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GLOBAL SMOOTH SOLUTION FOR THE COMPRESSIBLE LANDAU-LIFSHITZ-BLOCH EQUATION*

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Abstract The Landau-Lifshitz-Bloch equation is often used to describe micromagnetic phenomenon under high temperature. In this paper, we establish the existence and uniqueness of global smooth solution for the initial problem of the compressible Landau-Lifshitz-Bloch equation in dimension one.

Keywords Landau-Lifshitz-Bloch equation, compressible model, global solution.

MSC(2010) 35K15, 35Q60, 82D40.

1. Introduction

As well known, the Landau–Lifshitz equation well describes the magnetization dynamics of ferromagnets at low temperature. It is famous and many important results has been obtained, see [7]. In order to describe the magnetization dynamics in a ferromagnetic body for a wide range of temperatures, Garanin [5, 6] derived the Landau-Lifshitz-Bloch equation from statistical mechanics with the mean field approximation in 1990s. The Landau-Lifshitz-Bloch equation is able to rule the time evolution of both the direction and the modulus of the vector Z, and has been recently applied to simulations of FePtin [9, 14]. It generalizes the well-known Landau-Lifshitz equation.

The Landau–Lifshitz–Bloch equation is given as follows

$$Z_t = -\gamma Z \times H^{\text{eff}} + \frac{L_1}{|Z|^2}(Z \cdot H^{\text{eff}})Z - \frac{L_2}{|Z|^2} Z \times (Z \times H^{\text{eff}}), \tag{1.1}$$

where $Z(x,t) = (Z_1(x,t), Z_2(x,t), Z_3(x,t))$ is a magnetization functional vector, $\gamma,\ L_1,\ L_2$ are constants, "$\times$" denotes the vector outer product, H^{eff} is the effective field. We can also rewrite (1.1) as follows

$$Z_t = -\gamma Z \times H^{\text{eff}} + \frac{\gamma a_\parallel}{|Z|^2}(H^{\text{eff}} \cdot Z)Z - \frac{\gamma a_\perp}{|Z|^2} Z \times (Z \times H^{\text{eff}})$$

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with $\gamma a_{\|} = L_1$, $\gamma a_{\perp} = L_2$. Here $a_{\|}$ and $a_{\perp}$ are dimensionless damping parameters that depend on the temperature and are defined as follows [1]

$$a_{\|}(\theta) = \frac{2\theta}{3\theta_c}\lambda, \quad a_{\perp}(\theta) = \begin{cases} \lambda\left(1 - \dfrac{\theta}{3\theta_c}\right), & \text{if } \theta < \theta_c, \\ a_{\|}(\theta), & \text{if } \theta \geq \theta_c, \end{cases}$$

where $\lambda > 0$ is a constant.

The effective field H^{eff} is given by

$$H^{\text{eff}} = \Delta Z - \frac{1}{\chi_{\|}}\left(1 + \frac{3}{5}\frac{T}{T - T_c}|Z|^2\right)Z, \tag{1.2}$$

where $\chi_{\|}$ is the longitudinal susceptibility, if $L_1 = L_2$, (1.1) can be reduced as follows [10]

$$Z_t = k_1 \Delta Z + \gamma Z \times \Delta Z - k_2(1 + \mu|Z|^2)Z, \tag{1.3}$$

where the coeffcients $k_1, k_2, \gamma, \mu > 0$ and the existence of global weak solution for the equation (1.3) has been obtained.

In 1982, Fivez derived the classical compressible Heisenberg chain equation

$$Z_t = \big(G(Z_x)Z \times Z_x\big)_x, \ x \in \mathbb{R}^1, \tag{1.4}$$

where $G(\xi) = A + B|\xi|^2$ and $A, B > 0$ are constants. Magyari obtained the solutions of equation (1.4) with $B = 0$ in [13]. When $B = 0, A = g(x)$ is some given function, the existence and uniqueness of the smooth solution of (1.4) were obtained in [15] with $g(x) \equiv 1$ and $g(x) \neq$ constant in [3]. The existence of measure-valued solution of the equation was establish in [2].

In this paper, we will consider the following generalized compressible Landau–Lifshitz–Bloch equation

$$Z_t = k_1 Z_{xx} + \big(G(Z)Z \times Z_x\big)_x - k_2(1 + \mu|Z|^2)Z \tag{1.5}$$

with the initial value

$$Z(x, 0) = Z_0(x), \ x \in \mathbb{R}^1, \tag{1.6}$$

where $G(\xi) = A + B|\xi|^q, q \geq 2$ and $A, B > 0$ are constants, which can be viewed as a generalization of Landau–Lifshitz–Bloch equation and compressible Heisenberg chain equation, no one has discussed the compressible LLB equation at present, we will concentrate on the existence and uniqueness of global smooth solution for the initial problem (1.5)-(1.6). Developing and extending the methods of [4, 8, 11], we obtain the following theorem:

Theorem 1.1. *Let the initial data $Z_0(x) \in H^m (m \geq 2)$, $k > 0, \mu > 0$, then the problem* (1.5)-(1.6) *admits a global smooth solution satisfying*

$$\partial_t^j \partial_x^\alpha Z \in L^\infty([0, T]; L^2(\mathbb{R}^1)), \ \partial_t^j \partial_x^\beta Z \in L^2([0, T]; L^2(\mathbb{R}^1)),$$

where $2j + |\alpha| \leq m$ and $2k + \beta \leq m + 1$.

The rest of this paper is as follows. In section 2, the proof of smooth local solution of (1.5)-(1.6) is proved in $\mathbb{R}^1$, a priori uniform estimates in H^m is established, and the existence and uniqueness of the global smooth solution of (1.5)-(1.6) is proved in $\mathbb{R}^1$.

2. The proof of Theorem 1.1

From [12] it can be shown that there exists $T > 0$ and a smooth solution of problem (1.5)-(1.6) in $[0, T]$. Indeed it is easy to check that $e^{tk_1\Delta}$ is a analytic semigroup generated by $k_1\Delta$ in $L^2(\mathbb{R}^1)$, let

$$X = \{Z | Z \in C([0,T]; H^m(\mathbb{R}^1)),\ t^\alpha Z \in C^\alpha([0,T]; H^m(\mathbb{R}^1)),\ Z(0) = Z_0\}$$

and

$$Y = \{Z | Z \in X,\ \|Z\|_{C([0,T];H^m(\mathbb{R}^1))} + [t^\alpha Z]_{C^\alpha([0,T];H^m(\mathbb{R}^1))} \le \delta\},$$

where $0 < \alpha < 1$ and $m \ge 2$. Define a nonlinear operator Γ on Y by $\Gamma(Z) = v$, where v is the solution of

$$v_t = k_1 v_{xx} + \big(G(Z)Z \times Z_x\big)_x - k(1 + \mu|Z|^2)Z.$$

By Theorem 4.3.5 of Reference [12] (pp.137-139), for every $Z \in Y, \Gamma(Z) \in C([0,T];$ $H^m(\mathbb{R}^1))$ and $t^\alpha\Gamma(Z) \in C^\alpha([0,T];\ H^m(\mathbb{R}^1))$, then, using the same arguments as in the proof of Theorem 8.1.1, there exists $T > 0$ and $\delta > 0$ such that $\Gamma : Y \to Y$ is a contraction, there exists a unique smooth local solution of problem (1.5)-(1.6).

The following Gagliardo-Nirenberg inequality will be used many times.

Lemma 2.1. *(Gagliardo-Nirenberg Inequality) Assume that* $u \in L^q(\Omega), D^m u \in L^r(\Omega)$, $\Omega \subset \mathbb{R}^n$, $1 \le q, r \le \infty$, $0 \le j \le m$. *Then*

$$\|D^j u\|_{L^p(\Omega)} \le C(j, m; p, r, q)\|u\|^a_{W^m_r(\Omega)}\|u\|^{1-a}_{L^q(\Omega)}, \tag{2.1}$$

where $C(j, m; p, r, q)$ *is a positive constant, and*

$$\frac{1}{p} = \frac{j}{n} + a(\frac{1}{r} - \frac{m}{n}) + (1 - a)\frac{1}{q}, \quad \frac{j}{m} \le a \le 1.$$

For simplicity, we denote

$$\|\cdot\|_{L^p(\mathbb{R}^1)} = \|\cdot\|_p, p \ge 2.$$

Lemma 2.2. *Assume that the initial data* $Z_0(x) \in H^m (m \ge 1)$, *for the smooth solution of problem* (1.5)-(1.6), *we have*

$$\|Z(\cdot,t)\|_2^2 + \int_0^t \|Z_x(\cdot,t)\|_2^2 dt \le C, \tag{2.2}$$

$$\|Z_x(\cdot,t)\|_2^2 + \int_0^t \|Z_{xx}(\cdot,t)\|_2^2 dt \le C. \tag{2.3}$$

Proof. Taking the scalar product of Z with equation (1.5), and integrating the result over $\mathbb{R}^1$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|Z(\cdot,t)\|_2^2 + k_1\|Z_x(\cdot,t)\|_2^2 = &- \int_{\mathbb{R}^1} G(Z)Z \times Z_x \cdot Z_x dx \\ &- k_2\int_{\mathbb{R}^1}(1 + \mu|Z|^2)|Z|^2 dx,\end{aligned}$$

then by the fact $A \times B \cdot B = 0, A, B$ are vectors, we find

$$\frac{1}{2}\frac{d}{dt}\|Z(\cdot,t)\|_2^2 + k_1\|Z_x(\cdot,t)\|_2^2 + k_2\int_{\mathbb{R}^1}(1+\mu|Z|^2)|Z|^2dx = 0.$$

Since $k_1 > 0, k_2 > 0, \mu > 0$, we get

$$\|Z(\cdot,t)\|_2^2 + \int_0^t \|Z_x(\cdot,t)\|_2^2 dt \le \|Z(\cdot,0)\|_2^2. \tag{2.4}$$

Taking the scalar product of $|Z|^{p-2}Z (p \ge 2)$ with equation (1.5), and integrating the result over $\mathbb{R}^1$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^1}|Z|^{p-2}Z\cdot Z_t dx\\
=&\int_{\mathbb{R}^1}|Z|^{p-2}Z\cdot Z_{xx}dx + \int_{\mathbb{R}^1}|Z|^{p-2}Z\cdot\big(G(Z)Z\times Z_x\big)_x dx\\
&-k_2\int_{\mathbb{R}^1}|Z|^{p-2}\cdot(1+\mu|Z|^2)Z^2dx\\
\le&-\int_{\mathbb{R}^1}|Z|^{p-2}Z_x\cdot Z_x dx - (p-2)\int_{\mathbb{R}^1}|Z|^{p-4}(Z\cdot Z_x)^2dx\\
\le&0,
\end{aligned}$$

so we have

$$\frac{1}{p}\frac{d}{dt}\|Z(\cdot,t)\|_{L^p}^p \le 0$$

which implies

$$\|Z(\cdot,t)\|_{L^p} \le \|Z_0(x)\|_{L^p}, \tag{2.5}$$

let $p \to \infty$, we have

$$\|Z(\cdot,t)\|_{L^\infty} \le \|Z_0(x)\|_{L^\infty},\ \ \forall t \ge 0. \tag{2.6}$$

Differentiating (1.5) with respect to x and multiplying it by Z_x, we have

$$Z_{xt}Z_x = k_1 Z_{xxx}Z_x + \big(G(Z)Z\times Z_x\big)_{xx}Z_x - k_2\big((1+\mu|Z|^2)Z\big)_x Z_x. \tag{2.7}$$

Integrating the result over $\mathbb{R}^1$, we find

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|Z_x\|_2^2 + k_1\|Z_{xx}\|_2^2\\
=&-\int_{\mathbb{R}^1}\big(G(Z)Z\times Z_x\big)_x Z_{xx}dx - \int_{\mathbb{R}^1}k_2\big((1+\mu|Z|^2)Z\big)_x Z_x dx\\
=&-\int_{\mathbb{R}^1}qB|Z|^{q-2}(Z\cdot Z_x)Z\times Z_x\cdot Z_{xx}dx\\
&-\int_{\mathbb{R}^1}k_2\big(2\mu Z^2Z_x^2 + (1+\mu|Z|^2)Z_x^2\big)dx\\
\le&C\|Z\|_\infty^q\|Z_x\|_4^2\|Z_{xx}\|_2 + C(\|Z\|_\infty^2+1)\|Z_x\|_2^2
\end{aligned}$$

$$\leq \frac{1}{2}\|Z_{xx}\|_2^2 + C(\|Z_x\|_2^4 + \|Z_x\|_2^2).$$

The generalized Gronwall's inequality says that if $f' = C(f \cdot g) + C$, $f \leq C \exp\big(\int_0^t g dt\big) + C$, by replacing f and g by $\|Z_x\|_2^2$ and $\|Z_x\|_2^2$ respectively, and the boundedness of $\int_0^t g dt$ from (2.4), we have the estimate (2.3). Thus the lemma is proved. □

Lemma 2.3. *Assume that the initial data $Z_0(x) \in H^m (m \geq 2)$, for the smooth solution of problem* (1.5)-(1.6), *we have the following estimate*

$$\|Z_{xx}(\cdot, t)\|_2^2 + \int_0^t \|Z_{xxx}(\cdot, t)\|_2^2 dt \leq C, \tag{2.8}$$

where the constant C depends on $k_1, k_2, \mu, \|Z_0(x)\|_2^2$.

Proof. Taking the scalar product of Z_{xxxx} with equation (1.5), and integrating the result over $\mathbb{R}^1$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|Z_{xx}\|_2^2 - k_1 \int_{\mathbb{R}^1} Z_{xx} Z_{xxxx} dx \\
=&\frac{1}{2}\frac{d}{dt}\|Z_{xx}\|_2^2 + \|Z_{xxx}\|_2^2 \\
=&-\int_{\mathbb{R}^1} \big(G(Z)Z \times Z_x\big)_{xx} Z_{xxx} dx - \int_{\mathbb{R}^1} k_2\big((1+\mu|Z|^2)Z\big) Z_{xxxx} dx,
\end{aligned} \tag{2.9}$$

where

$$\begin{aligned}
&\int_{\mathbb{R}^1} \big(G(Z)Z \times Z_x\big)_{xx} Z_{xxx} dx \\
=&\int_{\mathbb{R}^1} Bq(q-2)|Z|^{q-4}(Z \cdot Z_x)^2 Z \times Z_x \cdot Z_{xxx} dx \\
&+\int_{\mathbb{R}^1} Bq|Z|^{q-2}(Z_x^2 + Z \cdot Z_{xx}) Z \times Z_x \cdot Z_{xxx} dx \\
&+\int_{\mathbb{R}^1} 2Bq|Z|^{q-2}(Z \cdot Z_x) Z \times Z_{xx} \cdot Z_{xxx} dx \\
&+\int_{\mathbb{R}^1} G(Z) Z_x \times Z_{xx} \cdot Z_{xxx} dx \\
\leq& C\|Z\|_\infty^{q-1}\|Z_x\|_6^3\|Z_{xxx}\|_2 + C\|Z\|_\infty^q \|Z_x\|_4 \|Z_x\|_4 \|Z_{xxx}\|_2 \\
&+ C(1 + \|Z\|_\infty^q)\|Z_x\|_4 \|Z_{xx}\|_4 \|Z_{xxx}\|_2.
\end{aligned}$$

By Gagliardo-Nirenlerg inequality, we get

$$\begin{aligned}
&\|Z_x\|_\infty \leq C\|Z_x\|_2^{\frac{3}{4}}\|Z_{xxx}\|_2^{\frac{1}{4}}, \ \|Z_x\|_4 \leq C\|Z_x\|_2^{\frac{3}{4}}\|Z_{xx}\|_2^{\frac{1}{4}}, \\
&\|Z_x\|_6 \leq C\|Z_x\|_2^{\frac{5}{6}}\|Z_{xxx}\|_2^{\frac{1}{6}}, \ \|Z_{xx}\|_4 \leq C\|Z_{xx}\|_2^{\frac{3}{4}}\|Z_{xxx}\|_2^{\frac{1}{4}},
\end{aligned}$$

so we have

$$\begin{aligned}
&\int_{\mathbb{R}^1} \big(G(Z)Z \times Z_x\big)_{xx} Z_{xxx} dx \\
\leq& C\|Z\|_\infty^{q-1}\|Z_x\|_2^{\frac{5}{2}}\|Z_{xxx}\|_2^{\frac{3}{2}} + C\|Z\|_\infty^q \|Z_x\|_2^{\frac{3}{4}}\|Z_{xx}\|_2 \|Z_{xxx}\|_2^{\frac{5}{4}}
\end{aligned}$$

$$
\begin{aligned}
&+C(1+\|Z\|_\infty^q)\|Z_x\|_2^{\frac{3}{4}}\|Z_{xx}\|_2\|Z_{xxx}\|_2^{\frac{5}{4}}\\
\leq& C(\|Z_{xx}\|_2^4)+\frac{1}{4}\|Z_{xxx}\|_2^2+C.
\end{aligned}
$$

By the Hölder inequality, it follows that

$$
\begin{aligned}
&-\int_{\mathbb{R}^1} k_2\big((1+\mu|Z|^2)Z\big)Z_{xxxx}dx\\
=&k_2\int_{\mathbb{R}^1}\big((1+\mu|Z|^2)Z\big)_x Z_{xxx}dx\\
=&k_2\int_{\mathbb{R}^1}(2\mu(Z\cdot Z_x)Z+(1+\mu|Z|^2)Z_x)Z_{xxx}dx\\
\leq& C\|Z\|_\infty^2\|Z_x\|_2\|Z_{xxx}\|_2\\
\leq&\frac{1}{4}\|Z_{xxx}\|_2^2+C,
\end{aligned}
$$

then

$$
\frac{d}{dt}\|Z_{xx}\|_2^2+\|Z_{xxx}\|_2^2\leq C\|Z_{xx}\|_2^4+C, \tag{2.10}
$$

by Gronwall's inequality, the (2.8) holds. □

Lemma 2.4. *Assume that the initial data $Z_0(x)\in H^m(m\geq 2)$, then for the smooth solution of problem* (1.5)-(1.6)*, we have the following estimate*

$$
\|Z_{xxx}(\cdot,t)\|_2^2+\int_0^t\|Z_{xxxx}(\cdot,t)\|_2^2dt\leq C \tag{2.11}
$$

where the constant C depends on $k_1,k_2,\mu,\|Z_0(x)\|_2^2$.

Proof. Taking the scalar product of Z_{xxxxxx} with equation (1.5) and integrating the result over $\mathbb{R}^1$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|Z_{xxx}\|_2^2-k_1\int_{\mathbb{R}^1}Z_{xxxxx}Z_{xxx}dx\\
=&\int_{\mathbb{R}^1}\big(G(Z)Z\times Z_x\big)_{xxxx}Z_{xxx}dx-\int_{\mathbb{R}^1}k_2\big((1+\mu|Z|^2)Z\big)_{xxx}Z_{xxx}dx\\
=&-\int_{\mathbb{R}^1}\big(G(Z)Z\times Z_x\big)_{xxx}Z_{xxxx}dx+\int_{\mathbb{R}^1}k_2\big((1+\mu|Z|^2)Z\big)_{xx}Z_{xxxx}dx,
\end{aligned}\tag{2.12}
$$

where

$$
\begin{aligned}
&\int_{\mathbb{R}^1}\big(G(Z)Z\times Z_x\big)_{xxx}Z_{xxxx}dx\\
=&\int_{\mathbb{R}^1}\Big\{2G_{xxx}(Z)Z\times Z_x+3G_{xx}(Z)Z\times Z_{xx}+3G_x(Z)Z_x\times Z_{xx}\\
&+3G_x(Z)Z\times Z_{xxx}+2G(Z)Z_x\times Z_{xxx}+G(Z)Z\times Z_{xxxx}\Big\}Z_{xxxx}dx\\
=&\int_{\mathbb{R}^1}\Big\{Bq(q-2)(q-4)|Z|^{q-6}(Z\cdot Z_x)^3
\end{aligned}
$$

$$
\begin{aligned}
&+3Bq(q-2)|Z|^{q-4}(Z\cdot Z_x)(Z_x^2+Z\cdot Z_{xx})\\
&+Bq|Z|^{q-2}(3Z_x\cdot Z_{xx}+Z\cdot Z_{xxxx})\Big\}Z\times Z_x\cdot Z_{xxxx}dx\\
&+\int_{\mathbb{R}^1}\Big\{Bq(q-2)|Z|^{q-4}(Z\cdot Z_x)^2\\
&+Bq|Z|^{q-2}(Z_x^2+Z\cdot Z_{xx})\Big\}Z\times Z_{xx}\cdot Z_{xxxx}dx\\
&+\int_{\mathbb{R}^1}3qB|Z|^{q-2}(Z\cdot Z_x)Z_x\times Z_{xx}\cdot Z_{xxxx}dx\\
&+\int_{\mathbb{R}^1}3qB|Z|^{q-2}(Z\cdot Z_x)Z\times Z_{xxx}\cdot Z_{xxxx}dx\\
&+\int_{\mathbb{R}^1}2G(Z)Z_x\times Z_{xxx}\Big\}Z_{xxxx}dx\\
\le& C\|Z\|_\infty^{q-2}\|Z_x\|_\infty^4\|Z_{xxxx}\|_2+C\|Z\|_\infty^{q-1}\|Z_x\|_\infty^2\|Z_{xx}\|_2\|Z_{xxxx}\|_2\\
&+C\|Z\|_\infty^q\|Z_x\|_\infty^2\|Z_{xxx}\|_2\|Z_{xxxx}\|_2\\
&+C(\|Z\|_\infty^{q-1}\|Z_x\|_\infty^2\|Z_{xx}\|_2+\|Z\|_\infty^q\|Z_{xx}\|_4^2)\|Z_{xxxx}\|_2\\
&+C\|Z\|_\infty^q\|Z_x\|_\infty\|Z_{xx}\|_2\|Z_{xxxx}\|_2\\
&+C(1+\|Z\|_\infty^q)\|Z_x\|_\infty\|Z_{xx}\|_2\|Z_{xxx}\|_2\|Z_{xxxx}\|_2.
\end{aligned}
$$

By Gagliardo-Nirenlerg inequality, we get

$$
\|Z_x\|_\infty\le C\|Z_x\|_2^{\frac34}\|Z_{xxx}\|_2^{\frac14},\ \|Z_{xx}\|_4\le C\|Z_{xx}\|_2^{\frac34}\|Z_{xxx}\|_2^{\frac14},
$$

thus,

$$
\begin{aligned}
&\int_{\mathbb{R}^1}\big(G(Z)Z\times Z_x\big)_{xxx}Z_{xxxx}dx\\
\le& C\|Z_x\|_2^3\|Z_{xxx}\|_2\|Z_{xxxx}\|_2+C\|Z_x\|_2^{\frac32}\|Z_{xxx}\|_2^{\frac12}\|Z_{xx}\|_2\|Z_{xxxx}\|_2\\
&+C\|Z_x\|_2^{\frac32}\|Z_{xxx}\|_2^{\frac32}\|Z_{xxxx}\|_2+C\|Z_x\|_2^{\frac34}\|Z_{xxx}\|_2^{\frac14}\|Z_{xx}\|_2\|Z_{xxxx}\|_2\\
&+C\|Z_{xx}\|_2^{\frac34}\|Z_{xxx}\|_2^{\frac14}\|Z_{xxxx}\|_2+C\|Z_x\|_2^{\frac34}\|Z_{xxx}\|_2^{\frac54}\|Z_{xx}\|_2\|Z_{xxxx}\|_2\\
\le&\frac14\|Z_{xxxx}\|_2^2+C(\|Z_{xxx}\|_2^2+\|Z_{xxx}\|_2^4).
\end{aligned}
$$

By the Hölder inequality, it follows that

$$
\begin{aligned}
&\int_{\mathbb{R}^1}k_2\big((1+\mu|Z|^2)Z\big)_{xx}Z_{xxxx}dx\\
=&k_2\int_{\mathbb{R}^1}(2\mu(Z_x\cdot Z_x)Z+2\mu Z\cdot Z_{xx}\cdot Z\\
&+5\mu Z\cdot Z_x\cdot Z_x+(1+\mu|Z|^2)Z_{xx})Z_{xxxx}dx\\
\le& C\|Z\|_\infty\|Z_x\|_4^2\|Z_{xxxx}\|_2+C\|Z\|_\infty^2\|Z_{xx}\|_2\|Z_{xxxx}\|_2\\
\le&\frac14\|Z_{xxx}\|_2^2+C,
\end{aligned}
$$

then,

$$
\frac{d}{dt}\|Z_{xxx}\|_2^2+\|Z_{xxxx}\|_2^2\le C(\|Z_{xxx}\|_2^2+\|Z_{xxx}\|_2^4)+C, \tag{2.13}
$$

by Gronwall's inequality, the (2.11) holds. □

By induction, we also have

Lemma 2.5. *Assume that the initial data $Z_0(x) \in H^m (m \geq 2)$, then for the smooth solution of problem* (1.5)-(1.6)*, we have the following estimate*

$$\sup_{0 \leq t \leq T} \|Z_{x^j}(\cdot, t)\|_2^2 + \int_0^t \|Z_{x^{j+1}}(\cdot, t)\|_2^2 dt \leq C \tag{2.14}$$

where the constant C depends on $k_1, k_2, \mu, \|Z_0(x)\|_2^2$, $j \geq 3$.

Now we will deal with the uniqueness of the solution in problem (1.5)-(1.6). Assume that there exist two solutions Z, Y. Let $W = Z - Y$, then W satisfies the following equation

$$\begin{aligned} W_t = {} & k_1 W_{xx} + \big(G(Z)W \times Z_x\big)_x + \big(B(R(Z,Y) \cdot W)Y \times Z_x\big)_x \\ & + \big(G(Y)Y \times W_x\big)_x - k_2(1 + \mu|Z|^2)W - k\mu(Z+Y) \cdot WY, \end{aligned} \tag{2.15}$$

where $R(Z,Y) = \sum_{i=1}^{q-1} |Z|^i |Y|^{q-1-i}$, making the scalar product of W with equation (2.15) and then integrating the result over $\mathbb{R}^1$, we have

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\|W\|_2^2 + k_1\|W_x\|_2^2 \\ = & -\int_{\mathbb{R}^1} G(Z)W \times Z_x \cdot W_x dx - \int_{\mathbb{R}^1} B((R(Z,Y) \cdot W)Y \times Z_x \cdot W_x dx \\ & + \int_{\mathbb{R}^1} k(1 + \mu|Z|^2)|W|^2 dx + k_2\mu \int_{\mathbb{R}^1} (Z+Y) \cdot WY \cdot W dx \\ \leq & C\|W\|_\infty (1 + \|Z\|_\infty^q)\|Z_x\|_2\|W_x\|_2 \\ & + C\|Y\|_\infty \|W\|_\infty (\|Z\|_\infty^{q-1} + \|Y\|_\infty^{q-1})\|Z_x\|_2\|W_x\|_2 \\ & + C\|Z\|_\infty^2 \|W\|_2^2 + C(\|Z\|_\infty + \|Y\|_\infty)\|Y\|_\infty \|W\|_2^2. \end{aligned} \tag{2.16}$$

By using Gagliardo-Nirenberg inequality, one has

$$\|W\|_{L^\infty} \leq C\|W\|_2^{\frac{3}{4}} \|W_{xx}\|_2^{\frac{1}{4}}. \tag{2.17}$$

Applying the estimate (2.14) and inequality (2.17), we get

$$\frac{1}{2}\frac{d}{dt}\|W\|_2^2 + k_1\|W_x\|_2^2 \leq \frac{1}{4}\|W_{xx}\|_2^2 + C(\|W_x\|_2^2 + \|W\|_2^2). \tag{2.18}$$

On the other hand, multiplying (2.15) by W_{xx} and then integrating the result over $\mathbb{R}^1$, we have

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\|W_x\|_2^2 + k_1\|W_{xx}\|_2^2 \\ = & \int_{\mathbb{R}^1} (G(Z)W \times Z_x)_x \cdot W_{xx} dx + \int_{\mathbb{R}^1} B\big((R(Z,Y) \cdot W)Y \times Z_x\big)_x \cdot W_{xx} dx \\ & + \int_{\mathbb{R}^1} (G(Y)Y \times W_x)_x \cdot W_{xx} dx + \int_{\mathbb{R}^1} k_2(1 + \mu|Z|^2)W \cdot W_{xx} dx \end{aligned}$$

$$
\begin{aligned}
&+ k_2\mu \int_{\mathbb{R}^1} (Z+Y)\cdot WY \cdot W_{xx} dx \\
\leq &\frac{1}{4}\|W_{xx}\|_2^2 + C(\|W_x\|_2^2 + \|W\|_2^2),
\end{aligned} \tag{2.19}
$$

where

$$
\begin{aligned}
&\int_{\mathbb{R}^1} (G(Z)W \times Z_x)_x \cdot W_{xx} dx \\
\leq &C\big(\|W\|_\infty \|Z\|_\infty^{q-1} \|Z_x\|_4^2 \|W_{xx}\|_2 + \|Z\|_\infty^q \|Z_x\|_4 \|W_x\|_4 \|W_{xx}\|_2 \\
&+ \|W\|_\infty \|Z\|_\infty^q \|Z_{xx}\|_2 \|W_{xx}\|_2\big),
\end{aligned}
$$

similarly,

$$
\begin{aligned}
&\int_{\mathbb{R}^1} B\big((R(Z,Y)\cdot W)Y \times Z_x\big)_x \cdot W_{xx} dx \\
= &B\bigg\{ \int_{\mathbb{R}^1} (R_x(Z,Y)\cdot W)Y \times Z_x \cdot W_{xx} dx + \int_{\mathbb{R}^1} (R(Z,Y)\cdot W_x)Y \times Z_x \cdot W_{xx} dx \\
&+ \int_{\mathbb{R}^1} (R(Z,Y)\cdot W)Y_x \times Z_x \cdot W_{xx} dx \\
&+ \int_{\mathbb{R}^1} (R(Z,Y)\cdot W)Y \times Z_{xx} \cdot W_{xx} dx \bigg\} \\
\leq &C\bar{R}(\|Z\|_\infty^i \|Y\|_\infty^{q-1-i}) \|W\|_\infty \big(\|Z_x\|_4^2 + \|Z_x\|_4 \|Y_x\|_4\big) \|W_{xx}\|_2 \\
&+ C\bar{R}(\|Z\|_\infty, \|Y\|_\infty) \|W\|_\infty \|Z_x\|_4 \|Y_x\|_4 \|W_{xx}\|_2 \\
&+ C\bar{R}(\|Z\|_\infty, \|Y\|_\infty) \|Z_x\|_4 \|W_x\|_4 \|W_{xx}\|_2 \\
&+ C\bar{R}(\|Z\|_\infty, \|Y\|_\infty) \|W\|_\infty \|Z_{xx}\|_2 \|W_{xx}\|_2,
\end{aligned}
$$

where $\bar{R}(\|Z\|_\infty, \|Y\|_\infty) = \sum_{i=1}^{q-1} \|Z\|_\infty^i \|Y\|_\infty^{q-i}$, and

$$
\begin{aligned}
&\int_{\mathbb{R}^1} (G(Y)Y \times W_x)_x \cdot W_{xx} dx \\
= &\int_{\mathbb{R}^1} G_x(Y)Y \times W_x \cdot W_{xx} dx + \int_{\mathbb{R}^1} G(Y)Y_x \times W_x \cdot W_{xx} dx \\
&+ \int_{\mathbb{R}^1} G(Y)Y \times W_{xx} \cdot W_{xx} dx \\
\leq &C(\|Y\|_\infty \|Z\|_\infty^q \|Y_x\|_4 \|W_x\|_4 \|W_{xx}\|_2 \\
&+ C(1 + \|Y\|_\infty^{q-1}) \|Y_x\|_4 \|W_x\|_4 \|W_{xx}\|_2 + C\|Y\|_\infty^{q+1} \|W_{xx}\|_2^2,
\end{aligned}
$$

then by (2.14) and the following imbedding inequality

$$
\|W\|_4 \leq C\|W\|_2^{\frac{3}{4}} \|W_x\|_2^{\frac{1}{4}},
$$

we have

$$
\frac{1}{2}\frac{d}{dt}\|W_x\|_2^2 + k_1\|W_{xx}\|_2^2 \leq \frac{1}{4}\|W_{xx}\|_2^2 + C(\|W\|_2^2 + \|W_x\|_2^2). \tag{2.20}
$$

Combining (2.18) and (2.20), we can get

$$
\frac{d}{dt}\{\|W\|_2^2 + \|W_x\|_2^2\} + \{\|W_x\|_2^2 + \|W_{xx}\|_2^2\} \leq C(\|W\|_2^2 + \|W_x\|_2^2),
$$

and then by Gronwall's inequality, we get the uniqueness.

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