49573.5	214.332	12.1946	164.719	119.736	9.14749
5	39794.2	30.1132	0.658967	20.0825	12.0315	0.0000
6	30450.6	4.15003	0.0000	2.36139	0.943144	0.0000
7	22263.8	0.0000	0.0000	0.0000	0.0000	0.0000
8	15587.4	0.0000	0.0000	0.0000	0.0000	0.0000

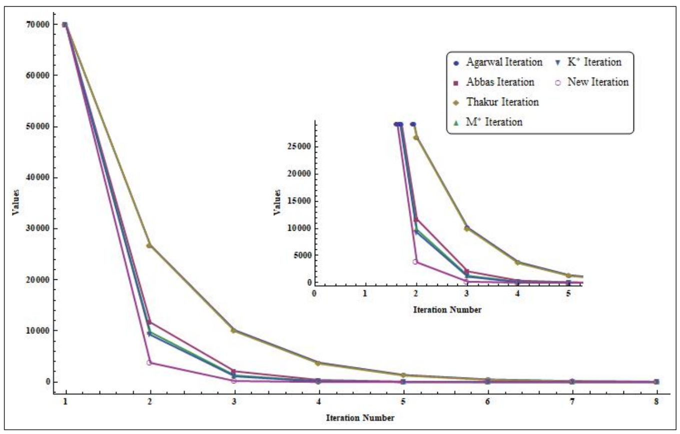

Figure 3. Graph corresponding to Table 3.

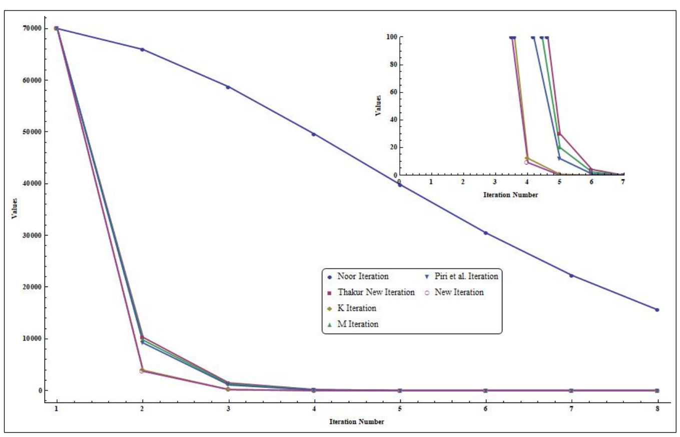

Figure 4. Graph corresponding to Table 4.

It is evident from above tables and graphs that our algorithm (1.2) converges at a better speed than the above mentioned schemes.

5. Application

In this section, we will use our iteration process (1.2) to find the solution of split feasibility problem.

Let H_1 and H_2 be two real Hilbert spaces, C and Q be closed, convex and nonempty subsets of H_1 and H_2, respectively and let $A : H_1 \to H_2$ be a bounded and linear operator. Then, the split feasibility problem (abbreviate SFP) can be mathematically described as finding a point $x \in C$ such that

$$x \in C, Ax \in Q. \tag{5.1}$$

We assume that the solution set Ω of the SFP (5.1) is nonempty, let

$$\Omega = \{x \in C : Ax \in Q\} = C \cap A^{-1}Q.$$

Then, Ω is closed, convex and nonempty set. Censor and Elfving [9] solved the class of inverse problems with the help of SFP. In 2002, Byrne [8] introduced the famous CQ-algorithm for solving the SFP. In this, the iterative step x_k is calculated as follows:

$$x_{k+1} = P_C[I - \gamma A^*(I - P_Q)A]x_k, \qquad k \geq 0, \tag{5.2}$$

where $0 < \gamma < \frac{2}{\|A\|^2}$, P_C and P_Q denote the projections onto sets C and Q, respectively and $A^* : {H_2}^* \to {H_1}^*$ is the adjoint of A.

We have the following important lemma due to Feng et al. [10]

Lemma 5.1. *Let operator* $T = P_C[I - \gamma A^*(I - P_Q)A]$, *where* $0 < \gamma < \frac{2}{\|A\|^2}$. *Then,* T *is a nonexpansive map.*

Also, since we have assumed that solution set Ω of SFP is nonempty, it is easy to see that any $x^* \in C$ is the solution of SFP if and only if it solves the following fixed point equation:

$$P_C[I - \gamma A^*(I - P_Q)A]x = x, \qquad x \in C.$$

So, the solution set Ω is equal to the fixed point set of T , i.e, $F(T) = \Omega = C \cap A^{-1}Q \neq \emptyset$. For details, one can refer [29, 30].

Now, we present our main results.

Theorem 5.1. *If* $\{x_n\}$ *is the sequence generated by the iterative algorithm* (1.2) *with* $T = P_C[I - \gamma A^*(I - P_Q)A]$ *then,* $\{x_n\}$ *converges weakly to the solution of SFP (5.1).*

Proof. By Lemma 5.1, T is a nonexpansive map and every nonexpansive mapping is a generalized 0-nonexpansive mapping, so the result follows from Theorem 4.1. □

Theorem 5.2. *If* $\{x_n\}$ *is the sequence generated by the iterative algorithm* (1.2) *with* $T = P_C[I - \gamma A^*(I - P_Q)A]$ *then,* $\{x_n\}$ *converges strongly to the solution of SFP (5.1) if and only if* $\liminf_{n\to\infty} d(x_n, \Omega) = 0$.

Proof. Proof follows from Theorem 4.2. □

Acknowledgment

We are very much thankful to the reviewers for their constructive comments and suggestions which have been useful for the improvement of this paper.

References

[1] M. Abbas and T. Nazir, *A new faster iteration process applied to constrained minimization and feasibility problems*, Matematicki Vesnik, 2014, 66(2), 223–234.

[2] R. P. Agarwal, D. Ó Regan and D. R. Sahu, *Iterative construction of fixed points of nearly asymptotically nonexpansive mappings*, Journal of Nonlinear and Convex Analysis, 2007, 8(1), 61–79.

[3] K. Aoyama and F. Kohsaka, *Fixed point theorem for α-nonexpansive mappings in Banach spaces*, Nonlinear Anal., 2011, 74, 4387–4391.

[4] S. Banach, *Sur les operations dans les ensembles abstraits et leurs applications*, Fundam. Math., 1922, 3, 133–181.

[5] V. Berinde, *On the convergence of the Ishikawa iteration in the class of quasi contractive operators*, Acta Math. Univ. Comen., 2004, 73, 119–126.

[6] V. Berinde, *Picard iteration converges faster than Mann iteration for a class of quasicontractive operators*, Fixed Point Theory and Applications, 2004, 2, 97–105.

[7] F. E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proc. Nat. Acad. Sci. U.S.A., 1965, 54, 1041–1044.

[8] C. Byrne, *Iterative oblique projection onto convex sets and the split feasibility problem*, Inverse Probl., 2002, 18(2), 441–453.

[9] Y. Censor and T. Elfving, *A multiprojection algorithm using Bregman projections in a product space*, Numer. Algorithms, 1994, 8(24), 221–239.

[10] M. Feng, L. Shi and R. Chen, *A new three-step iterative algorithm for solving the split feasibility problem*, U. P. B. Sci. Bull., Series A, 2019, 81(1), 93–102.

[11] D. Göhde, *Zum Prinzip der kontraktiven abbildung. Math,* Nachr., 1965, 30, 251–258.

[12] N. Hussain, K. Ullah and M. Arshad, *Fixed point approximation for Suzuki generalized nonexpansive mappings via new iteration process*, Journal of Nonlinear and Convex Analysis, 2018, 19(8), 1383–1393.

[13] C. O. Imoru and M. O. Olantiwo, *On the stability of the Picard and Mann iteration processes*, Carpath. J. Math., 2003, 19, 155–160.

[14] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc., 1974, 44, 147–150.

[15] W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly, 1965, 72, 1004–1006.

[16] W. R. Mann, *Mean value methods in iteration*, Proc. Am. Math. Soc., 1953, 4, 506–510.

[17] M. A. Noor, *New approximation schemes for general variational inequalities*, Journal of Mathematical Analysis and Applications, 2000, 251(1), 217–229.

[18] R. Pant and R. Shukla, *Approximating Fixed Points of Generalized α-Nonexpansive Mappings in Banach Spaces*, Numerical Functional Analysis and Optimization, 2017, 38(2), 248–266.

[19] H. Piri, B. Daraby, S. Rahrovi and M. Ghasemi, *Approximating fixed points of generalized α-nonexpansive mappings in Banach spaces by new faster iteration process*, Numerical Algorithm, 2018.

[20] J. Schu, *Weak and strong convergence to fixed points of asymptotically nonexpansive mappings*, Bulletin of the Australian Mathematical Society, 1991, 43, 153–159.

[21] H. F. Senter and W. G. Dotson, *Approximating fixed points of nonexpansive mappings*, Proceedings of the American Mathematical Society, 1974, 44(2), 375–380.

[22] T. Suzuki, *Fixed point theorems and convergence theorems for some generalized non-expansive mapping*, J. Math. Anal. Appl., 2008, 340, 1088–1095.

[23] K. Tan and H. Xu, *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, Journal of Mathematical Analysis and Applications, 1993, 178, 301–308.

[24] B. S. Thakur, D. Thakur and M. Postolache, *A New iteration scheme for approximating fixed points of nonexpansive mappings*, Filomat, 2016, 30(10), 2711–2720.

[25] D. Thakur, B. S. Thakur and M. Postolache, *A new iterative scheme for numerical reckoning fixed points of Suzuki's generalized nonexpansive mappings*, Applied Mathematics and Computation, 2016, 275, 147–155.

[26] K. Ullah and M. Arshad, *New iteration process and numerical reckoning fixed points in Banach spaces*, U. P. B. Sci. Bull., Series A, 2017, 79(4), 113–122.

[27] K. Ullah and M. Arshad, *Numerical Reckoning Fixed Points for Suzuki's Generalized Nonexpansive Mappings via New Iteration Process*, Filomat, 2018, 32(1), 187–196.

[28] K. Ullah and M. Arshad, *New three step iteration process and fixed point approximation in Banach spaces*, Journal of Linear and Topological Algebra, 2018, 7(2), 87–100.

[29] H. Xu, *A variable Krasnosel'skii-Mann algorithm and the multiple-set split feasibility problem*, Inverse Probl., 2006, 22(6), 2021–2034.

[30] H. Xu, *Iterative methods for the split feasibility problem in infinite-dimensional Hilbert spaces*, Inverse Probl., 26, 105018 (2010), 17pp.

[31] T. Zamfirescu, *Fix point theorems in metric spaces*, Arch. Math., 1972, 23, 292–298.