THE APPROXIMATE SOLUTION OF RIEMANN TYPE PROBLEMS FOR DIRAC EQUATIONS BY NEWTON EMBEDDING METHOD*

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Abstract We study an existence and uniqueness for the nonlinear Riemann type problem and also give an error estimation for the approximate solutions in the Newton embedding procedure in higher dimensional spaces. Clifford analysis plays a key role in our approach.

Keywords Dirac equations, Riemann type problems, Newton embedding method, Clifford analysis.

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1. Introduction

The Riemann-Hilbert boundary value problem is also called Riemann type problem and is a boundary value problem for analytic functions in plane which was first formulated by Hilbert during his investigations of a set of problems mentioned by Riemann in his dissertation. The results of linear Riemann-Hilbert type problems on the complex plane in the classical sense were studied. What is the nonlinear Riemann-Hilbert approach in higher dimensional space? Is this nonlinear approach exists? Clifford algebras were introduced over one hundred years ago in attempt by William Kingdon Clifford to develop higher dimensional number system analogous to the real and complex numbers. Clifford analysis generalized complex analysis to a higher dimension in a natural and elegant way is systematically studied, see [4,7]. Thus, it is natural to consider Riemann-Hilbert problems within the frame work of Clifford analysis setting. We refer to [1–3, 8–12, 14]. From pure mathematics, mathematical physics and engineering applications, we need to research the theory of nonlinear Riemann type problems in higher dimensional spaces. The nonlinear problems are not easy to be solved. Due to the Hilbert transform which plays an important role in Riemann-Hilbert problems in Clifford analysis is not a compact operator, the classical method of functional analysis fail to solve the problems. To solve the nonlinear Riemann type problems, we use Clifford analysis and Newton embedding method.

The structure of this article is the following. In Section 2 some basic notations of Clifford algebras and Clifford analysis need in the sequel are introduced. Section

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3 is dedicated to our main result, where the nonlinear Riemann-Hilbert problem is investigated in Clifford Hölder spaces. Section 4 gives an error estimation for the approximate solutions in the Newton embedding procedure.

2. Preliminaries

Let $\mathcal{A} := \mathbb{R}(e_1, \ldots, e_n)$ denote the free $\mathbb{R}$-algebra with n indeterminants $\{e_1, \ldots, e_n\}$. Let J be the two-sided ideal in $\mathcal{A}$ generated by the elements

$$\{e_i^2 - 1, i = 1, \ldots, s; e_i^2 + 1, i = s+1, \ldots, n; e_i e_j + e_j e_i, 1 \leq i < j \leq n\}.$$

The quotient algebra $Cl(V_{n,s}) := \mathcal{A}/J$ is called the Clifford algebra with parameters n, s. Without risk of ambiguity, we take the usual practice of using the same symbol to denote an indeterminant e_i in $\mathcal{A}$ and its equivalent class in $\mathcal{A}/J$. Therefore, $e_1, \cdots, e_n$ considered as elements of $\mathcal{A}/J$ have the following relations:

$$\begin{cases} e_i^2 = 1, & i = 1, \ldots, s, \\ e_i^2 = -1, & i = s+1, \ldots, n, \\ e_i e_j + e_j e_i = 0, \, i \neq j. \end{cases} \tag{2.1}$$

Set
$$e_{l_1 \ldots l_r} := e_{l_1} \cdots e_{l_r}, \quad \text{while } 1 \leq l_1 < \cdots < l_r \leq n.$$

For more information on $Cl(V_{n,s})$, we refer to [4–7]. In this article, we only consider $s = n$. Thus $Cl(V_{n,n})$ is a real linear non-commutative algebra. An involution is defined by

$$\begin{cases} \overline{e}_A = (-1)^{\frac{n(A)(n(A)+3)}{2}} e_A, \text{ if } A \in \mathcal{P}N, \\ \overline{\lambda} = \sum_{A \in \mathcal{P}N} \lambda_A \overline{e_A}, & \text{if } \lambda = \sum_{A \in \mathcal{P}N} \lambda_A e_A, \end{cases} \tag{2.2}$$

where

$$\{e_A, A = \{l_1, \ldots, l_r\} \in \mathcal{P}N, 1 \leq l_1 < \cdots < l_r \leq n\},$$

$n(A)$ is the cardinal number of the set A, N stands for the set $\{1, 2, \cdots, n\}$ and $\mathcal{P}N$ denotes the family of all order-preserving subsets of N in the above way. The $Cl(V_{n,n})$-value n-1-differential form

$$d\sigma = \sum_{i=1}^{n} (-1)^{i-1} e_i d\widehat{x}_i^N$$

is exact, where

$$d\widehat{x}_i^N = dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n.$$

If dS stands for the classical surface element and

$$\mathbf{n} = \sum_{i=1}^{n} e_i \mathbf{n}_i,$$

where $\mathbf{n}_i$ is the i-th component of the outward pointing normal, then the Clifford-valued surface element $d\sigma$ can be written as

$$d\sigma = \mathbf{n}dS. \tag{2.3}$$

The norm of λ is defined by $|\lambda| = (\sum_{A \in \mathcal{P}N} |\lambda_A|^2)^{\frac{1}{2}}$.

Suppose Ω be an open non-empty subset of $\mathbb{R}^n$ ($n \geq 3$), denote $\Omega^+ = \Omega$ and $\Omega^- = \mathbb{R}^n \setminus \Omega$. We introduce the Dirac operator $D = \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i}$. In particular, we have that $DD = \Delta$ where Δ is the Laplacian over $\mathbb{R}^n$. A function $u : \Omega \mapsto Cl(V_{n,n})$ is said to be left monogenic if it satisfies the equation $D[u](\mathbf{x}) = 0$ for each $\mathbf{x} \in \Omega$. A similar definition can be given for right monogenic functions. For more information as regards the Clifford algebra can be found in [4,7].

3. The Clifford Hölder spaces and a non-linear Riemann type Problems

In order to solve nonlinear Riemann type boundary value problem, we need to introduce the theory of Clifford Hölder space and define a new function space.

Let Ω be an nonempty subset of $\mathbb{R}^n$, $u(\mathbf{x}) = \sum_A e_A u_A(\mathbf{x})$, where $u_A(\mathbf{x})$ are real functions. $u(\mathbf{x})$ is called a Hölder continuous functions on Ω if the following condition is satisfied

$$|u(\mathbf{x}_1) - u(\mathbf{x}_2)| = \left[\sum_A |u_A(\mathbf{x}_1) - u_A(\mathbf{x}_2)|^2 \right]^{\frac{1}{2}} \leq C|\mathbf{x}_1 - \mathbf{x}_2|^\alpha,$$

where for any $\mathbf{x}_1, \mathbf{x}_2 \in \Omega$, $\mathbf{x}_1 \neq \mathbf{x}_2$, $0 < \alpha \leq 1$, C is a positive constant independent of $\mathbf{x}_1, \mathbf{x}_2$. Denote by $H^\alpha(\Omega)$ the set of Hölder continuous functions with values in $Cl(V_{n,n})$ on Ω (the Hölder exponent is α, $0 < \alpha \leq 1$). Define the norm of u in $H^\alpha(\Omega)$ as

$$\|u\|_{H^\alpha(\Omega)} = \|u\|_\infty + \|u\|_{h^\alpha} \tag{3.1}$$

where $\|u\|_\infty := \sup_{\mathbf{x} \in \Omega} |u(\mathbf{x})|$, $\|u\|_{h^\alpha} := \sup_{\substack{\mathbf{x}_1, \mathbf{x}_2 \in \Omega \\ \mathbf{x}_1 \neq \mathbf{x}_2}} \frac{|u(\mathbf{x}_1) - u(\mathbf{x}_2)|}{|\mathbf{x}_1 - \mathbf{x}_2|^\alpha}$. It is clear that $H^\alpha(\Omega)$ is a Banach space with norm (3.1).

We study the following Riemann type problems with respect to a given boundary $\partial\Omega$ which is a Lyapunov surface of an open bounded nonempty subset Ω in $\mathbb{R}^n$.

In what follows, we denote

$$u^\pm(\mathbf{x}) = \lim_{\substack{\mathbf{y} \to \mathbf{x} \in \partial\Omega \\ \mathbf{y} \in \Omega^\pm}} u(\mathbf{y}).$$

For $u \in H^\alpha(\partial\Omega)$, $0 < \alpha \leq 1$. Its Cauchy transform $\mathcal{C}u$ and Hilbert transform $\mathcal{H}u$ by

$$\mathcal{C}u(\mathbf{x}) := \frac{1}{\omega_n} \int_{\partial\Omega} \frac{\mathbf{y} - \mathbf{x}}{|\mathbf{y} - \mathbf{x}|^n} d\sigma_\mathbf{y} u(\mathbf{y}), \; \mathbf{x} \in \mathbb{R}^n \setminus \partial\Omega, \tag{3.2}$$

and

$$\mathcal{H}u(\mathbf{x}) := \frac{1}{\omega_n} \int_{\partial\Omega} \frac{\mathbf{y} - \mathbf{x}}{|\mathbf{y} - \mathbf{x}|^n} d\sigma_{\mathbf{y}} u(\mathbf{y}), \, \mathbf{x} \in \partial\Omega, \tag{3.3}$$

respectively. Here ω_n is the area of the unit sphere in $\mathbb{R}^n$. In the articles [13, 15], the authors established the relationship between the Cauchy transform (3.2) and the Hilbert transform (3.3).

Lemma 3.1 ([13, 15]). *For $u \in H^\alpha(\partial\Omega)$, $0 < \alpha \leq 1$. Then*

$$\mathcal{C}u^\pm(\mathbf{x}) = \pm\frac{u(\mathbf{x})}{2} + \frac{1}{2}\mathcal{H}u(\mathbf{x}). \tag{3.4}$$

Furthermore, the Cauchy transform $\mathcal{C}u$ can be Hölder continuously extended from Ω into $\overline{\Omega}$ and from $\mathbb{R}^n \setminus \overline{\Omega}$ into $\mathbb{R}^n \setminus \Omega$ with limiting values in (3.4) and we have the inequalities

$$\|\mathcal{C}u\|_{H^\alpha(\overline{\Omega})} \leq \widetilde{C}\|u\|_{H^\alpha(\partial\Omega)} \tag{3.5}$$

and

$$\|\mathcal{C}u\|_{H^\alpha(\mathbb{R}^n\setminus\Omega)} \leq \widetilde{C}\|u\|_{H^\alpha(\partial\Omega)}, \tag{3.6}$$

for some constant $\widetilde{C}$ depending on α and $\partial\Omega$.

Theorem 3.1. *Let u be the solution of the following Riemann type problem:*

$$\begin{cases} D[u] = 0, & in \ \mathbb{R}^n \setminus \partial\Omega, \\ u^+(x) = u^-(x) + g(x), \ x \in \partial\Omega, \\ \lim_{|\mathbf{x}|\to\infty} u(\mathbf{x}) = 0, \end{cases}$$

where $g(x)$ is Clifford value function in $H^\alpha(\partial\Omega)$ $0 < \alpha \leq 1$. Then

$$\|u\|_{H^\alpha(\overline{\Omega})} \leq \widetilde{C}\|g\|_{H^\alpha(\partial\Omega)} \tag{3.7}$$

and

$$\|u\|_{H^\alpha(\mathbb{R}^n\setminus\Omega)} \leq \widetilde{C}\|g\|_{H^\alpha(\partial\Omega)}. \tag{3.8}$$

Proof.   In view of Lemma 3.1, we can directly prove the result.   □

Remark 3.1. If a bounded u in $H^\alpha(\overline{\Omega}) \bigcap H^\alpha(\mathbb{R}^n \setminus \Omega)$, we define the norm

$$\|u\|_\alpha := \|u\|_{H^\alpha(\overline{\Omega})} + \|u\|_{H^\alpha(\mathbb{R}^n\setminus\Omega)},$$

then (3.7) and (3.8) in Theorem 3.1 can be written in the form

$$\|u\|_\alpha \leq C\|g\|_{H^\alpha(\partial\Omega)}$$

where $C = 2\widetilde{C}$.

In this sequence, we consider the following nonlinear Riemann type problem now:

$$\begin{cases} D[u] = 0, & \text{in } \mathbb{R}^n \setminus \partial\Omega, \\ u^+(\mathbf{x}) = u^-(\mathbf{x}) + g(\mathbf{x}, u^+, u^-), \ \mathbf{x} \in \partial\Omega, \\ \lim_{|\mathbf{x}| \to \infty} u(\mathbf{x}) = 0, \end{cases} \tag{3.9}$$

we assume the following conditions to be fulfilled:

(**C**1) For each u_1, u_2 in $H^\alpha(\partial\Omega)$ $0 < \alpha \le 1$, the function

$$g(\mathbf{x}, u_1, u_2) = g(\mathbf{x}, u_1(\mathbf{x}), u_2(\mathbf{x}))$$

is in $H^\alpha(\partial\Omega)$ as a function of $\mathbf{x}$. Moreover there exists a nonnegative constant N such that $CN < 1$ where C is the same as in Remark 3.1, and for all $u_1, \widetilde{u}_1, u_2, \widetilde{u}_2$ in $H^\alpha(\partial\Omega)$ we have

$$\|g(\cdot, u_1, u_2) - g(\cdot, \widetilde{u}_1, \widetilde{u}_2)\|_{H^\alpha(\partial\Omega)}$$
$$\le N[\|u_1 - \widetilde{u}_1\|_{H^\alpha(\partial\Omega)} + \|u_2 - \widetilde{u}_2\|_{H^\alpha(\partial\Omega)}].$$

We shall prove the existence of solution for the boundary value problem (3.9).

Theorem 3.2. *Suppose g satisfies the above conditions* (**C**1). *Then the problem* (3.9) *has exactly one solution provided that the constant N in* (**C**1).

Proof.   Firstly, for each $t(0 \le t \le 1)$, we consider the problem

$$\begin{cases} D[u_t] = 0, & \text{in } \mathbb{R}^n \setminus \partial\Omega, \\ u_t^+(x) = u_t^-(x) + tg(x, u^+, u^-), \ x \in \partial\Omega, \\ \lim_{|\mathbf{x}| \to \infty} u_t(\mathbf{x}) = 0. \end{cases} \tag{3.10}$$

When $t = 1$, the problem (3.10) is just (3.9). For $t = 0$, $u_0(\mathbf{x})$ is a monogenic in $\mathbb{R}^n$ vanishing at infinity so that $u_0(\mathbf{x}) \equiv 0$ is the unique solution. We now assume $u_{t_0}(\mathbf{x})$ to be a solution of (3.10) for a given t_0 with $0 \le t_0 < 1$. With the help of a combination an imbedding method with a Newton's method, we will show the existence of a solution of (3.10) for all t in $t_0 \le t \le t_0 + \delta$ for some $\delta > 0$ that is independent of t_0. Then we can conclude there is a solution for $t = 1$.

We denote $u_t^0(\mathbf{x}) \triangleq u_{t_0}(\mathbf{x})$ and let $u_t^{k+1}(\mathbf{x})$ to be the solution of the linear problem

$$\begin{cases} D[u_t^{k+1}] = 0, & \text{in } \mathbb{R}^n \setminus \partial\Omega, \\ (u_t^{k+1})^+(\mathbf{x}) = (u_t^{k+1})^-(\mathbf{x}) + tg(\mathbf{x}, (u_t^k)^+, (u_t^k)^-), \ \mathbf{x} \in \partial\Omega, \\ \lim_{|\mathbf{x}| \to \infty} u_t(\mathbf{x}) = 0. \end{cases} \tag{3.11}$$

Thus the linear problem (3.11) is uniquely solvable. The differences

$$f_t^k(\mathbf{x}) \triangleq u_t^{k+1}(\mathbf{x}) - u_t^k(\mathbf{x}), \ k \in \mathbb{N},$$

fulfill

$$\begin{cases} D[f_t^k] = 0, & \text{in } \mathbb{R}^n \setminus \partial\Omega, \\ (f_t^k)^+(\mathbf{x}) = (f_t^k)^-(\mathbf{x}) + h_k(\mathbf{x}), \ \mathbf{x} \in \partial\Omega, \\ \lim_{|\mathbf{x}| \to \infty} f_t^n(\mathbf{x}) = 0, \end{cases}$$

where

$$h_0(\mathbf{x}) \triangleq (t - t_0)g(\mathbf{x}, (u_t^0)^+, (u_t^0)^-)$$

and

$$h_k(\mathbf{x}) \triangleq t[g(\mathbf{x}, (u_t^k)^+, (u_t^k)^-) - g(\mathbf{x}, (u_t^{k-1})^+, (u_t^{k-1})^-)], \quad k \in \mathbb{N} \setminus \{0\}.$$

In view of Theorem 3.1, we obtain that

$$\|f_t^k\|_\alpha \le C\|h_k\|_{H^\alpha(\partial\Omega)} \ k \in \mathbb{N} \setminus \{0\}. \tag{3.12}$$

From the condition (**C**1), it is easy to check that

$$\begin{aligned} \|h_0\|_{H^\alpha(\partial\Omega)} \le& (t - t_0)N\|(u_t^0)^+\|_{H^\alpha(\partial\Omega)} + (t - t_0)N\|(u_t^0)^-\|_{H^\alpha(\partial\Omega)} \\ &+ (t - t_0)\|g(\cdot, 0, 0)\|_{H^\alpha(\partial\Omega)} \\ \le& (t - t_0)N\|u_t^0\|_\alpha + (t - t_0)\|g(\cdot, 0, 0)\|_{H^\alpha(\partial\Omega)} \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} \|h_k\|_{H^\alpha(\partial\Omega)} \le& tN\|(f_t^{k-1})^+\|_{H^\alpha(\partial\Omega)} + tN\|(f_t^{k-1})^-\|_{H^\alpha(\partial\Omega)} \\ \le& tN\|f_t^{k-1}\|_\alpha. \end{aligned} \tag{3.14}$$

Combining (3.12), (3.13) with (3.14), we have the following inequalities

$$\|f_t^0\|_\alpha \le C(t - t_0)N\|u_t^0\|_\alpha + C(t - t_0)\|g(\cdot, 0, 0)\|_{H^\alpha(\partial\Omega)} \tag{3.15}$$

and

$$\|f_t^k\|_\alpha \le CtN\|f_t^{k-1}\|_\alpha, \tag{3.16}$$

where $k \in \mathbb{N} \setminus \{0\}$.

Since $u_t^0(\mathbf{x})$ is a solution of (3.11) for $t = t_0$, applying Theorem 3.1, the apriori estimate gives

$$\|u_t^0\|_\alpha \le t_0 C\|g(\cdot, (u_t^0)^+, (u_t^0)^-)\|_{H^\alpha(\partial\Omega)},$$

using the condition (**C**1), we obtain that

$$\|u_t^0\|_\alpha \le t_0 CN\|u_t^0\|_\alpha + t_0 C\|g(\cdot, 0, 0)\|_{H^\alpha(\partial\Omega)}. \tag{3.17}$$

Denote

$$\kappa \triangleq CN$$

and

$$\kappa_0 \triangleq C\|g(\cdot, 0, 0)\|_{H^\alpha(\partial\Omega)}.$$

We then have

$$\|u_t^0\|_\alpha \leq \frac{t_0 \kappa_0}{1 - t_0 \kappa}, \tag{3.18}$$

and rewrite (3.15) as

$$\|f_t^0\|_\alpha \leq (t - t_0)(\frac{t_0 \kappa_0}{1 - t_0 \kappa} + \kappa_0). \tag{3.19}$$

We have

$$\sum_{j=0}^{k}(\|u_t^{j+1}\|_\alpha - \|u_t^j\|_\alpha) \leq \sum_{k=0}^{n}\|f_t^k\|_\alpha$$

and use the inequalities (3.15), (3.16), (3.17) and $CN < 1$, when n tend to $+\infty$, these imply the convergence of $\{u_t^k\}_{k=0}^{+\infty}$ in the $\|\cdot\|_\alpha$.

Secondly, we now prove that the limit function $u_t(\mathbf{x})$ satisfies (3.11). To do so we let n tend to $+\infty$ in (3.11). Due to convergence is with respect to the $\|\cdot\|_\alpha$ norm it follows that $u_t(\mathbf{x})$ belongs to $H^\alpha(\overline{\Omega}) \bigcap H^\alpha(\mathbb{R}^n \setminus \Omega)$, and that the transmission condition of (3.10) is satisfied. Moreover, $\|u_t^k\|_\alpha$ are uniformly bounded, by Weierstrass' Theorem (See [4, 7]), we conclude that $D[u_t^{k+1}]$ converge to $D[u_k]$ uniformly on compact subsets of $\mathbb{R}^n \setminus \partial\Omega$ such that $D[u_t] = 0$ in $\mathbb{R}^n \setminus \partial\Omega$. It is clear that $\lim_{|\mathbf{x}|\to\infty} u_t(\mathbf{x}) = 0$. Hence we have completed to show that $u_t(\mathbf{x})$ satisfies all of (3.10). It follows that after finitely many steps one ends up with a solution of (3.11) for $t = 1$, which is the problem (3.10).

Finally, to finish the proof of Theorem 3.2, we need to show the uniqueness. Let u_1 and u_2 be two solutions of (3.9). Then $u = u_1 - u_2$ is a solution of the linear

$$\begin{cases} D[u] = 0, & \text{in } \mathbb{R}^n \setminus \partial\Omega, \\ u^+(\mathbf{x}) = u^-(\mathbf{x}) + \widetilde{g}(\mathbf{x}), \mathbf{x} \in \partial\Omega, \\ \lim_{|\mathbf{x}|\to\infty} u(\mathbf{x}) = 0. \end{cases}$$

where

$$\widetilde{g}(\mathbf{x}) \triangleq g(\mathbf{x}, u_1^+(\mathbf{x}), u_1^-(\mathbf{x})) - g(\mathbf{x}, u_2^+(\mathbf{x}), u_2^-(\mathbf{x})).$$

Using Theorem 3.1 and the condition (**C**1) again, we get

$$\|u\|_\alpha \leq CN\|u\|_\alpha,$$

since $CN < 1$, we conclude that $u_1 = u_2$. The proof is done. $\square$

4. Error Estimation

In this section, we shall compute the difference of the solution of (3.9) and its approximation $u_t^k(\mathbf{x})$. Let

$$v_k(\mathbf{x}) \triangleq u_t(\mathbf{x}) - u_t^k(\mathbf{x}), v(\mathbf{x}) \triangleq u(\mathbf{x}) - u_t(\mathbf{x})$$

where $u(\mathbf{x}) = u_1(\mathbf{x})$.

In view of (3.10) and (3.11), we have

$$
\begin{cases}
D[v_{k+1}] = 0, & \text{in } \mathbb{R}^n \setminus \partial\Omega, \\
v_{k+1}^+(\mathbf{x}) = v_{k+1}^-(\mathbf{x}) + \widetilde{g}_k(\mathbf{x}), \ \mathbf{x} \in \partial\Omega, \\
\lim_{|\mathbf{x}| \to \infty} v_{k+1}(\mathbf{x}) = 0,
\end{cases}
$$

where

$$
\widetilde{g}_k(\mathbf{x}) \triangleq t[g(\mathbf{x}, u_t^+, u_t^-) - g(x, (u_t^k)^+, (u_t^k)^-)].
$$

According to Theorem 3.1, we get

$$
\begin{aligned}
\|v_{n+1}\|_\alpha &\le C\|\widetilde{g}_k\|_{H^\alpha(\partial\Omega)} \\
&\le tCN\|u_t^+ - (u_t^k)^+\|_{H^\alpha(\partial\Omega)} + tCN\|u_t^- - (u_t^n)^-\|_{H^\alpha(\partial\Omega)} \\
&\le tCN\|u_t - u_t^k\|_\alpha \\
&\le tCN[\|v_{k+1}\|_\alpha + \|u_t^{k+1} - u_t^k\|_\alpha] \\
&= t\kappa[\|v_{k+1}\|_\alpha + \|f_t^k\|_\alpha],
\end{aligned}
\tag{4.1}
$$

moreover we have

$$
\|v_{k+1}\|_\alpha \le \frac{t\kappa}{1 - t\kappa}\|f_t^k\|_\alpha,
$$

by (3.15) and (3.16), we deduce that

$$
\|v_{k+1}\|_\alpha \le c(t\kappa)^{k+1}
\tag{4.2}
$$

where $c = \kappa_0 \frac{2-\kappa}{(1-\kappa)^2}$.

On the other hand, the function v is a solution of

$$
\begin{cases}
D[v] = 0, & \text{in } \mathbb{R}^n \setminus \partial\Omega, \\
v^+ = v^- + \widehat{g}, & \text{on } \partial\Omega, \\
\lim_{|\mathbf{x}| \to \infty} v(\mathbf{x}) = 0,
\end{cases}
$$

where

$$
\widehat{g} \triangleq g(\mathbf{x}, u^+, u^-) - g(\mathbf{x}, u_t^+, u_t^-) + (1-t)g(\mathbf{x}, u_t^+, u_t^-).
$$

Applying Theorem 3.1 and the condition (**C1**) again, we obtain that

$$
\begin{aligned}
\|v\|_\alpha &\le \kappa\|v\|_\alpha + (1-t)\kappa\|u_t\|_\alpha + (1-t)C\|g(\cdot, 0, 0)\|_{H^\alpha(\partial\Omega)} \\
&= \kappa\|v\|_\alpha + (1-t)\kappa\|u_t\|_\alpha + (1-t)\kappa_0,
\end{aligned}
$$

furthermore

$$
\begin{aligned}
\|v\|_\alpha &\le \frac{1-t}{1-\kappa}(\kappa_0 + \kappa\|u_t\|_\alpha) \\
&\le (1-t)\frac{2-\kappa}{(1-\kappa)^2}\kappa_0 \\
&= (1-t)c.
\end{aligned}
\tag{4.3}
$$

Combining (4.2) with (4.3), we have the following result:

Theorem 4.1. *The error between the solution $u(x)$ of (3.9) and its approximation $u_t^k(x)$ can be estimated by*

$$\|u - u_t^k\|_\alpha \le c[(t\kappa)^{k+1} + (1 - t)],$$

where $c = \kappa_0 \frac{2-\kappa}{(1-\kappa)^2}$.

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