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FRACTIONAL HAMILTONIAN SYSTEMS WITH POSITIVE SEMI-DEFINITE MATRIX

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Abstract We study the existence of solutions for the following fractional Hamiltonian systems

$$\begin{cases} -{}_tD^{\alpha}_{\infty}({}_{-\infty}D^{\alpha}_{t}u(t)) - \lambda L(t)u(t) + \nabla W(t,u(t)) = 0, \\ u \in H^{\alpha}(\mathbb{R},\mathbb{R}^n), \end{cases} \tag{FHS$_\lambda$}$$

where $\alpha \in (1/2,1)$, $t \in \mathbb{R}$, $u \in \mathbb{R}^n$, $\lambda > 0$ is a parameter, $L \in C(\mathbb{R},\mathbb{R}^{n^2})$ is a symmetric matrix, $W \in C^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$. Assuming that $L(t)$ is a positive semi-definite symmetric matrix, that is, $L(t) \equiv 0$ is allowed to occur in some finite interval T of $\mathbb{R}$, $W(t,u)$ satisfies some superquadratic conditions weaker than Ambrosetti-Rabinowitz condition, we show that (FHS)$_\lambda$ has a solution which vanishes on $\mathbb{R}\setminus T$ as $\lambda \to \infty$, and converges to some $\tilde{u} \in H^{\alpha}(\mathbb{R},\mathbb{R}^n)$. Here, $\tilde{u} \in E_0^{\alpha}$ is a solution of the Dirichlet BVP for fractional systems on the finite interval T. Our results are new and improve recent results in the literature even in the case $\alpha = 1$.

Keywords Fractional Hamiltonian systems, fractional Sobolev space, critical point theory, concentration phenomena.

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1. Introduction

Fractional derivatives are nonlocal operators and are historically applied in the study of nonlocal time-dependent processes. The first and well established application of fractional calculus in physics was in the framework of anomalous diffusion, which is related to features observed in many physical systems, for example; in dispersive transport in amorphous semiconductor, liquid crystals, polymers, proteins, etc. [6, 10–12].

The fractional calculus of variations is a beautiful and useful field of mathematics that deals with the problems of determining extrema (maxima or minima) of functionals whose Lagrangians contain fractional integrals and/or derivatives. It was born in 1996-1997, when Riewe derived Euler-Lagrange fractional differential equations and showed how nonconservative systems in mechanics can be described using fractional derivatives [22]. More precisely, for $y : [a,b] \to \mathbb{R}^n$ and $\alpha_j, \beta_j \in [0,1]$,

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$i = 1 \cdots N$, $j = 1, \cdots, \tilde{N}$, he considered the energy functional

$$J(y) = \int_a^b F({}_aD_t^{\alpha_1} y(t), \cdots, {}_aD_t^{\alpha_N} y(t), {}_tD_b^{\beta_1} y(t), \cdots, {}_tD_b^{\beta_{\tilde{N}}} y(t), y(t), t)dt,$$

with $n, N, \tilde{N} \in \mathbb{N}$. Using the fractional variational principle he obtained the following Euler-Lagrange equation

$$\sum_{i=1}^{N} {}_tD_b^{\alpha_i}[\partial_i F] + \sum_{i=1}^{\tilde{N}} {}_aD_t^{\beta_i}[\partial_{i+N} F] + \partial_{\tilde{N}+N+1} F = 0. \tag{1.1}$$

In particular, if

$$F = \frac{1}{2} m \dot{y}^2 - V(y) + \frac{1}{2} \gamma i \left({}_aD_t^{\frac{1}{2}}[y] \right)^2, \tag{1.2}$$

he got the Euler-Lagrange equation

$$m \ddot{y} = -\gamma i \left({}_tD_b^{\frac{1}{2}} \circ {}_aD_t^{\frac{1}{2}} \right)[y] - \frac{\partial V(y)}{\partial y}. \tag{1.3}$$

Recently, several different approaches have been developed to generalize the least action principle and the Euler-Lagrange equations to include fractional derivatives, for more details see [14,15].

On the other hand, it should be noted that critical point theory and variational methods have also turned out to be very effective tools in determining the existence of solutions for integer order differential equations. The idea behind them is try to find solutions of a given boundary value problem by looking for critical points of a suitable energy functional defined on an appropriate function space. In the last years, the critical point theory has become a wonderful tool in studying the existence of solutions to differential equations with variational structures (see Ekeland [5], Mawhin and Willem [16], Rabinowitz [20], Schechter [23], and the references therein).

Motivated by the aforementioned classical works and equation (1.3), Jiao and Zhou [9], for the first time, showed that the critical point theory is an effective approach to tackle the existence of nontrivial solutions for the following fractional boundary value problem

$$\begin{aligned} {}_tD_T^{\alpha}({}_0D_t^{\alpha} u(t)) &= \nabla F(t, u(t)), \quad t \in [0, T], \\ u(0) &= u(T) = 0. \end{aligned} \tag{1.4}$$

From then on, there is growing interest in using this wonderful tool to study fractional differential equations with variational structure. Recently, Torres [25], considered the following fractional systems

$$\begin{aligned} & {}_tD_{\infty}^{\alpha}({}_{-\infty}D_t^{\alpha} u(t)) + L(t)u(t) = \nabla W(t, u(t)), \\ & u \in H^{\alpha}(\mathbb{R}, \mathbb{R}^n), \end{aligned} \tag{1.5}$$

where $\alpha \in (1/2, 1)$, $t \in \mathbb{R}$, $u \in \mathbb{R}^n$, $L \in C(\mathbb{R}, \mathbb{R}^{n^2})$ is a symmetric and positive definite matrix, $W \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$. Assuming that $L(t)$ satisfied the following condition

(L) there exists an $l \in C(\mathbb{R},(0,\infty))$ with $l(t) \to \infty$ as $|t| \to \infty$ such that

$$(L(t)u,u) \geq l(t)|u|^2 \quad \text{for all } t \in \mathbb{R} \text{ and } u \in \mathbb{R}^n \tag{1.6}$$

and $W(t,u)$ satisfies the following conditions:

(FHS_1) There is a constant $\theta > 2$ such that

$$0 < \theta W(t,u) \leq (\nabla W(t,u),u) \quad \text{for all } t \in \mathbb{R} \text{ and } u \in \mathbb{R}^n \backslash \{0\},$$

(FHS_2) $|\nabla W(t,u)| = o(|u|)$ as $|u| \to 0$ uniformly with respect to $t \in \mathbb{R}$.

(FHS_3) There exists $\overline{W} \in C(\mathbb{R}^n,\mathbb{R})$ such that

$$|W(t,u)| + |\nabla W(t,u)| \leq |\overline{W}(u)| \quad \text{for every } t \in \mathbb{R} \text{ and } u \in \mathbb{R}^n,$$

the author showed that (1.5) possesses at least one nontrivial solution via Mountain Pass Theorem. After these interesting works, many researchers deal with the existence and multiplicity of solutions for (1.4) and (1.5) via different tools from critical point theory; see for instance [13, 17, 18, 27–29, 31, 33, 34, 36]. In addition, some perturbed fractional systems are discussed in [26, 31].

In this work we deal with the following fractional Hamiltonian systems

$$\begin{aligned} &{}_tD_\infty^\alpha({}_{-\infty}D_t^\alpha u(t)) + \lambda L(t)u(t) = \nabla W(t,u(t)), \\ &u \in H^\alpha(\mathbb{R},\mathbb{R}^n), \end{aligned} \tag{FHS$_\lambda$}$$

where $\alpha \in (1/2,1)$, $t \in \mathbb{R}$, $u \in \mathbb{R}^n$, $\lambda > 0$ is a parameter, $W \in C^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$ and L satisfies the following conditions

$(\mathcal{L})_1$ $L \in C(\mathbb{R},\mathbb{R}^{n\times n})$ is a symmetric matrix and there exists a nonnegative continuous function $l : \mathbb{R} \to \mathbb{R}$ and a constant $c > 0$ such that

$$(L(t)u,u) \geq l(t)|u|^2.$$

The set $\{l < c\} := \{t \in \mathbb{R} \,|\, l(t) < c\}$ is nonempty with $|\{l < c\}| < \frac{1}{C_\infty^2}$, where $|\cdot|$ is the Lebesgue measure and C_∞ is the best Sobolev constant for the embedding of X^α into $L^\infty(\mathbb{R})$;

$(\mathcal{L})_2$ $J = int(l^{-1}(0))$ is a nonempty finite interval and $\overline{J} = l^{-1}(0)$;

$(\mathcal{L})_3$ there exists an open interval $T \subset J$ such that $L(t) \equiv 0$ for all $t \in \overline{T}$.

When $\alpha = 1$, (FHS)$_\lambda$ reduces to the following well-known second order Hamiltonian systems

$$\ddot{u} - \lambda L(t)u + \nabla W(t,u) = 0. \tag{HS}$$

Recently a second order Hamiltonian systems like (HS) with positive semi-definite matrix was considered in [24]. Assuming that $W \in C^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$ is an indefinite potential satisfying asymptotically quadratic condition at infinity on u, Sun and Wu, have proved the existence of two homoclinic solutions of (HS). For more related works, we refer the reader to [2, 4, 7, 8, 19, 21] and the references therein.

Here we must point out that, to obtain the existence or multiplicity of solutions for Hamiltonian systems, all the papers mentioned above need the assumption that the symmetric matrix $L(t)$ be positive definite. Recently this condition was removed in [1, 27, 30, 35], that is, the authors considered the case that $L(t)$ is positive semi-definite symmetry matrix satisfying $(\mathcal{L})_1$. In [1], the author dealt with $(\mathrm{FHS})_\lambda$ for the case that $(\mathcal{L})_1$ is satisfied and $W(t,u)$ involves a combination of superquadratic and subquadratic terms and is allowed to be sign-changing. In [27], the author has considered the existence, multiplicity and concentration of solutions of $(\mathrm{FHS})_\lambda$ when $(\mathcal{L})_1$-$(\mathcal{L})_3$ are satisfied with $T=(0,\mathbb{T})$ and $W(t,u)$ satisfies the following subquadratic assumptions as $|u|\to\infty$:

$(\mathrm{FHS})_4$ $W\in C^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$ and there exist a constant $p\in(1,2)$ and a positive function $\xi\in L^{\frac{2}{2-p}}(\mathbb{R})$ such that

$$|\nabla W(t,u)|\le \xi(t)|u|^{p-1},\ \ \forall (t,u)\in\mathbb{R}\times\mathbb{R}^n.$$

$(\mathrm{FHS})_5$ There exist three constant $\eta,\delta>0$ and $\nu\in(1,2)$ such that

$$|W(t,u)|\ge \eta|u|^{\nu}\ \ \forall t\in\mathbb{I}\ \text{ and }\ |u|\le\delta.$$

Furthermore in [35], the authors have complemented the previous work by considering the superquadratic potential when $|u|\to\infty$, namely they considered $(\mathrm{FHS})_\lambda$ when $(\mathcal{L})_1$-$(\mathcal{L})_3$ are satisfied and $W(t,u)$ satisfies $(\mathrm{FHS})_1$ and

$(\mathrm{FHS})_5$ There is a positive continuous function $a:\mathbb{R}\to\mathbb{R}$ with

$$\lim_{|t|\to\infty} a(t)=0$$

such that

$$|\nabla W(t,u)|\le a(t)|u|^{\theta-1}\ \ \forall (t,u)\in\mathbb{R}\times\mathbb{R}^n.$$

By using mountain pass theorem, the authors have proved the existence of at least one nontrivial weak solution u_λ for $(\mathrm{FHS})_\lambda$. Moreover, they analyzed the behavior of u_λ when $\lambda\to+\infty$. We also mention the recent work [30], where the authors have considered $(\mathrm{FHS})_\lambda$ when $W(t,u)$ satisfies $(\mathrm{FHS})_1$-$(\mathrm{FHS})_3$ and

$(\mathrm{FHS})_6$ The function $s\to\frac{\langle\nabla W(t,su),u\rangle}{s^{\theta-1}}$ is strictly increasing for all $u\neq0$ and $s>0$, θ is given by $(\mathrm{FHS})_\lambda$.

By combining mountain pass arguments and Nehari manifold method, the authors have proved the existence of a ground state solution u_λ for $(\mathrm{FHS})_\lambda$ and analyzed the behavior when $\lambda\to+\infty$.

It is worth pointing out that $(\mathrm{FHS})_1$ is the well-known Ambrosetti-Rabinowitz condition. This condition play a key role to ensure the boundedness of the Palais-Smale sequences of the energy functional. This is very crucial in applying the critical point theory. However, there are many functions that are superquadratic at infinity but do not satisfy $(\mathrm{FHS})_1$ for any $\theta>2$. In fact, $(\mathrm{FHS})_1$ implies that

$$\lim_{|u|\to\infty}\frac{W(t,u)}{|u|^2}=+\infty.$$

Thus, for example the superquadratic function

$$\mathcal{W}(t,u)=g(t)(|u|^p+(p-2)|u|^{p-\epsilon}\sin^2(\frac{|u|^\epsilon}{\epsilon})),$$

where $g(t) > 0$ is periodic in t, $0 < \epsilon < p - 2$ and $p > 2$, does not satisfy $(\mathrm{FHS})_1$, for more details see [3].

Motivated by these previous results and the function $\mathcal{W}$, the main purpose of this paper is to investigate $(\mathrm{FHS})_\lambda$ without Ambrosetti-Rabinowitz condition (FHS_1), more precisely, we consider functions W that satisfy the following assumptions

(W_1) $|\nabla W(t,u)| = o(|u|)$ as $|u| \to 0$ uniformly in $t \in \mathbb{R}$.

(W_2) $W(t,u) \geq 0$ for all $(t,u) \in \mathbb{R} \times \mathbb{R}^N$ and $H(t,u) \geq 0$ for all $(t,u) \in \mathbb{R} \times \mathbb{R}^N$, where

$$H(t,u) := \frac{1}{2}\langle \nabla W(t,u), u\rangle - W(t,u).$$

(W_3) $\frac{W(t,u)}{|u|^2} \to +\infty$ as $|u| \to +\infty$ uniformly in $t \in \mathbb{R}$.

(W_4) There exist $C_0, R > 0$, and $\sigma > 1$ such that

$$\frac{|\nabla W(t,u)|^\sigma}{|u|^\sigma} \leq C_0 H(t,u) \ \text{ if } \ |u| \geq R.$$

Note that $\mathcal{W}$ is an example of function satisfying $(W_1) - (W_4)$.

Now we are in the position to state our main existence result.

Theorem 1.1. *Suppose that* $(\mathcal{L})_1$-$(\mathcal{L})_3$ *and* $(W_1) - (W_4)$ *hold. Then, there exists* $\Lambda_* > 0$ *such that for every* $\lambda > \Lambda_*$, $(\mathrm{FHS})_\lambda$ *has at least one nontrivial solution.*

To state our second result considering the concentration phenomena of the solution obtained by Theorem 1.1, we consider $T = [-\varrho, \varrho]$ for some $0 < \varrho < +\infty$, where T is given by $(\mathcal{L})_3$, then we have.

Theorem 1.2. *Let* u_λ *be a solution of problem* $(\mathrm{FHS})_\lambda$ *given by Theorem 1.1, then* $u_\lambda \to \tilde{u}$ *strongly in* $H^\alpha(\mathbb{R})$ *as* $\lambda \to \infty$, *where* $\tilde{u}$ *is a nontrivial solution of the following boundary value problem*

$$\begin{aligned} &{}_tD_\varrho^\alpha({}_{-\varrho}D_t^\alpha)u = \nabla W(t,u), t \in (-\varrho, \varrho), \\ &u(-\varrho) = u(\varrho) = 0, \end{aligned} \tag{1.7}$$

where ${}_{-\varrho}D_t^\alpha$ *and* ${}_tD_\varrho^\alpha$ *are left and right Riemann-Liouville fractional derivatives of order* α *on* $[-\varrho, \varrho]$ *respectively.*

Remark 1.1. In Theorems 1.1, 1.2, we give some new superquadratic conditions on $W(t,u)$ to guarantee the existence and concentration of solutions of $(\mathrm{FHS})_\lambda$. However, we must point out that the ideas used in [1, 27, 30, 35] are not applicable for our new assumptions. To overcome this difficulty we apply the Mountain Pass Theorem with Cerami condition, however, the direct application of the mountain pass theorem is not enough since the Cerami sequences might lose compactness in the whole space $\mathbb{R}$. Then it is necessary to introduce a new compactness result to recover the convergence of Cerami sequence, for more details see Lemma 3.2. Moreover, the main difficulty to proof Theorem 1.2, is to show that c_λ is bounded from above independent of λ. In the process to overcome this difficulty, we note that, the election of $T = [-\varrho, \varrho]$ play a key role which is very different from the previous works, where was enough to consider $T = [0, \mathbb{T}]$, for more details see section 3.

The remaining part of this paper is organized as follows. Some preliminary results are presented in Section 2. In Section 3, we are devoted to accomplishing the proof of Theorem 1.1 and in Section 4 we present the proof of Theorem 1.2.

2. Preliminary Results

In this section, for the reader's convenience, firstly we introduce some basic definitions of fractional calculus, for more details see [11]. The Liouville-Weyl fractional derivatives of order $0 < \alpha < 1$ are defined as

$$_{-\infty}D_x^\alpha u(x) = \frac{d}{dx}\,{}_{-\infty}I_x^{1-\alpha}u(x) \quad \text{and} \quad {}_xD_\infty^\alpha u(x) = -\frac{d}{dx}\,{}_xI_\infty^{1-\alpha}u(x), \tag{2.1}$$

where $_{-\infty}I_x^\alpha$ and $_xI_\infty^\alpha$ are the left and right Liouville-Weyl fractional integrals of order $0 < \alpha < 1$ defined as

$$_{-\infty}I_x^\alpha u(x)=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^x (x-\xi)^{\alpha-1}u(\xi)d\xi \text{ and } {}_xI_\infty^\alpha u(x)=\frac{1}{\Gamma(\alpha)}\int_x^\infty (\xi-x)^{\alpha-1}u(\xi)d\xi.$$

Furthermore, for $u \in L^p(\mathbb{R})$, $p \geq 1$, we have

$$\mathcal{F}({}_{-\infty}I_x^\alpha u(x)) = (i\omega)^{-\alpha}\widehat{u}(\omega) \qquad \text{and} \qquad \mathcal{F}({}_xI_\infty^\alpha u(x)) = (-i\omega)^{-\alpha}\widehat{u}(\omega),$$

and for $u \in C_0^\infty(\mathbb{R})$, we have

$$\mathcal{F}({}_{-\infty}D_x^\alpha u(x)) = (i\omega)^{\alpha}\widehat{u}(\omega) \qquad \text{and} \qquad \mathcal{F}({}_xD_\infty^\alpha u(x)) = (-i\omega)^{\alpha}\widehat{u}(\omega).$$

In order to establish the variational structure which enables us to reduce the existence of solutions of $(\mathrm{FHS})_\lambda$ to find critical points of the corresponding functional, it is necessary to consider some appropriate function spaces. Denote by $L^p(\mathbb{R},\mathbb{R}^n)$ $(1 \leq p < \infty)$ the Banach spaces of functions on $\mathbb{R}$ with values in $\mathbb{R}^n$ under the norms

$$\|u\|_{L^p} = \Big(\int_{\mathbb{R}} |u(t)|^p dt\Big)^{1/p},$$

and $L^\infty(\mathbb{R},\mathbb{R}^n)$ is the Banach space of essentially bounded functions from $\mathbb{R}$ into $\mathbb{R}^n$ equipped with the norm

$$\|u\|_\infty = \operatorname{ess\,sup}_{t\in\mathbb{R}}|u(t)|.$$

Let $-\infty < a < b < +\infty$, $0 < \alpha \leq 1$ and $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is defined by the closure of $C_0^\infty([a,b],\mathbb{R}^n)$ with respect to the norm

$$\|u\|_{\alpha,p} = \left(\int_a^b |u(t)|^p dt + \int_a^b |{}_aD_t^\alpha u(t)|^p dt\right)^{1/p}. \tag{2.2}$$

Next $(E_0^{\alpha,p}, \|.\|_{\alpha,p})$ is a reflexive and separable Banach space and for $\alpha \in (\frac{1}{2}, 1]$, $E_0^{\alpha,p}$ can be characterized as

$$E_0^{\alpha,p} = \{u \in L^p([a,b],\mathbb{R}^n)|{}_aD_t^\alpha u \in L^p([a,b],\mathbb{R}^n) \text{ and } u(a) = u(b) = 0\}.$$

Proposition 2.1 ([9]). *Let $0 < \alpha \leq 1$ and $1 < p < \infty$. For all $u \in E_0^{\alpha,p}$, we have*

$$\|u\|_{L^p} \leq \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}\|{}_aD_t^\alpha u\|_{L^p}. \tag{2.3}$$

If $\alpha > 1/p$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\|u\|_\infty \leq \frac{(b-a)^{\alpha-1/p}}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}}\|{}_aD_t^\alpha u\|_{L^p}. \tag{2.4}$$

By (2.3), we can consider in $E_0^{\alpha,p}$ the following norm

$$\|u\|_{\alpha,p} = \|{}_aD_t^\alpha u\|_{L^p}, \tag{2.5}$$

which is equivalent to (2.2).

Proposition 2.2 ([9]). *Let $0 < \alpha \leq 1$ and $1 < p < \infty$. Assume that $\alpha > \frac{1}{p}$ and $\{u_k\} \rightharpoonup u$ in $E_0^{\alpha,p}$. Then $u_k \to u$ in $C[a,b]$, i.e.*

$$\|u_k - u\|_\infty \to 0,\ k \to \infty.$$

For $\alpha > 0$, consider the Liouville-Weyl fractional spaces

$$I_{-\infty}^\alpha = \overline{C_0^\infty(\mathbb{R},\mathbb{R}^n)}^{\|\cdot\|_{I_{-\infty}^\alpha}},$$

where

$$\|u\|_{I_{-\infty}^\alpha} = \Big(\int_{\mathbb{R}} u^2(x)dx + \int_{\mathbb{R}} |{}_{-\infty}D_x^\alpha u(x)|^2 dx\Big)^{1/2}. \tag{2.6}$$

Furthermore, the classical fractional Sobolev space $H^\alpha(\mathbb{R},\mathbb{R}^n)$ is defined as

$$H^\alpha = \overline{C_0^\infty(\mathbb{R},\mathbb{R}^n)}^{\|\cdot\|_\alpha}, \tag{2.7}$$

where

$$\|u\|_\alpha = \Big(\int_{\mathbb{R}} u^2(x)dx + \int_{\mathbb{R}} |w|^{2\alpha}\widehat{u}^2(w)dw\Big)^{1/2}.$$

Note that, a function $u \in L^2(\mathbb{R},\mathbb{R}^n)$ belongs to $I_{-\infty}^\alpha$ if and only if

$$|w|^\alpha \widehat{u} \in L^2(\mathbb{R},\mathbb{R}^n).$$

Therefore, $I_{-\infty}^\alpha$ and H^α are equivalent with equivalent norm, for more details see [25].

Lemma 2.1 ([25, Theorem 2.1]). *If $\alpha > 1/2$, then $H^\alpha \subset C(\mathbb{R},\mathbb{R}^n)$ and there is a constant $C_\infty = C_{\alpha,\infty}$ such that*

$$\|u\|_\infty = \sup_{x\in\mathbb{R}} |u(x)| \leq C_\infty \|u\|_\alpha. \tag{2.8}$$

Remark 2.1. From Lemma 2.1, we know that if $u \in H^\alpha$ with $1/2 < \alpha < 1$, then $u \in L^p(\mathbb{R},\mathbb{R}^n)$ for all $p \in [2,\infty)$, since

$$\int_{\mathbb{R}} |u(x)|^p dx \leq \|u\|_\infty^{p-2}\|u\|_{L^2}^2.$$

For $\lambda > 0$, consider the fractional space $X^{\alpha,\lambda}$ given by

$$X^{\alpha,\lambda} = \Big\{u \in H^\alpha : \int_{\mathbb{R}} [|{}_{-\infty}D_t^\alpha u(t)|^2 + \lambda(L(t)u(t),u(t))]dt < \infty\Big\}.$$

$X^{\alpha,\lambda}$ is a reflexive and separable Hilbert space with the inner product

$$\langle u,v\rangle_{X^\alpha} = \int_{\mathbb{R}} [{}_{-\infty}D_t^\alpha u(t) \cdot {}_{-\infty}D_t^\alpha v(t) + \lambda(L(t)u(t),v(t))]dt$$

and the corresponding norm is

$$\|u\|_{X^\alpha}^2 = \langle u,u\rangle_{X^{\alpha,\lambda}}.$$

Remark 2.2. Suppose $L(t)$ satisfies $(\mathcal{L})_1$ and $(\mathcal{L})_2$, then, for all $\lambda \geq \frac{1}{cC_\infty^2 |\{l<c\}|}$, we get

$$\int_{\mathbb{R}} |u(t)|^2 dt \leq \frac{C_\infty^2 |\{l<c\}|}{1-C_\infty^2 |\{l<c\}|} \|u\|_{X^{\alpha,\lambda}} = \frac{1}{\Theta}\|u\|^2_{X^{\alpha,\lambda}} \tag{2.9}$$

and

$$\|u\|_\alpha^2 \leq \Big(1+\frac{C_\infty^2 |\{l<c\}|}{1-C_\infty^2 |\{l<c\}|}\Big)\|u\|^2_{X^\alpha} = (1+\frac{1}{\Theta})\|u\|^2_{X^{\alpha,\lambda}}. \tag{2.10}$$

Then $X^{\alpha,\lambda}$ is continuously embedded in H^α. Furthermore, for every $\lambda \geq \frac{1}{cC_\infty^2 |\{l<c\}|}$ and $p \in (2,\infty)$, we have

$$\int_{\mathbb{R}} |u(t)|^p dt \leq \mathcal{K}_p^p \|u\|^p_{X^{\alpha,\lambda}}. \tag{2.11}$$

where $\mathcal{K}_p^p = \frac{1}{\Theta^{\frac{p}{2}} |\{l<c\}|^{\frac{p-2}{2}}}$. For more details, see [27,35].

3. Proof of Theorem 1.1

The aim of this section is to establish the proof of Theorem 1.1. Consider the functional $I : X^{\alpha,\lambda} \to \mathbb{R}$ defined as

$$I_\lambda(u) = \frac{1}{2}\|u\|^2_{X^{\alpha,\lambda}} - \int_{\mathbb{R}} W(t,u(t))dt. \tag{3.1}$$

Under the conditions of Theorem 1.1, we can show that $I \in C^1(X^{\alpha,\lambda},\mathbb{R})$, and

$$I'_\lambda(u)v = \int_{\mathbb{R}} \Big[({}_{-\infty}D_t^\alpha u(t), {}_{-\infty} D_t^\alpha v(t)) + (\lambda L(t)u(t),v(t)) - (\nabla W(t,u(t)),v(t))\Big]dt \tag{3.2}$$

for all u, $v \in X^\alpha$. In particular we have

$$I'_\lambda(u)u = \|u\|^2_{X^{\alpha,\lambda}} - \int_{\mathbb{R}} (\nabla W(t,u(t)),u(t))dt. \tag{3.3}$$

Remark 3.1. By (W_1) and (W_4), for any $\epsilon > 0$, there is $C_\epsilon > 0$ such that

$$|\nabla W(t,u)| \leq \epsilon|u| + C_\epsilon |u|^{p-1}, \ \ \forall (t,u) \in \mathbb{R}\times\mathbb{R}^N \tag{3.4}$$

and

$$|W(t,u)| \leq \frac{\epsilon}{2}|u|^2 + \frac{C_\epsilon}{p}|u|^p \ \ \forall (t,u) \in \mathbb{R}\times\mathbb{R}^N, \tag{3.5}$$

where $p = \frac{2\sigma}{\sigma-1} > 2$.

We start our analysis by considering the following compactness results which is important to recover the Cerami condition for I_λ.

Lemma 3.1. *Suppose that $(\mathcal{L})_1 - (\mathcal{L})_3$ and $(W_1) - (W_4)$ hold. If $u_n \rightharpoonup u$ in $X^{\alpha,\lambda}$, then*

$$I_\lambda(u_n - u) = I_\lambda(u_n) - I_\lambda(u) + o(1) \ \ \text{as} \ \ n \to +\infty \tag{3.6}$$

and

$$I'_\lambda(u_n - u) = I'_\lambda(u_n) - I'_\lambda(u) + o(1) \ \ \text{as} \ \ n \to +\infty. \tag{3.7}$$

In particular, if $I_\lambda(u_n) \to c$ and $I'_\lambda(u_n) \to 0$, then, up to a subsequence $I'_\lambda(u) = 0$.

Proof. To show (3.6) and (3.7) it suffices to check that

$$\int_{\mathbb{R}}[W(t,u_n)-W(t,u_n-u)-W(t,u)]dt=o(1) \tag{3.8}$$

and

$$\sup_{\varphi\in X^{\alpha,\lambda},\|\varphi\|_{\alpha,\lambda}=1}\int_{\mathbb{R}}\langle\nabla W(t,u_n)-\nabla W(t,u_n-u)-\nabla W(t,u),\varphi\rangle dt=o(1), \tag{3.9}$$

because the weak convergence of u_n to u in $X^{\alpha,\lambda}$ implies that

$$\|u_n\|^2_{X^{\alpha,\lambda}}=\|u_n-u\|^2_{X^{\alpha,\lambda}}+\|u\|^2_{X^{\alpha,\lambda}}+o(1).$$

We are going to prove (3.9), since (3.8) is proved in a similar way. In fact, let

$$\Pi:=\lim_{n\to\infty}\sup_{\varphi\in X^{\alpha,\lambda},\|\varphi\|_{\alpha,\lambda}=1}\int_{\mathbb{R}}\langle\nabla W(t,u_n)-\nabla W(t,u_n-u)-\nabla W(t,u),\varphi\rangle dt. \tag{3.10}$$

If $\Pi>0$, then, there exists $\varphi_0\in X^{\alpha,\lambda}$ with $\|\varphi_0\|_{X^{\alpha,\lambda}}=1$ such that

$$\left|\int_{\mathbb{R}}\langle\nabla W(t,u_n)-\nabla W(t,u_n-u)-\nabla W(t,u),\varphi_0\rangle dt\right|\geq\frac{\Pi}{2},\quad\text{for } n \text{ large enough.} \tag{3.11}$$

From (3.4) and Young's inequality, there exist C_1, C_2 and $C_3>0$ such that

$$\begin{aligned}&|\langle\nabla W(t,u_n)-\nabla W(t,u_n-u),\varphi_0\rangle|\\ \leq&C_1\left(\epsilon|u|^2+\epsilon|u_n-u|^2+\epsilon|\varphi_0|^2+C_2|u|^p+\epsilon|u_n-u|^p+C_3|\varphi_0|^p\right).\end{aligned}$$

Hence, there exists $C_4,C_5,C_6>0$ such that

$$\begin{aligned}&|\langle\nabla W(t,u_n)-\nabla W(t,u_n-u)-\nabla W(t,u),\varphi_0\rangle|\\ \leq&C_4\left(\epsilon|u|^2+\epsilon|u_n-u|^2+\epsilon|\varphi_0|^2+C_5|u|^p+\epsilon|u_n-u|^p+C_6|\varphi_0|^p\right).\end{aligned}$$

Let

$$h_n(t)=\max\{|\langle\nabla W(t,u_n)-\nabla W(t,u_n-u)-\nabla W(t,u),\varphi_0\rangle|-C_4\epsilon(|u_n-u|^2+|u_n-u|^p),0\}.$$

So

$$0\leq h_n(t)\leq C_4(\epsilon|u|^2+\epsilon|\varphi_0|^2+C_5|u|^p+C_6|\varphi_0|^p).$$

By the Lebesgue dominated convergence theorem and the fact $u_n\to u$ a.e. in $\mathbb{R}$, we obtain

$$\int_{\mathbb{R}}h_n(t)dt\to0\ \text{ as }\ n\to\infty.$$

From where

$$\int_{\mathbb{R}}|\langle\nabla W(t,u_n(t))-\nabla W(t,u_n(t)-u(t))-\nabla W(t,u(t)),\varphi_0(t)\rangle|dt\to0\ \text{ as }\ n\to\infty,$$

which contradict (3.11).

On the other hand, if $I_\lambda(u_n)\to c$ and $I'_\lambda(u_n)\to0$ as $n\to\infty$, by (3.6) and (3.7), we get

$$I_\lambda(u_n-u)\to c-I_\lambda(u)+o(1)$$

and

$$I'_\lambda(u_n - u) = -I'_\lambda(u) \text{ as } n \to +\infty.$$

So, for every $\varphi \in C_0^\infty(\mathbb{R}, \mathbb{R}^n)$ we have

$$I'_\lambda(u)\varphi = \lim_{n\to\infty} I'_\lambda(u_n)\varphi = 0.$$

Consequently, $I'_\lambda(u) = 0$. $\square$

Lemma 3.2. *Suppose that $(\mathcal{L})_1 - (\mathcal{L})_3$ and $(W_1) - (W_4)$ hold. Let $c \in \mathbb{R}^+$, then each $(Ce)_c$-sequence of I_λ is bounded in $X^{\alpha,\lambda}$.*

Proof. Let $(u_n)_{n\in\mathbb{N}} \subset X^{\alpha,\lambda}$ a sequence such that

$$I_\lambda(u_n) \to c, \quad (1 + \|u_n\|_{X^{\alpha,\lambda}})I'_\lambda(u_n) \to 0 \text{ as } n \to \infty. \tag{3.12}$$

Then

$$\begin{aligned} c - o_n(1) = I_\lambda(u_n) - \frac{1}{2}I'_\lambda(u_n)u_n &= \int_{\mathbb{R}} H(t, u_n(t))dt \\ &= \int_{\Omega_n(0,a)} H(t,u_n)dt + \int_{\Omega_n(a,b)} H(t,u_n)dt + \int_{\Omega_n(b,+\infty)} H(t,u_n)dt, \end{aligned} \tag{3.13}$$

where

$$\Omega_n(a,b) := \{t \in \mathbb{R} : \ a \le |u_n(t)| < b\}, \text{ with } 0 \le a < b.$$

Suppose by contradiction, there is a subsequence, still denoted by (u_n), such that $\|u_n\|_{X^{\alpha,\lambda}} \to +\infty$ as $n \to +\infty$. Taking $v_n = \frac{u_n}{\|u_n\|_{X^{\alpha,\lambda}}}$, then $(v_n)_{n\in\mathbb{N}}$ is bounded in $X^{\alpha,\lambda}$ and $\|v_n\|_{X^{\alpha,\lambda}} = 1$. Moreover, if $n \to \infty$ we obtain

$$o(1) = \frac{\langle I'_\lambda(u_n), u_n\rangle}{\|u_n\|^2_{X^{\alpha,\lambda}}} = 1 - \int_{\mathbb{R}} \frac{\langle \nabla W(t,u_n), u_n\rangle}{\|u_n\|^2_{X^{\alpha,\lambda}}},$$

which implies

$$\int_{\mathbb{R}} \frac{\langle \nabla W(t,u_n), v_n\rangle}{|u_n|}|v_n|dt = \int_{\mathbb{R}} \frac{\langle \nabla W(t,u_n), u_n\rangle}{\|u_n\|^2_{X^{\alpha,\lambda}}} \to 1. \tag{3.14}$$

For $r \ge 0$, let

$$h(r) := \inf\{H(t,u) : \ t \in \mathbb{R}, \ |u| \ge r\}.$$

From (W_2) we have $h(r) > 0$ for all $r > 0$. Furthermore, by (W_2) and (W_4), for $|u| \ge r$,

$$C_0 H(t,u) \ge \frac{|\nabla W(t,u)|^\sigma}{|u|^\sigma} \ge \left(\frac{\langle \nabla W(t,u), u\rangle}{|u|^2}\right)^\sigma \ge \left(\frac{2W(t,u)}{|u|^2}\right)^\sigma. \tag{3.15}$$

By (W_3), (3.15) and the definition of $h(r)$ we obtain

$$h(r) \to \infty \text{ as } r \to \infty.$$

Let

$$C_a^b := \inf\left\{\frac{H(t,u)}{|u|^2} : \ t \in \mathbb{R} \text{ and } u \in \mathbb{R}^N \text{ with } a \le |u| < b\right\}.$$

So

$$H(t,u_n) \geq C_a^b |u_n|^2 \quad \forall t \in \Omega_n(a,b),$$

consequently, by (3.13) we get

$$\begin{aligned} c - o_n(1) &\geq \int_{\Omega_n(0,a)} H(t,u_n)dt + C_a^b \int_{\Omega_n(a,b)} |u_n|^2 dt + \int_{\Omega_n(b,+\infty)} H(t,u_n)dt \\ &= \int_{\Omega_n(0,a)} H(t,u_n)dt + C_a^b \int_{\Omega_n(a,b)} |u_n|^2 dt + h(b) meas(\Omega_n(b,+\infty)). \end{aligned} \tag{3.16}$$

Since $h(r) \to +\infty$ as $r \to +\infty$, for $2 < \frac{2\sigma}{\sigma-1} = p < q < \infty$ it follows from (3.16) that

$$\begin{aligned} \int_{\Omega_n(b,+\infty)} |v_n|^p dt &\leq \left(\int_{\Omega_n(b,+\infty)} |v_n|^q dt \right)^{\frac{p}{q}} meas(\Omega_n(b+\infty))^{\frac{q-p}{q}} \\ &\leq \|v_n\|_{L^q}^p \left(\frac{c - o_n(1)}{h(b)} \right)^{\frac{q-p}{p}} \leq \mathcal{K}_q^p \left(\frac{c - o_n(1)}{h(b)} \right)^{\frac{q-p}{p}} \to 0 \end{aligned} \tag{3.17}$$

as $b \to +\infty$. Furthermore, by (W_4) and the Hölder inequality, for any $\epsilon > 0$, we can choose $R > 0$ large enough such that

$$\begin{aligned} &\left| \int_{\Omega_n(R,+\infty)} \frac{\langle \nabla W(t,u_n), u_n \rangle}{\|u_n\|_{X^{\alpha,\lambda}}^2} dt \right| \\ &\leq \int_{\Omega_n(R,+\infty)} \frac{|\nabla W(t,u_n)|}{|u_n|} |v_n|^2 dt \\ &\leq \left(\int_{\Omega_n(R,+\infty)} \frac{|\nabla W(t,u_n)|^\sigma}{|u_n|^\sigma} \right)^{1/\sigma} \left(\int_{\Omega_n(R,+\infty)} |v_n|^p dt \right)^{\frac{\sigma-1}{\sigma}} \\ &\leq \left(\int_{\Omega_n(R,+\infty)} C_0 H(t,u_n) dt \right)^{1/\sigma} \left(\int_{\Omega_n(R,+\infty)} |v_n|^p dt \right)^{\frac{\sigma-1}{\sigma}} \\ &\leq C_0^{1/\sigma} (c - o_n(1))^{1/\sigma} \left(\int_{\Omega_n(R,+\infty)} |v_n|^p dt \right)^{\frac{\sigma-1}{\sigma}} < \epsilon. \end{aligned} \tag{3.18}$$

and by (W_1), there is $\delta > 0$ such that

$$\int_{\Omega_n(0,\delta)} \frac{|\nabla W(t,u_n)|}{|u_n|} |v_n|^2 dt \leq \int_{\Omega_n(0,\delta)} \frac{\epsilon}{\mathcal{K}_2^2} |v_n|^2 dt \leq \frac{\epsilon}{\mathcal{K}_2^2} \|v_n\|_{L^2}^2 \leq \epsilon, \quad \forall n. \tag{3.19}$$

Now, by using (3.16) again, we get

$$\int_{\Omega_n(\delta,R)} |v_n|^2 dt = \frac{1}{\|u_n\|_{X^{\alpha,\lambda}}^2} \int_{\Omega_n(\delta,R)} |u_n|^2 dt \leq \frac{c - o_n(1)}{C_\delta^R \|u_n\|_{X^{\alpha,\lambda}}^2} \to 0$$

as $n \to \infty$. Then, for n large enough, by the continuity of ∇W one has

$$\int_{\Omega_n(\delta,R)} \frac{|\nabla W(t,u_n)|}{|u_n|} |v_n|^2 dt \leq K \int_{\Omega_n(\delta,R)} |v_n|^2 dt < \epsilon. \tag{3.20}$$

Hence, by (3.19), (3.18) and (3.20), for n large enough we have

$$\int_{\mathbb{R}} \frac{\langle \nabla W(t,u_n), v_n\rangle}{|u_n|}|v_n|dt \leq \int_{\mathbb{R}} \frac{|\nabla W(t,u_n)|}{|u_n|}|v_n|^2 dt \leq 3\epsilon < 1,$$

which is a contradiction with (3.14). Then $(u_n)_{n\in\mathbb{N}}$ is bounded in $X^{\alpha,\lambda}$. $\square$

Lemma 3.3. *Suppose that $(\mathcal{L})_1-(\mathcal{L})_3$ and $(W_1)-(W_4)$ hold. Then, for any $\mathfrak{C} > 0$, there exists $\Lambda_1 = \Lambda(\mathfrak{C}) > 0$ such that I_λ satisfies $(Ce)_c$ condition for all $c \leq \mathfrak{C}$ and $\lambda > \Lambda_1$.*

Proof. For any $\mathfrak{C} > 0$, suppose that $(u_n)_{n\in\mathbb{N}} \subset X^{\alpha,\lambda}$ is a $(Ce)_c$ sequence for $c \leq \mathfrak{C}$, namely

$$I_\lambda(u_n) \to c \;\text{ and } \quad (1+\|u_n\|_{X^{\alpha,\lambda}})I'_\lambda(u_n) \to 0 \;\text{ as }\; n\to\infty.$$

By Lemma 3.2, $(u_n)_{n\in\mathbb{N}}$ is bounded. Therefore, there exists $u \in X^{\alpha,\lambda}$ such that

$$\begin{aligned} &u_n \rightharpoonup u \text{ in } X^{\alpha,\lambda} \text{ and}\\ &u_n \to u \text{ a.e. in } \mathbb{R}. \end{aligned}$$

Let $w_n := u_n - u$. By Lemma 3.1 we get

$$I'_\lambda(u) = 0, \quad I_\lambda(w_n) \to c - I_\lambda(u) \quad \text{and} \quad I'_\lambda(w_n) \to 0 \;\text{ as }\; n\to\infty.$$

Next

$$I_\lambda(u) = I_\lambda(u) - \frac{1}{2}I'_\lambda(u)u = \int_{\mathbb{R}} H(t,u)dt \geq 0, \tag{3.21}$$

and

$$\int_{\mathbb{R}} H(t,w_n)dt \to c - I_\lambda(u). \tag{3.22}$$

Therefore, for $c \leq \mathfrak{C}$, we get

$$\int_{\mathbb{R}} H(t,w_n)dt \leq \mathfrak{C} + o_n(1). \tag{3.23}$$

On the other hand, by $(\mathcal{L})_1$ and since $w_n \to 0$ in $L^2_{loc}(\mathbb{R},\mathbb{R}^N)$, we have

$$\|w_n\|^2_{L^2} \leq \frac{1}{\lambda c}\int_{\{l\geq c\}} \lambda\langle L(t)w_n, w_n\rangle dt + o_n(1) \leq \frac{1}{\lambda c}\|w_n\|^2_{X^{\alpha,\lambda}} + o_n(1). \tag{3.24}$$

Let $p < q < \infty$, where $p = \frac{2\sigma}{\sigma-1}$. Using Remark 2.2 and Hölder inequality we obtain

$$\begin{aligned}\int_{\mathbb{R}} |w_n|^p dt &= \int_{\mathbb{R}} |w_n|^{\frac{2(q-p)}{q-2}}|w_n|^{\frac{q(p-2)}{q-2}}dt \leq \|w_n\|_{L^2}^{\frac{2(q-p)}{q-2}}\|w_n\|_{L^q}^{\frac{q(p-2)}{q-2}}\\ &\leq \mathcal{K}_q^{\frac{q(p-2)}{q-2}}\left(\frac{1}{\lambda c}\right)^{\frac{q-p}{q-2}}\|w_n\|^p_{X^{\alpha,\lambda}} + o_n(1).\end{aligned} \tag{3.25}$$

Furthermore, for $|u| \leq R$ (where R is defined in (W_4)), from (3.4), we get

$$|\nabla W(t,u)| \leq (\epsilon + C_\epsilon R^{p-2})|u| = \tilde{C}|u|.$$

It follows from (3.24) that

$$\begin{aligned}
&\int_{\{t\in\mathbb{R}:\ |w_n(t)|\le R\}} \langle \nabla W(t,w_n), w_n\rangle dt \\
&\le \int_{\{t\in\mathbb{R}:\ |w_n(t)|\le R\}} |\nabla W(t,w_n)||w_n| dt \\
&\le \tilde{C}\int_{\{t\in\mathbb{R}:\ |w_n(t)|\le R\}} |w_n|^2 dt \le \frac{\tilde{C}}{\lambda c}\|w_n\|^2_{X^{\alpha,\lambda}} + o_n(1).
\end{aligned}$$

On the other hand, from (3.25) and the Hölder inequality we obtain

$$\begin{aligned}
&\int_{\{t\in\mathbb{R}:\ |w_n(t)|> R\}} \langle \nabla W(t,w_n), w_n\rangle dt \le \int_{\{t\in\mathbb{R}:\ |w_n(t)|> R\}} |\nabla W(t,w_n)||w_n| dt \\
&\le \int_{\{t\in\mathbb{R}:\ |w_n(t)|> R\}} \frac{|\nabla W(t,w_n)|}{|w_n|}|w_n|^2 dt \\
&\le \left(\int_{\{t\in\mathbb{R}:\ |w_n(t)|> R\}} \frac{|\nabla W(t,w_n)|^\sigma}{|w_n|^\sigma} dt\right)^{1/\sigma} \left(\int_{\{t\in\mathbb{R}:\ |w_n(t)|> A\}} |w_n|^p\right)^{\frac{2}{p}} \\
&\le \left(C_0\int_{\mathbb{R}} H(t,w_n)dt\right)^{1/\sigma} \|w_n\|_p^2 \\
&\le (C_0\mathfrak{C})^{1/\sigma}\mathcal{K}_q^{\frac{2q(p-2)}{p(q-2)}}\left(\frac{1}{\lambda c}\right)^{\frac{2(q-p)}{p(q-2)}} \|w_n\|^2_{X^{\alpha,\lambda}} + o_n(1).
\end{aligned}$$

Therefore

$$\begin{aligned}
o_n(1) = \langle I'_\lambda(w_n), w_n\rangle &= \|w_n\|^2_{X^{\alpha,\lambda}} - \int_{\mathbb{R}} \langle \nabla W(t,w_n), w_n\rangle dt \\
&= \|w_n\|^2_{X^{\alpha,\lambda}} - \int_{\{t\in\mathbb{R}:\ |w_n(t)|\le R\}} \langle \nabla W(t,w_n), w_n\rangle dt \\
&\quad - \int_{\{t\in\mathbb{R}:\ |w_n(t)|> R\}} \langle \nabla W(t,w_n), w_n\rangle dt \\
&\ge \left(1 - \frac{\tilde{C}}{\lambda c} - C^*\left(\frac{1}{\lambda c}\right)^{\frac{2(q-p)}{p(q-2)}}\right)\|w_n\|^2_{X^{\alpha,\lambda}} + o_n(1),
\end{aligned}$$

where $C^* = (C_0\mathfrak{C})^{1/\sigma}\mathcal{K}_q^{\frac{2q(q-p)}{p(q-2)}}$. Now, we choose $\Lambda_1 = \Lambda(\mathfrak{C}) > 0$ large enough such that

$$1 - \frac{\tilde{C}}{\lambda c} - C^*\left(\frac{1}{\lambda c}\right)^{\frac{2(q-p)}{p(q-2)}} > 0 \quad \text{for all } \ \lambda > \Lambda_1.$$

Then $w_n \to 0$ in $X^{\alpha,\lambda}$ for all $\lambda > \Lambda_1$. □

Proof of Theorem 1.1. By Lemma 3.3, I_λ satisfies the $(Ce)_c$-condition. In order to apply the mountain pass theorem with Cerami condition we just need to show that I_λ has the mountain pass geometry. In fact, by (3.5) and Remark 2.2, we obtain

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{2}\|u\|^2_{X^{\alpha,\lambda}} - \frac{\epsilon}{2}\int_{\mathbb{R}} |u(t)|^2 dt - \frac{C_\epsilon}{p}\int_{\mathbb{R}} |u(t)|^p dt \\ &\geq \frac{1}{2}\left(1 - \frac{\epsilon}{\Theta}\right)\|u\|^2_{X^{\alpha,\lambda}} - \frac{C_\epsilon}{p\Theta^{\frac{p}{2}}(meas\{l<c\})^{\frac{p-2}{2}}}\|u\|^p_{X^{\alpha,\lambda}}. \end{aligned}$$

Let $\epsilon > 0$ small enough such that $1 - \frac{\epsilon}{\Theta} > 0$ and $\|u\|_{X^{\alpha,\lambda}} = \rho$. Since $p > 2$, taking ρ small enough such that

$$\frac{1}{2}\left(1 - \frac{\epsilon}{\Theta}\right) - \frac{C_\epsilon}{p\Theta^{\frac{p}{2}}(meas\{l<c\})^{\frac{p-2}{2}}}\rho^{p-2} > 0.$$

Then

$$I_\lambda(u) \geq \rho^2\left[\frac{1}{2}\left(1 - \frac{\epsilon}{\Theta}\right) - \frac{C_\epsilon}{p\Theta^{\frac{p}{2}}(meas\{l<c\})^{\frac{p-2}{2}}}\rho^{p-2}\right] := \eta > 0.$$

On the other hand, let $T = (-\varrho, \varrho) \subset J$ such that $L(t) \equiv 0$. Let $\psi \in C_0^\infty(\mathbb{R}, \mathbb{R}^n)$ such that $supp(\psi) \subset (-\tau, \tau)$, for some $\tau < \varrho$. Hence

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}} \langle L(t)\psi, \psi\rangle dt = \int_{supp(\psi)} \langle L(t)\psi, \psi\rangle dt \\ &\leq \int_{-\tau}^{\tau} \langle L(t)\psi, \psi\rangle dt \leq \int_T \langle L(t)\psi, \psi\rangle dt = 0. \end{aligned} \tag{3.26}$$

Furthermore, by (W_3), for any $\epsilon > 0$, there exists $R > 0$ such that

$$W(t,u) > \frac{|u|^2}{\epsilon} - \frac{R^2}{\epsilon} \text{ for all } |u| \geq R.$$

Then, by taking $\epsilon \to 0$ we get

$$\lim_{|\sigma|\to\infty} \int_{supp(\psi)} \frac{W(t,\sigma\psi)}{|\sigma|^2} dt = +\infty. \tag{3.27}$$

Hence, by (3.26) and (3.27) we obtain

$$\frac{I_\lambda(\sigma\psi)}{|\sigma|^2} = \frac{1}{2}\int_{\mathbb{R}} |{}_{-\infty}D_t^\alpha \psi(t)|^2 dt - \int_{\mathbb{R}} \frac{W(t,\sigma\psi)}{|\sigma|^2} dt \to -\infty, \tag{3.28}$$

as $|\sigma| \to \infty$. So, if σ_0 is large enough and $e = \sigma_0\psi$ one gets $I_\lambda(e) < 0$. Therefore, by using mountain pass lemma with Cerami condition [5], for any $c_\lambda > 0$ defined as follows

$$c_\lambda = \inf_{g\in\Gamma} \max_{s\in[0,1]} I_\lambda(g(s)),$$

where

$$\Gamma = \{g \in C([0,1], X^{\alpha,\lambda}) \,|\, g(0) = 0, g(1) = e\},$$

there exists $u_\lambda \in X^{\alpha,\lambda}$ such that

$$I_\lambda(u_\lambda) = c_\lambda \quad \text{and} \quad I'_\lambda(u_\lambda) = 0. \tag{3.29}$$

That is, $(\mathrm{FHS})_\lambda$ has at least one nontrivial solution for $\lambda > \Lambda(c_\lambda)$. □

4. Concentration phenomena

In this section, we study the concentration of solutions for problem $(\mathrm{FHS})_\lambda$ as $\lambda \to \infty$. That is, we focus our attention on the proof of Theorem 1.2. The main difficulty to proof Theorem 1.2, is to show that c_λ is bounded form above independent of λ. To overcome this problem, choose ψ as in the proof of Theorem 1.1, then by definition of c_λ, we have

$$\begin{aligned} c_\lambda &\leq \max_{\sigma\geq 0} I_\lambda(\sigma\psi) \\ &= \max_{\sigma\geq 0}\left(\frac{\sigma^2}{2}\int_{\mathbb{R}} |{}_{-\infty}D_t^\alpha \psi(t)|^2 - \int_{\mathbb{R}} W(t,\sigma\psi)dt\right) \\ &= \tilde{c}, \end{aligned}$$

where $\tilde{c} < +\infty$ is independent of λ.

As a consequence of the above estimates, we have that $\Lambda(c_\lambda)$ is bounded from below. That is, there exists $\Lambda_* > 0$ such that the conclusion of Theorem 1.1 is satisfied for $\lambda > \Lambda_*$.

Consider the following fractional boundary value problem

$$\begin{cases} {}_tD_{\varrho}^\alpha\, {}_{-\varrho}D_t^\alpha u = \nabla W(t,u), \quad t \in (-\varrho,\varrho), \\ u(-\varrho) = u(\varrho) = 0. \end{cases} \tag{4.1}$$

Associated to (4.1) we have the energy functional $I : E_0^\alpha \to \mathbb{R}$ given by

$$I(u) := \frac{1}{2}\int_{-\varrho}^{\varrho} |{}_{-\varrho}D_t^\alpha u(t)|^2 dt - \int_{-\varrho}^{\varrho} W(t,u(t))dt$$

and we have that $I \in C^1(E_0^\alpha,\mathbb{R})$ with

$$I'(u)v = \int_{-\varrho}^{\varrho} \langle {}_{-\varrho}D_t^\alpha u(t), {}_{-\varrho}D_t^\alpha v(t)\rangle dt - \int_{-\varrho}^{\varrho} \langle \nabla W(t,u(t)), v(t)\rangle dt.$$

Following the ideas of the proof of Theorem 1.1, we can get the following existence result

Theorem 4.1. *Suppose that W satisfies $(W_1) - (W_4)$, then (4.1) has at least one weak nontrivial solution.*

Proof of Theorem 1.2. We follow the argument in [35]. For any sequence $\lambda_k \to \infty$, let $u_k = u_{\lambda_k}$ be the critical point of I_{λ_k}, namely

$$c_{\lambda_k} = I_{\lambda_k}(u_k) \quad \text{and} \quad I'_{\lambda_k}(u_k) = 0,$$

and, by (3.5), we get

$$\begin{aligned} c_{\lambda_k} = I_{\lambda_k}(u_k) &= \frac{1}{2}\|u_k\|^2_{X^{\alpha,\lambda}} - \int_{\mathbb{R}} W(t,u_k(t))dt \\ &\geq \frac{1}{2}\|u_k\|^2_{X^{\alpha,\lambda}} - \frac{\epsilon}{2}\int_{\mathbb{R}} |u_k|^2 dt - \frac{C_\epsilon}{p}\int_{\mathbb{R}} |u_k|^p dt, \end{aligned}$$

which implies that (u_k) is bounded, due to Remarks 2.1 and 2.2. Therefore, we may assume that $u_k \rightharpoonup \tilde{u}$ weakly in X^{α,λ_k}. Moreover, by Fatou's lemma, we have

$$\begin{aligned}\int_{\mathbb{R}} l(t)|\tilde{u}(t)|^2dt &\leq \liminf_{k\to\infty}\int_{\mathbb{R}} l(t)|u_k(t)|^2dt \\ &\leq \liminf_{k\to\infty}\int_{\mathbb{R}}(L(t)u_k(t),u_k(t))dt \leq \liminf_{k\to\infty}\frac{\|u_k\|^2_{X^{\alpha,\lambda_k}}}{\lambda_k} = 0.\end{aligned}$$

Thus, $\tilde{u} = 0$ a.e. in $\mathbb{R}\setminus J$. Now, for any $\varphi \in C_0^\infty((-\varrho,\varrho),\mathbb{R}^n)$, since $I'_{\lambda_k}(u_k)\varphi = 0$, it is easy to see that

$$\int_{-\varrho}^{\varrho}({}_{-\varrho}D_t^\alpha\tilde{u}(t), {}_{-\varrho}D_t^\alpha\varphi(t))dt - \int_{-\varrho}^{\varrho}(\nabla W(t,\tilde{u}(t)),\varphi(t))dt = 0,$$

that is, $\tilde{u}$ is a solution of (4.1) by the density of $C_0^\infty(T,\mathbb{R}^n)$ in E^α.

Now we show that $u_k \to \tilde{u}$ in X^α. Since $I'_{\lambda_k}(u_k)u_k = I'_{\lambda_k}(u_k)\tilde{u} = 0$, we have

$$\|u_k\|^2_{X^{\alpha,\lambda_k}} = \int_{\mathbb{R}}(\nabla W(t,u_k(t)),u_k(t))dt \tag{4.2}$$

and

$$\langle u_k,\tilde{u}\rangle_{\lambda_k} = \int_{\mathbb{R}}(\nabla W(t,u_k(t)),\tilde{u}(t))dt, \tag{4.3}$$

which implies that

$$\lim_{k\to\infty}\|u_k\|^2_{X^{\alpha,\lambda_k}} = \lim_{k\to\infty}\langle u_k,\tilde{u}\rangle_{X^{\alpha,\lambda_k}} = \lim_{k\to\infty}\langle u_k,\tilde{u}\rangle_{X^\alpha} = \|\tilde{u}\|^2_{X^\alpha}.$$

Furthermore, by the weak semi-continuity of norms we obtain

$$\|\tilde{u}\|^2_{X^\alpha} \leq \liminf_{k\to\infty}\|u_k\|^2_{X^\alpha} \leq \limsup_{k\to\infty}\|u_k\|^2_{X^\alpha} \leq \lim_{k\to\infty}\|u_k\|^2_{X^{\alpha,\lambda_k}}.$$

So $u_k \to \tilde{u}$ in X^α, and $u_k \to \tilde{u}$ in $H^\alpha(\mathbb{R},\mathbb{R}^n)$ as $k\to\infty$. □

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