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GLOBAL RELAXED MODULUS-BASED SYNCHRONOUS BLOCK MULTISPLITTING MULTI-PARAMETERS METHODS FOR LINEAR COMPLEMENTARITY PROBLEMS

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Abstract Recently, Bai and Zhang [Numerical Linear Algebra with Applications, 2013, 20, 425–439] constructed modulus-based synchronous multisplitting methods by an equivalent reformulation of the linear complementarity problem into a system of fixed-point equations and studied the convergence of them; Li et al. [Journal of Nanchang University (Natural Science), 2013, 37, 307–312] studied synchronous block multisplitting iteration methods; Zhang and Li [Computers and Mathematics with Application, 2014, 67, 1954–1959] analyzed and obtained the weaker convergence results for linear complementarity problems. In this paper, we generalize their algorithms and further study global relaxed modulus-based synchronous block multisplitting multi-parameters methods for linear complementarity problems. Furthermore, we give the weaker convergence results of our new method in this paper when the system matrix is a block H_+-matrix. Therefore, new results provide a guarantee for the optimal relaxation parameters, please refer to [A. Hadjidimos, M. Lapidakis and M. Tzoumas, SIAM Journal on Matrix Analysis and Applications, 2012, 33, 97–110, (dx.doi.org/10.1137/100811222)], where optimal parameters are determined.

Keywords Global relaxed modulus-based method, linear complementarity problem, block multisplitting, block H_+-matrix, synchronous multisplitting.

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1. Introduction

Consider the linear complementarity problem, abbreviated as LCP(q, A), for finding a pair of real vectors r and $z \in R^n$ such that

$$r := Az + q \geq 0, z \geq 0 \text{ and } \mathrm{z^T(Az+q)} = 0, \tag{1.1}$$

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where $A = (a_{ij}) \in R^{n\times n}$ is a given large, sparse and real matrix and $q = (q_1, q_2, ..., q_n)^T \in R^n$ is a given real vector. Here, z^T and $\geq$ denote the transpose of the vector z and the componentwise defined partial ordering between two vectors, respectively.

Many problems in scientific computing and engineering applications may lead to solutions of LCPs of the form (1.1). For example, the linear complementarity problem may arise from application problems such as the convex quadratic programming, the Nash equilibrium point of the bimatrix game, the free boundary problems of fluid dynamics etc. (e.g. see [15, 17] and the references therein). Some solvers for LCP(q, A) with a special matrix A were proposed [2–8, 14, 16, 20]. Recently, many people have focused the solver of LCP(q, A) with an algebra equation [7–9, 11–14, 16, 20, 29, 33–42]. In particular Bai proposed a modulus-based matrix multisplitting iteration method for solving LCP(q, A) and presented convergence analysis for the proposed methods; see [7, 8]. Zhang and Ren [33] extended the condition of a compatible $H-$splitting to that of an H-splitting. Li [27] extended the modulus-based matrix splitting iteration method to more general cases. Bai [10] presented parallel matrix block multisplitting relaxation iteration methods and established the convergence theory of these new methods in a thorough manner. Li et al. [28] studied synchronous block multisplitting iteration methods. Zhang and Li [35] generalized Bai and Zhang's methods [1] and studied modulus-based synchronous multisplitting multi-parameters methods for linear complementarity problems.

In this paper, we generalize the methods of Bai and Zhang's [1] and Zhang and Li's [35] from point form to block form according to the modulus-based synchronous multisplitting iteration methods and consider global relaxed modulus-based synchronous block multisplitting multi-parameters method for solving LCP(q, A). Moreover, we give some theoretical analysis and improve some existing convergence results in [1, 28].

The rest of this paper is organized as follows: In section 2, we give some notations and lemmas. In section 3, we propose global relaxed modulus-based synchronous block multisplitting multi-parameters method for solving LCP(q, A). In section 4, we give the convergence analysis for the proposed method.

2. Notations and Lemmas

In order to study mudulus-based synchronous block multisplitting iteration methods for solving LCP(q, A), let us introduce some definitions and lemmas.

A matrix $A = (a_{ij})$ is called an *M-matrix* if $a_{ij} \leq 0$ for $i \neq j$ and $A^{-1} \geq 0$. The *comparison matrix* $\langle A\rangle = (\alpha_{ij})$ of matrix $A = (a_{ij})$ is defined by: $\alpha_{ij} = |a_{ij}|$, if $i = j$; $\alpha_{ij} = -|a_{ij}|$, if $i \neq j$. A matrix A is called an *H-matrix* if $\langle A\rangle$ is an M-matrix and is called an *H_+-matrix* if it is an H-matrix with positive diagonal entries [29]. Let $\rho(A)$ denote the spectral radius of A. A representation $A = M - N$ is called a *splitting* of A when M is nonsingular. Let A and B be M-matrices. If $A \leq B$, then $A^{-1} \geq B^{-1}$. Finally, we define by $R^n_+ = \{x | x \geq 0, x \in R^n\}$.

Definition 2.1 ([10]). Define the set:
$(1) L_{n,I}(n_1, n_2, ..., n_p) = \{A = A_{ij} \in L_n(n_1, n_2, ..., n_p) | A_{ii} \in L(R^{n_i})$ is nonsingular $(i = 1, 2, ..., p)\}$;

(2) $L^d_{n,I}(n_1,n_2,...,n_p) = \{A = \text{diag}(A_{11},A_{22},...,A_{pp})|A_{ii} \in L(R^{n_i})$ is nonsingular $(i = 1,2,...,p)$

Definition 2.2 ([30]). Let $A \in L_{n,I}(n_1,n_2,...,n_p)$, and (I)-type block comparison matrix $\langle M\rangle = (\langle M\rangle_{ij}) \in L(R^n)$ and (II)-type block comparison matrix $\langle\langle M\rangle\rangle = (\langle\langle M\rangle\rangle_{ij}) \in L(R^n)$ are defined as

$$\langle M\rangle_{ij} = \begin{cases} \|M_{ii}^{-1}\|^{-1}, & i = j \\ -\|M_{ij}\|, & i \neq j \end{cases} \quad i,j = 1,2,...,p$$

$$\langle\langle M\rangle\rangle_{ij} = \begin{cases} 1, & i = j \\ -\|M_{ii}^{-1}M_{ij}\|, & i \neq j \end{cases} \quad i,j = 1,2,...,p$$

respectively.

Moreover, based on block matrix $A\in L_{n,I}(n_1,n_2,...,n_p)$ and $L\in L_{n,I}(n_1,n_2,...,n_p)$, let $D(L) = \text{diag}(L_{11},L_{22},...,L_{pp}), B(L) = D(L)-L, J(A) = D(A)^{-1}B(A), \mu_1(A) = \rho(J_{\langle A\rangle}), \mu_2(A) = \rho(I - \langle\langle A\rangle\rangle)$, using definition 2.2, then we easily verify

$$\langle I - J(A)\rangle = \langle\langle I - J(A)\rangle\rangle = \langle\langle A\rangle\rangle, \mu_2(A) \leq \mu_1(A).$$

Definition 2.3 ([30]). Let $A\in L_{n,I}(n_1,n_2,...,n_p)$, if there exist $P,Q\in L^d_{n,I}(n_1,n_2,...,n_p)$, such that $\langle PAQ\rangle$ is M-matrix, then A is called (I)-type block H-matrix($H_B^{(I)}(P,Q)$-matrix) about nonsingular block matrices P,Q; such that $\langle\langle PAQ\rangle\rangle$ is M-matrix, then A is called (II)-type block H-matrix($H_B^{(II)}(P,Q)$-matrix) about nonsingular block matrices P,Q.

Definition 2.4 ([30]). Let $A \in L_{n,I}(n_1,n_2,...,n_p)$, then $[A] = (\|M_{ij}\|) \in L(R^p)$ is called block absolute value of block matrix A. Similarly, we may define block absolute value of block vector $x \in V_n(n_1,n_2,...,n^p)$ as $[x] \in R^n$.

Lemma 2.5 ([10]). *Let $A,B \in L_{n,I}(n_1,n_2,...,n_p), x,y \in V_n(n_1,n_2,...,n_p), \gamma \in R^1$, then*

(1) $|[A]-[B]| \leq [A+B] \leq [A]+[B](|[x]-[y]| \leq [x+y] \leq [x]+[y])$;
(2) $[AB] \leq [A][B]([Ax] \leq [A][x])$;
(3) $[\gamma A] \leq |\gamma|[A](\gamma[x] \leq |\gamma|[x])$;
(4) $\rho(A) \leq \rho(|A|) \leq \rho([A])$.

Lemma 2.6([10]). *Let $A,B \in L_{n,I}(n_1,n_2,...,n_p)$ is $H_B^{(I)}(P,Q)$-matrix, then*

(1) A is nonsingular;
(2) $[(PAQ)^{-1}] \leq \langle PAQ\rangle^{-1}$;
(3) $\mu_1(PAQ) < 1$.

Lemma 2.7 ([10]). *Let $A,B \in L_{n,I}(n_1,n_2,...,n_p)$ is $H_B^{(II)}(P,Q)$-matrix, then*

(1) A is nonsingular;
(2) $[(PAQ)^{-1}] \leq \langle\langle PAQ\rangle\rangle^{-1}[D(PAQ)^{-1}]$;
(3) $\mu_2(PAQ) < 1$.

Definition 2.8 ([10]). Define the set:

(1) $\Omega_B^{(I)}(M) = \{F = (F_{ij}) \in L_{n,I}(n_1,n_2,...,n_p) \,|\, \| F_{ii}^{-1} \| = \| M_{ii}^{-1} \|, \| F_{ij} \| = \| M_{ij} \| \ (i \neq j), i,j = 1,2,...,p\}$;

(2) $\Omega_B^{(II)}(M) = \{F = (F_{ij}) \in L_{n,I}(n_1, n_2, ..., n_p) \mid \| F_{ii}^{-1}F_{ij} \| = \| M_{ii}^{-1}M_{ij} \|, i, j = 1, 2, ..., p\}$,
express the same mode set of (I)-type and (II)-type associated with the matrix $M = (M_{ij}) \in L_{n,I}(n_1, n_2, ..., n_p)$, respectively.

Lemma 2.9 ([18]). *Let A be an H-matrix. Then A is nonsingular, and $|A^{-1}| \leq \langle A \rangle^{-1}$.*

Lemma 2.10 ([32]). *Let $A = (a_{ij}) \in Z^{n\times n}$ which has all positive diagonal entries. A is an M-matrix if and only if $\rho(B) < 1$, where $B = D^{-1}C,\ D = \mathrm{diag}(A),\ A = D - C$.*

Lemma 2.11 ([4]). *$A \in R^{n\times n}$ be an H_+-matrix. Then, the LCP(q, A) has a unique solution for any $q \in R^n$.*

Lemma 2.12 ([7]). *Let $A = M - N$ be a splitting of the matrix $A \in R^{n\times n}, \Omega$ be a positive diagonal matrix, and γ a positive constant. Then, for the LCP(q, A) the following statements hold true:*

(i) if (z, r) is a solution of the LCP(q, A), then $x = \frac{1}{2}\gamma(z - \Omega^{-1}r)$ satisfies the implicit fixed-point equation

$$(\Omega + M)x = Nx + (\Omega - A)|x| - \gamma q; \tag{2.1}$$

(ii) if x satisfies the implicit fixed-point equation (2), then

$$z = \gamma^{-1}(|x| + x) \text{ and } r = \gamma^{-1}\Omega(|x| - x) \tag{2.2}$$

is a solution of the LCP(q, A).

3. GRMSBMMAOR methods

At first, we introduce the concept of multisplitting method and the detailed process of parallel iterative method.

$\{M_k, N_k, E_k\}_{k=1}^{l}$ is a *multisplitting* of block matrix A if
1) $A = M_k - N_k, \det(M_k) \neq 0$ is a splitting for $k = 1, 2, ..., l$;
2) $E_k = \mathrm{diag}(E_{11}^k, ..., E_{pp}^k), k = 1, 2, ..., l$, and $\sum_{k=1}^{l} \|E_{ii}^{(k)}\| = 1, i = 1, 2, ..., p$,
where the block matrices $M_k, N_k, E_k \in L_n(n_1, n_2, ..., n_p)$, and $\| \bullet \|$ expresses consistent matrix norm satisfying $\|I\| = 1$ ($I \in L(R^m)$ is an unit matrix).

Given a positive diagonal matrix Ω and a positive constant γ, form Lemma 2.13, we know that if x satisfies either of the implicit fixed-point equations

$$(\Omega + M_k)x = N_k x + (\Omega - A)|x| - \gamma q, k = 1, 2, ..., l, \tag{3.1}$$

then

$$z = \gamma^{-1}(|x| + x) \text{ and } r = \gamma^{-1}\Omega(|x| - x) \tag{3.2}$$

is a solution of the LCP(q, A).

Based on block matrix $A \in R^{m\times m}$, the corresponding block diagonal matrix is $D = \mathrm{diag}(A_{11}, A_{22}, ..., A_{pp})$, and L_k is block strictly triangular matrix, $U_k = D -$

$L_k - A$, then $(D - L_k, U_k, E_k)$ is a *multisplitting* of block matrix $A \in R^{m\times m}$. With the equivalent reformulations (4), (5) and accelerated over-relaxation (AOR) of the LCP(q, A), we can establish the following global relaxed modulus-based synchronous block multisplitting multi-parameters AOR method (GRMSBMMAOR), which is similar to Method 3.1 in [19] and Method 3.1 in [28].

Method 3.1 (The GRMSBMMAOR method for LCP(q, A))**.**
Let $(M_k, N_k, E_k)(k = 1, 2, ...l)$ be a multisplitting of the system matrix $A \in R^{n\times n}$. Given an initial vector $x^{(0)} \in R^n$ for $m = 0, 1, ...$ until the iteration sequence $\{z^{(m)}\}_{m=0}^{\infty} \subset R^n_+$ is convergent, compute $z^{(m+1)} \in R^n_+$ and $x^{(m+1)} \in R^n_+$ by

$$z^{(m+1)} = \frac{1}{\gamma}(|x^{(m+1)}| + x^{(m+1)})$$

and $x^{(m,k)} \in R^n$ according to

$$x^{(m+1)} = \omega \sum_{k=1}^{l} E_k x^{(m,k)} + (1-\omega)x^{(m)},$$

where $x^{(m,k)}, k = 1, 2, ..., l$, are obtained by solving the linear systems

$$(\alpha_k\Omega + D - \beta_k L_k)x^{(m,k)} = [(1-\alpha_k)D + (\alpha_k - \beta_k)L_k + \alpha_k U_k]x^{(m)} + \alpha_k[(\Omega - A)|x^{(m)}| - \gamma q], \quad k = 1, 2, ..., l, \tag{3.3}$$

respectively.

Remark 3.1. In Method 3.1, when the coefficient matrix A is point form and $\alpha_k = \alpha, \beta_k = \beta, \omega = 1$, the GRMSBMMAOR method reduces to the modulus-based synchronous multisplitting AOR method (MSMAOR) [1]; When the coefficient matrix A is point form and $\omega = 1$, the GRMSBMMAOR method reduces to the modulus-based synchronous multisplitting multi-parameters AOR method (MSMMAOR) [35]; When $\omega = 1$, the GRMSBMMAOR method reduces to the modulus-based synchronous block multisplitting multi-parameters AOR method (MSBMMAOR) [28]; When the parameters $(\alpha_k, \beta_k, \omega) = (\alpha_k, \alpha_k, 1), (1, 1, 1)$ and $(1, 0, 1)$, the GRMSBMMAOR method reduces to the modulus-based synchronous block multisplitting multi-parameters successive over-relaxation (MSBMMSOR), modulus-based synchronous block multisplitting Gauss-Seidel (MSBMGS) and modulus-based synchronous block multisplitting Jacobi (MSBMJ) methods, respectively; When the parameters $(\alpha_k, \beta_k, \omega) = (\alpha_k, \alpha_k, \omega), (1, 1, \omega)$ and $(1, 0, \omega)$, the GRMSBMMAOR method reduces to the global relaxed modulus-based synchronous block multisplitting multi-parameters successive over-relaxation (GRMSBMMSOR), global relaxed modulus-based synchronous block multisplitting G-S (GRMSBMMGS) and global relaxed modulus-based synchronous block multisplitting Jacobi (GRMSBMMJ) methods, respectively.

4. Convergence analysis

In 2013, based on the modulus-based synchronous multisplitting AOR method, Bai and Zhang [1] obtained the following results.

Theorem 4.1 ([1]). *Let $A \in R^{n\times n}$ be an H_+-matrix, with $D =$diag(A) and $B = D - A$, and let $(M_k, N_k, E_k)(k = 1, 2, ..., l)$ and $(D - L_k, U_k, E_k)(k = 1, 2, ..., l)$ be a multisplitting and a triangular multisplitting of the matrix A, respectively. Assume that $\gamma > 0$ and the positive diagonal matrix Ω satisfies $\Omega \geq D$. If $A = D - L_k - U_k(k = 1, 2, ..., l)$ satisfies $\langle A\rangle = D - |L_k| - |U_k|(k = 1, 2, ..., l)$, then the iteration sequence $\{z^{(m)}\}_{m=0}^{\infty}$ generated by the MSMAOR iteration method converges to the unique solution z_* of LCP(q, A) for any initial vector $z^{(0)} \in R^n_+$, provided the relaxation parameters α and β satisfy*

$$0 < \beta \leq \alpha < \frac{1}{\rho(D^{-1}|B|)}.$$

In 2013, based on the modulus-based synchronous block multisplitting AOR method, Li et al. [28] analyzed the following results.

Theorem 4.2([28]). *Let $A \in L_{n,I}(n_1, n_2, ..., n_p)$ be a block $H_B^{(I)}(P, Q)$-matrix, with $H \in \Omega_B^{(I)}(PAQ)$, and let $(\bar{M}_k, \bar{N}_k, E_k)(k = 1, 2, ..., l)$ and $(\bar{D} - \bar{L}_k, \bar{U}_k, E_k)(k = 1, 2, ..., l)$ be a block mulisplitting and a block triangular multisplitting of block H matrix, respectively. Assume that $\gamma > 0$ and the positive matrix Ω satisfies $\Omega \geq D(H)$ and* $\mathrm{diag}(\Omega) = \mathrm{diag}(D(H))$. *If $H = \bar{D} - \bar{L}_k - \bar{U}_k(k = 1, 2, ..., l)$ satisfies $\langle H\rangle = \langle \bar{D}\rangle - [\bar{L}_k] - [\bar{U}_k] = D_{\langle H\rangle} - B_{\langle H\rangle}(k = 1, 2, ..., l)$, then the iteration sequence $\{z^{(m)}\}_{m=0}^{\infty}$ generated by the MSBMAOR iteration method converges to the unique solution z_* of LCP(q, A) for any initial vector $z^{(0)} \in R^n_+$, provided the relaxation parameters α_k and β_k satisfy*

$$0 < \beta \leq \alpha < \frac{1}{\mu_1(PAQ)}.$$

In 2014, based on the modulus-based synchronous multisplitting multi-parameters AOR method, Zhang and Li [35] studied the following results.

Theorem 4.3 ([35]). *Let $A \in R^{n\times n}$ be an H_+-matrix, with $D =$diag(A) and $B = D - A$, and let $(M_k, N_k, E_k)(k = 1, 2, ..., l)$ and $(D - L_k, U_k, E_k)(k = 1, 2, ..., l)$ be a multisplitting and a triangular multisplitting of the matrix A, respectively. Assume that $\gamma > 0$ and the positive diagonal matrix Ω satisfies $\Omega \geq D$. If $A = D - L_k - U_k(k = 1, 2, ..., l)$ satisfies $\langle A\rangle = D - |L_k| - |U_k|(k = 1, 2, ..., l)$, then the iteration sequence $\{z^{(m)}\}_{m=0}^{\infty}$ generated by the MSMMAOR iteration method converges to the unique solution z_* of LCP(q, A) for any initial vector $z^{(0)} \in R^n_+$, provided the relaxation parameters α_k and β_k satisfy*

$$0 < \beta_k \leq \alpha_k \leq 1 \text{ or } 0 < \beta_k < \frac{1}{\rho(D^{-1}|B|)}, 1 < \alpha_k < \frac{1}{\rho(D^{-1}|B|)}.$$

Based global relaxed modulus-based synchronous block multisplitting multi-parameters AOR method, we will present a weaker convergence results of the multi-splitting methods for the linear complementarity problem when the system matrix is a block H_+-matrix, which is as follows:

Theorem 4.4. *Let $A \in L_{n,I}(n_1, n_2, ..., n_p)$ be a block $H_B^{(I)}(P, Q)$-matrix, with $H \in \Omega_B^{(I)}(PAQ)$, and let $(\bar{M}_k, \bar{N}_k, E_k)(k = 1, 2, ..., l)$ and $(\bar{D} - \bar{L}_k, \bar{U}_k, E_k)(k =$*

$1, 2, ..., l$) *be a block mulisplitting and a block triangular multisplitting of block H matrix, respectively. Assume that* $\gamma > 0$ *and the positive matrix* Ω *satisfies* $\Omega \geq D(H)$ *and* $\mathrm{diag}(\Omega) = \mathrm{diag}(D(H))$. *If* $H = \bar{D} - \bar{L}_k - \bar{U}_k (k = 1, 2, ..., l)$ *satisfies* $\langle H\rangle = \langle \bar{D}\rangle - [\bar{L}_k] - [\bar{U}_k] = D_{\langle H\rangle} - B_{\langle H\rangle}(k = 1, 2, ..., l)$, *then the iteration sequence* $\{z^{(m)}\}_{m=0}^{\infty}$ *generated by the GRMSBMMAOR iteration method converges to the unique solution* z_* *of* $LCP(q, A)$ *for any initial vector* $z^{(0)} \in R^n_+$, *provided the relaxation parameters* α_k *and* β_k *satisfy*

$$\begin{aligned} &0 < \beta_k \leq \alpha_k \leq 1, 0 < \omega < \tfrac{2}{1+\rho'} \text{ or} \\ &0 < \beta_k < \tfrac{1}{\mu_1(PAQ)}, 1 < \alpha_k < \tfrac{1}{\mu_1(PAQ)}, 0 < \omega < \tfrac{2}{1+\rho'}, \end{aligned} \tag{4.1}$$

where $\mu_1(PAQ) = \rho(D^{-1}_{\langle H\rangle} B_{\langle H\rangle}) = \rho(J_{\langle H\rangle}), \rho' = \max_{1\leq k\leq l}\{1 - 2\alpha_k + 2\alpha_k\rho_\epsilon, 2\beta_k\rho_\epsilon - 1, 2\alpha_k\rho_\epsilon - 1\}$.

Proof. From Lemma 2.11 and (3.3), for the GRMSBMMAOR method, it holds that

$$(\alpha_k\Omega + \bar{D} - \beta_k\bar{L}_k)x_* = [(1 - \alpha_k)\bar{D} + (\alpha_k - \beta_k)\bar{L}_k + \alpha_k\bar{U}_k]x_* + \alpha_k[(\Omega - H)|x_*| - \gamma q], \quad k = 1, 2, ..., l. \tag{4.2}$$

By subtracting (4.2) from (3.3), we have

$$\begin{aligned} x^{(m+1)} - x_* = &\omega \sum_{k=1}^{l} E_k(\alpha_k\Omega + \bar{D} - \beta_k\bar{L}_k)^{-1}[(1-\alpha_k)\bar{D} + (\alpha_k - \beta_k)\bar{L}_k + \alpha_k\bar{U}_k](x^{(m)} - x_*) \\ &+\omega \sum_{k=1}^{l} E_k(\alpha_k\Omega + \bar{D}v - \beta_k\bar{L}_k)^{-1}\alpha_k(\Omega - H)(|x^{(m)}| - |x_*|) + (1-\omega)(x^{(m)} - x_*). \end{aligned} \tag{4.3}$$

By taking absolute values on both sides of the equality (4.3), estimating $|[x^{(m)}] - [x_*]| \leq [x^{(m)} - x_*]$ and amplifying, we may obtain

$$\begin{aligned} [x^{(m+1)} - x_*] \leq \omega \sum_{k=1}^{l} &[E_k][(\alpha_k\Omega + \bar{D} - \beta_k\bar{L}_k)^{-1}][|1 - \alpha_k|[\bar{D}] + |\alpha_k - \beta_k|[\bar{L}_k] \\ &+\alpha_k[\bar{U}_k]][x^{(m)} - x_*] + \omega \sum_{k=1}^{l} [E_k][(\alpha_k\Omega + \bar{D} - \beta_k\bar{L}_k)^{-1}]\alpha_k[\Omega - H]([x^{(m)} - x_*]) \\ &+|1 - \omega|[x^{(m)} - x_*]. \end{aligned}$$

Since $[\Omega - H] = \langle\Omega\rangle - (D_{\langle H\rangle} - B_{\langle H\rangle})$ and $B_{\langle H\rangle} = [\bar{L}_k] + [\bar{U}_k], [\bar{D}] \geq D_{\langle H\rangle}$, so we have

$$\begin{aligned} [x^{(m+1)} - x_*] &\leq \omega \sum_{k=1}^{l} [E_k][(\alpha_k\Omega + \bar{D} - \beta_k\bar{L}_k)^{-1}][|1 - \alpha_k|[\bar{D}] + |\alpha_k - \beta_k|[\bar{L}_k] + \alpha_k[\bar{U}_k] \\ &\quad +\alpha_k[\Omega - H]][x^{(m)} - x_*] + |1 - \omega|[x^{(m)} - x_*] \\ &\leq \omega \sum_{k=1}^{l} [E_k](\alpha_k\langle\Omega\rangle + D_{\langle H\rangle} - \beta_k[\bar{L}_k])^{-1}[(|1 - \alpha_k| - \alpha_k)D_{\langle H\rangle} \\ &\quad +(|\alpha_k - \beta_k| + \alpha_k)[\bar{L}_k] + 2\alpha_k[\bar{U}_k] + \alpha_k\langle\Omega\rangle][x^{(m)} - x_*] + |1 - \omega|[x^{(m)} - x_*] \\ &= \omega \sum_{k=1}^{l} [E_k]H_{GRMSBMMAOR}[x^{(m)} - x_*] + |1 - \omega|[x^{(m)} - x_*] \\ &= \{\omega \sum_{k=1}^{l} [E_k]H_{GRMSBMMAOR} + |1 - \omega|I\}[x^{(m)} - x_*] \\ &= \mathcal{H}_{GRMSBMMAOR}[x^{(m)} - x_*]. \end{aligned} \tag{4.4}$$

where

$$\begin{aligned}\mathcal{H}_{GRMSBMMAOR} &= \omega\sum_{k=1}^{l}[E_k]H_{GRMSBMMAOR} + |1-\omega|I,\\ H_{GRMSBMMAOR} &= (\alpha_k\langle\Omega\rangle + D_{\langle H\rangle} - \beta_k[\bar{L}_k])^{-1}[(|1-\alpha_k|-\alpha_k)D_{\langle H\rangle}\\ &\quad +(|\alpha_k-\beta_k|+\alpha_k)[\bar{L}_k] + 2\alpha_k[\bar{U}_k] + \alpha_k\langle\Omega\rangle].\end{aligned} \tag{4.5}$$

The error relationship (4.4) is the base for proving the convergence of GRMSB-MMAOR method. By making use of Lemmas 2.5 and 2.6, defining $\epsilon^{(m)} = x^{(m)} - x_*$ and arranging similar terms together, we can obtain

$$\begin{aligned}[\epsilon^{(m+1)}] &= [x^{(m+1)} - x_*]\\ &\le \mathcal{H}_{GRMSBMMAOR}[x^{(m)} - x_*]\\ &= \{\omega\sum_{k=1}^{l}[E_k]H_{GRMSBMMAOR} + |1-\omega|I\}[x^{(m)} - x_*].\end{aligned} \tag{4.6}$$

Case 1: Let $0 < \beta_k \le \alpha_k \le 1, 0 < \omega < \frac{2}{1+\rho'}$. Define

$$\begin{aligned}M_k &= \alpha_k\langle\Omega\rangle + D_{\langle H\rangle} - \beta_k[\bar{L}_k],\\ N_k^1 &= (|1-\alpha_k|-\alpha_k)D_{\langle H\rangle} + (|\alpha_k-\beta_k|+\alpha_k)[\bar{L}_k] + 2\alpha_k[\bar{U}_k] + \alpha_k\langle\Omega\rangle\\ &= (1-2\alpha_k)D_{\langle H\rangle} + (2\alpha_k-\beta_k)[\bar{L}_k] + 2\alpha_k[\bar{U}_k] + \alpha_k\langle\Omega\rangle\\ &= M_k - 2\alpha_k D_{\langle H\rangle} + 2\alpha_k B_{\langle H\rangle}.\end{aligned} \tag{4.7}$$

So, we have

$$\begin{aligned}H_{GRMSBMMAOR} &= M_k^{-1}N_k^1 = M_k^{-1}(M_k - 2\alpha_k D_{\langle H\rangle} + 2\alpha_k B_{\langle H\rangle})\\ &= I - 2\alpha_k M_k^{-1}(D_{\langle H\rangle} - B_{\langle H\rangle}).\end{aligned}$$

Through further analysis, we have

$$\begin{aligned}[H_{GRRMSBMMAOR}] &\le M_k^{-1}[M_k - 2\alpha_k(D_{\langle H\rangle} - B_{\langle H\rangle})]\\ &\le I - 2\alpha_k M_k^{-1}D_{\langle H\rangle}(I - D_{\langle H\rangle}^{-1}B_{\langle H\rangle}).\end{aligned}$$

Since $A \in L_{n,I}(n_1, n_2, ..., n_p)$ is a block $H_B^{(I)}(P,Q)$-matrix, by Lemmas 2.6 and 2.7 we know $\mu_1(PAQ) = \rho(D_{\langle H\rangle}^{-1}B_{\langle H\rangle}) = \rho(J_{\langle H\rangle}) < 1, J_\epsilon = J_{\langle H\rangle} + \epsilon ee^T$, where e denotes the vector $e = (1,1,...,1)^T \in R^n$. Since J_ϵ is nonnegative, the matrix $J_{\langle H\rangle} + \epsilon ee^T$ has only positive entries and irreducible for any $\epsilon > 0$. By the Perron-Frobenius theorem for any $\epsilon > 0$, there is a vector $x_\epsilon > 0$ such that

$$(J_{\langle H\rangle} + \epsilon ee^T)x_\epsilon = \rho_\epsilon x_\epsilon,$$

where $\rho_\epsilon = \rho(J_{\langle H\rangle} + \epsilon ee^T) = \rho(J_\epsilon)$. Moreover, if $\epsilon > 0$ is small enough, we have $\rho_\epsilon < 1$ by continuity of the spectral radius. Because of $0 < \alpha_k \le 1$, we also have $1 - 2\alpha_k + 2\alpha_k\rho < 1$, and $1 - 2\alpha_k + 2\alpha_k\rho_\epsilon < 1$. So

$$\begin{aligned}[H_{GRRMSBMMAOR}] &\le I - 2\alpha_k M_k^{-1}D_{\langle H\rangle}[I - (D_{\langle H\rangle}^{-1}B_{\langle H\rangle} + \epsilon ee^T)]\\ &= I - 2\alpha_k M_k^{-1}D_{\langle H\rangle}[I - J_\epsilon].\end{aligned}$$

Multiplying x_ϵ in two sides of the above inequality, and $M_k^{-1} \geq D_{\langle H\rangle}^{-1}$, we can obtain

$$\begin{aligned}[H_{GRMSBMMAOR}]x_\epsilon &\leq x_\epsilon - 2\alpha_k M_k^{-1} D_{\langle H\rangle}[1-\rho(J_\epsilon)]x_\epsilon \\ &\leq x_\epsilon - 2\alpha_k D_{\langle H\rangle}^{-1} D_{\langle H\rangle}[1-\rho(J_\epsilon)]x_\epsilon \\ &= (1-2\alpha_k + 2\alpha_k\rho(J_\epsilon))x_\epsilon.\end{aligned}$$

Based on E_k and the definition of $[\bullet]$, we know that $\sum_{k=1}^{l}[E_k] = I$. By (4.5), we have

$$\begin{aligned}[\mathcal{H}_{GRMSMMAOR}]x_\epsilon &\leq \omega \sum_{k=1}^{l}[E_k](1-2\alpha_k+2\alpha_k\rho(J_\epsilon))x_\epsilon + |1-\omega|x_\epsilon \\ &\leq \omega(1-2\alpha_k+2\alpha_k\rho_\epsilon)x_\epsilon + |1-\omega|x_\epsilon \\ &\leq (\omega\rho' + |1-\omega|)x_\epsilon \\ &= \theta_1 x_\epsilon(\epsilon \to 0^+),\end{aligned}$$

where $\theta_1 = \omega\rho' + |1-\omega| < 1$.
Case 2: $0 < \beta_k < \frac{1}{\mu_1(PAQ)}, 1 < \alpha_k < \frac{1}{\mu_1(PAQ)}, 0 < \omega < \frac{2}{1+\rho'}$.
Subcase 1: $\alpha_k \geq \beta_k$. Define

$$\begin{aligned}N_k^2 &= (|1-\alpha_k| - \alpha_k)D_{\langle H\rangle} + (|\alpha_k - \beta_k| + \alpha_k)[\bar{L}_k] + 2\alpha_k[\bar{U}_k] + \alpha_k\langle\Omega\rangle \\ &= M_k - 2D_{\langle H\rangle} + 2\alpha_k B_{\langle H\rangle}.\end{aligned} \tag{4.8}$$

So

$$\begin{aligned}[H_{GRMSBMMAOR}] &\leq M_k^{-1}[M_k - 2(D_{\langle H\rangle} - \alpha_k B_{\langle H\rangle})] \\ &\leq I - 2M_k^{-1}D_{\langle H\rangle}(I - \alpha_k D_{\langle H\rangle}^{-1}B_{\langle H\rangle}).\end{aligned}$$

Similar to the *Case 1*, let e denote the vector $e = (1,1,...,1)^T \in R^n$, and $x_\epsilon > 0$ such that $J_\epsilon x_\epsilon = (J_{\langle H\rangle} + \epsilon ee^T)x_\epsilon = \rho(J_\epsilon)x_\epsilon$. Moreover, if $\epsilon > 0$ is small enough, we have $\rho_\epsilon < 1$ by continuity of the spectral radius. Because of $1 < \alpha_k < \frac{1}{\mu_1(PAQ)}$, we also have

$$2\alpha_k\rho - 1 < 1 \text{ and } 2\alpha_k\rho_\epsilon - 1 < 1.$$

So

$$\begin{aligned}[H_{GRMSBMMAOR}] &\leq I - 2M_k^{-1}D_{\langle H\rangle}[I - \alpha_k(D_{\langle H\rangle}^{-1}D_{\langle H\rangle} + \epsilon ee^T)] \\ &= I - 2M_k^{-1}D_{\langle H\rangle}[I - \alpha_k J_\epsilon].\end{aligned}$$

Multiplying x_ϵ in two sides of the above inequality, and $M_k^{-1} \geq D_{\langle H\rangle}^{-1}$, we can obtain

$$\begin{aligned}[H_{GRMSBMMAOR}]x_\epsilon &\leq x_\epsilon - 2M_k^{-1}D_{\langle H\rangle}[1 - \alpha_k\rho(J_\epsilon)]x_\epsilon \\ &\leq x_\epsilon - 2(1-\alpha_k\rho(J_\epsilon))x_\epsilon \\ &= (2\alpha_k\rho(J_\epsilon) - 1)x_\epsilon.\end{aligned}$$

Based on E_k and the definition of $[\bullet]$, we know that $\sum\limits_{k=1}^{l}[E_k] = I$. By (4.5), we have

$$\begin{aligned}[\mathcal{H}_{GRMSBMMAOR}]x_\epsilon &\leq \omega\sum_{k=1}^{l}[E_k](2\alpha_k\rho(J_\epsilon)-1)x_\epsilon + |1-\omega|x_\epsilon \\ &\leq \omega(2\alpha_k\rho_\epsilon - 1)x_\epsilon + |1-\omega|x_\epsilon \\ &\leq (\omega\rho^{'} + |1-\omega|)x_\epsilon \\ &= \theta_2 x_\epsilon(\epsilon \to 0^+),\end{aligned}$$

where $\theta_2 = \omega\rho^{'} + |1-\omega| < 1$.
Subcase 2: $\alpha_k < \beta_k$. Define

$$\begin{aligned}N_k^3 &= (|1-\alpha_k| - \alpha_k)D_{\langle H\rangle} + (|\alpha_k - \beta_k| + \alpha_k)[\bar{L}_k] + 2\alpha_k[\bar{U}_k] + \alpha_k\langle\Omega\rangle \\ &= M_k - 2D_{\langle H\rangle} + 2\beta_k[\bar{L}_k] + 2\alpha_k[\bar{U}_k] \\ &\leq M_k - 2D_{\langle H\rangle} + 2\beta_k B_{\langle H\rangle}.\end{aligned} \tag{4.9}$$

So

$$\begin{aligned}[H_{GRMSBMMAOR}] &\leq M_k^{-1}[M_k - 2(D_{\langle H\rangle} - \beta_k B_{\langle H\rangle})] \\ &\leq I - 2M_k^{-1}D_{\langle H\rangle}(I - \beta_k D_{\langle H\rangle}^{-1}B_{\langle H\rangle}).\end{aligned}$$

Similar to the *Case 1*, let e denote the vector $e = (1,1,...,1)^T \in R^n$, and $x_\epsilon > 0$ such that $J_\epsilon x_\epsilon = (J_{\langle H\rangle} + \epsilon ee^T)x_\epsilon = \rho(J_\epsilon)x_\epsilon$. Moreover, if $\epsilon > 0$ is small enough, we have $\rho_\epsilon < 1$ by continuity of the spectral radius. Because of $0 < \beta_k < \frac{1}{\mu_1(PAQ)}$, we also have

$$2\beta_k\rho - 1 < 1 \ \text{ and } \ 2\beta_k\rho_\epsilon - 1 < 1.$$

So

$$\begin{aligned}[H_{GRMSBMMAOR}] &\leq I - 2M_k^{-1}D_{\langle H\rangle}[I - \beta_k(D_{\langle H\rangle}^{-1}D_{\langle H\rangle} + \epsilon ee^T)] \\ &= I - 2M_k^{-1}D_{\langle H\rangle}[I - \beta_k J_\epsilon].\end{aligned}$$

Multiplying x_ϵ in two sides of the above inequality, and $M_k^{-1} \geq D_{\langle H\rangle}^{-1}$, we can obtain

$$\begin{aligned}[H_{GRMSBMMAOR}]x_\epsilon &\leq x_\epsilon - 2(1 - \beta_k\rho(J_\epsilon))x_\epsilon \\ &= (2\beta_k\rho(J_\epsilon) - 1)x_\epsilon.\end{aligned}$$

Based on E_k and the definition of $[\bullet]$, we know that $\sum\limits_{k=1}^{l}[E_k] = I$. By (11), we have

$$\begin{aligned}[\mathcal{H}_{GRMSBMMAOR}]x_\epsilon &\leq \omega\sum_{k=1}^{l}[E_k](2\beta_k\rho(J_\epsilon)-1)x_\epsilon + |1-\omega|x_\epsilon \\ &\leq \omega(2\beta_k\rho_\epsilon - 1)x_\epsilon + |1-\omega|x_\epsilon \\ &\leq (\omega\rho^{'} + |1-\omega|)x_\epsilon \\ &= \theta_3 x_\epsilon(\epsilon \to 0^+),\end{aligned}$$

where $\theta_3 = \omega\rho^{'} + |1-\omega| < 1$.

Remark 4.1. Obviously, from Figure 1, we can find that the conditions of Theorem 4.4 (when $\omega = 1$) in this paper are weaker than those of Theorem 1 in [23]. Moreover, the parameters can be adjusted suitably so that the convergence property of method can be substantially improved. That is to say, we have more choices for the splitting $A = B - C$ which makes the multisplitting iteration methods converge. Therefore, our convergence theories extend the scope of multisplitting iteration methods in applications.

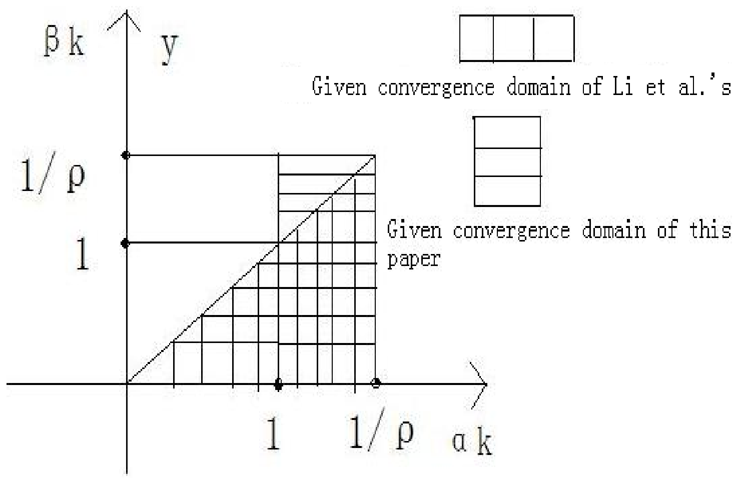

Figure 1. Comparison of convergence domains in Li et al.'s paper and in this paper. Here, $\rho = \mu_1(PAQ)$.

Based on the similar proving process of Theorem 4.4, we can obtain the following convergence results.

Theorem 4.5. *Let $A \in L_{n,I}(n_1, n_2, ..., n_p)$ be a block $H_B^{(II)}(P,Q)$-matrix, with $H \in \Omega_B^{(II)}(PAQ)$, and let $(\bar{M}_k, \bar{N}_k, E_k)(k = 1, 2, ..., l)$ and $(\bar{D} - \bar{L}_k, \bar{U}_k, E_k)(k = 1, 2, ..., l)$ be a block mulisplitting and a block triangular multisplitting of block H matrix, respectively. Assume that $\gamma > 0$ and the positive matrix Ω satisfies $\Omega \geq D(H)$ and $\mathrm{diag}(\Omega) = \mathrm{diag}(D(H))$. If $\langle\langle H\rangle\rangle = I - [\bar{D}^{-1}\bar{L}_k] - [\bar{D}^{-1}\bar{U}_k] = I - B_{\langle\langle H\rangle\rangle}(k = 1, 2, ..., l)$, then the iteration sequence $\{z^{(m)}\}_{m=0}^{\infty}$ generated by the GRMSBMMAOR iteration method converges to the unique solution z_* of LCP(q, A) for any initial vector $z^{(0)} \in R_+^n$, provided the relaxation parameters α_k and β_k satisfy*

$$\begin{aligned} &0 < \beta_k \leq \alpha_k \leq 1, 0 < \omega < \tfrac{2}{1+\rho^{'}} \text{ or} \\ &0 < \beta_k < \tfrac{1}{\mu_2(PAQ)}, 1 < \alpha_k < \tfrac{1}{\mu_2(PAQ)}, 0 < \omega < \tfrac{2}{1+\rho^{'}}, \end{aligned} \tag{4.10}$$

where $\mu_2(PAQ) = \rho(J_{\langle\langle H\rangle\rangle}), \rho^{'} = \max_{1\leq k\leq l}\{1 - 2\alpha_k + 2\alpha_k\rho_\epsilon, 2\beta_k\rho_\epsilon - 1, 2\alpha_k\rho_\epsilon - 1\}$.

Remark 4.2. From Table 1, we obviously see that the MSMMAOR method in [1] and the MSBMAOR method in [28] use the same parameters α, β in different processors, but the GRMSBMMAOR method in this paper uses different parameters

$\alpha_k, \beta_k(k = 1, 2, ..., l)$ in different processors. Moreover, when computing $x^{(m+1)}$ in Method 3.1, we utilize relaxation extrapolation technique and add a relaxation parameter ω. Therefore, we may choose proper relaxation parameters to increase convergence speed and reduce the computation time when doing numerical experiments. On the other hand, the convergence results in [1] and [28] are $0 < \beta \le \alpha < \frac{1}{\rho(D^{-1}|B|)}$ and $0 < \beta \le \alpha < \frac{1}{\mu_1(PAQ)}$, respectively, but the convergence results in this paper are $0 < \beta_k \le \alpha_k \le 1, 0 < \omega < \frac{2}{1+\rho'}$ or $0 < \beta_k < \frac{1}{\mu_1(PAQ)}, 1 < \alpha_k < \frac{1}{\mu_1(PAQ)}, 0 < \omega < \frac{2}{1+\rho'}$, where $\rho' = \max_{1\le k\le l}\{1-2\alpha_k+2\alpha_k\rho_\epsilon, 2\beta_k\rho_\epsilon-1, 2\alpha_k\rho_\epsilon-1\}$. So, our method is not only the generalization of MSMAOR and MSBMAOR methods, but also convergence results of new method are weaker than those of Bai and Zhang's [1] and Li et al.'s [28]. In GRMSBMMAOR method, we may choose proper E_k to balance the load of each processor and avoid synchronization.

Table 1. The global relaxed modulus-based synchronous (block) multisplitting multi-parameters method and corresponding convergence results.

Method	$\alpha_k, \beta_k, \omega$	Description	Ref
MSMJ	$\alpha_k = 1, \beta_k = 0, \omega = 1$	Modulus-based synchronous multisplitting Jacobi method	[1]
MSMGS	$\alpha_k = \beta_k = 1, \omega = 1$	Modulus-based synchronous multisplitting Gauss-Seidel method	[1]
MSMSOR	$0 < \alpha(\alpha_k) = \beta(\beta_k) < \frac{1}{\rho(D^{-1}\vert B\vert)}, \omega = 1$	Modulus-based synchronous multisplitting SOR method	[1]
MSMAOR	$0 < \beta(\beta_k) \le \alpha(\alpha_k) < \frac{1}{\rho(D^{-1}\vert B\vert)}, \omega = 1$	Modulus-based synchronous multisplitting AOR method	[1]
GRMSMMAOR	$0 < \beta_k \le \alpha_k \le 1, 0 < \omega < \frac{2}{1+\rho'}$ or $0 < \beta_k < \frac{1}{\rho(D^{-1}\vert B\vert)}, 1 < \alpha_k < \frac{1}{\rho(D^{-1}\vert B\vert)}$ $0 < \omega < \frac{2}{1+\rho'}$ where $\rho' = \max_{1\le k\le l}\{1-2\alpha_k+2\alpha_k\rho_\epsilon,$ $2\alpha_k\rho_\epsilon-1, 2\alpha_k\rho_\epsilon-1\}$	Global relaxed modulus-based synchronous multisplitting multi-parameters AOR method	[32]
MBRI	$0 < \beta < \frac{1}{\mu_1(PAQ)}, \omega = 1$	Parallel matrix block multisplitting relaxation iteration method	[10]
MSBMAOR	$0 < \beta \le \alpha < \frac{1}{\mu_1(PAQ)}, \omega = 1$	Modulus-based synchronous block multisplitting AOR method	[25]
GRMSBMMAOR	$0 < \beta_k \le \alpha_k \le 1, 0 < \omega < \frac{2}{1+\rho'}$ or $0 < \beta_k < \frac{1}{\mu_1(PAQ)}, 1 < \alpha_k < \frac{1}{\mu_1(PAQ)}$ $0 < \omega < \frac{2}{1+\rho'}$ where $\rho' = \max_{1\le k\le l}\{1-2\alpha_k+2\alpha_k\rho_\epsilon,$ $2\beta_k\rho_\epsilon-1, 2\alpha_k\rho_\epsilon-1\}$	Global relaxed modulus-based synchronous block multisplitting multi-parameters AOR method	this paper

The authors declare that they have no competing interests.

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