Journal of Applied Analysis and Computation
Volume 10, Number 3, June 2020, 920–934

Website:http://jaac-online.com/
DOI:10.11948/20190153

EQUIVALENT PROPERTY OF A MORE ACCURATE HALF-DISCRETE HILBERT'S INEQUALITY*

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Abstract By using the weight functions, the idea of introducing parameters, and Hermite-Hadamard's inequality, a more accurate half-discrete Hilbert's inequality with the nonhomogeneous kernel and its equivalent form are given. The equivalent statements of the best possible constant factor related to parameters, the operator expressions and some particular cases are considered. The cases of the relating homogeneous kernel are also deduced.

Keywords Weight function, half-discrete Hilbert's inequality, equivalent statement, Hermite-Hadamard's inequality, operator expression.

MSC(2010) 26D15.

1. Introduction

Assuming that $0 < \sum_{m=1}^{\infty} a_m^2 < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, we have the following Hilbert's inequality with the best possible constant factor π (cf. [3], Theorem 315):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \pi\left(\sum_{m=1}^{\infty} a_m^2 \sum_{n=1}^{\infty} b_n^2\right)^{1/2}. \tag{1.1}$$

If $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(y)dy < \infty$, then we still have the following Hilbert's integral inequality:

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \pi\left(\int_0^\infty f^2(x)dx\int_0^\infty g^2(y)dy\right)^{\frac{1}{2}}, \tag{1.2}$$

with the same best possible constant factor π (cf. [3], Theorem 316). Inequalities (1), (2) and their extensions with $(p,q)(p>1,\frac{1}{p}+\frac{1}{q}=1)$ are important in analysis and its applications (cf. [1, 2, 7, 11–17, 20]).

We still have the following half-discrete Hilbert-type inequalities (cf. [3], Theorem 351): If $K(x)(x>0)$ is a decreasing function, $p>1, \frac{1}{p}+\frac{1}{q}=1, 0<\phi(s)=$

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*The authors were supported by National Natural Science Foundation (No. 61772140), and Science and Technology Planning Project Item of Guangzhou City (No. 201707010229).

$\int_0^\infty K(x)x^{s-1}dx < \infty$, then

$$\int_0^\infty x^{p-2}\left(\sum_{n=1}^\infty K(nx)a_n\right)^p < \phi^p(\frac{1}{q})\sum_{n=1}^\infty a_n^p, \tag{1.3}$$

$$\sum_{n=1}^\infty n^{p-2}\left(\int_0^\infty K(nx)f(x)dx\right)^p < \phi^p(\frac{1}{q})\int_0^\infty f^p(x)dx. \tag{1.4}$$

In the last years, some new extensions of (1.3) and (1.4) with their applications were provided by [8–10, 18, 19].

In 2016, by the use of the technique of real analysis, Hong [4] and [5] considered some equivalent statements of the extensions of (1.1) and (1.2) with a few parameters.

In this paper, following the way of [4] and [5], by the use of the weight functions, the idea of introducing parameters and Hermite-Hadamard's inequality, a more accurate half-discrete Hilbert's inequality with the nonhomogeneous kernel and its equivalent form are given. The equivalent statements of the best possible constant factor related to a few parameters, the operator expressions and some particular cases are considered. The cases of the relating homogeneous kernel are also deduced.

2. Some Lemmas

In what follows, we assume that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \xi \in [0, \frac{1}{2}], 0 < \lambda \leq 1, \sigma, \sigma_1 \in (0, \lambda), f(x)$ is a nonnegative measurable function in $\mathbf{R}_+ = (0, \infty)$, $a_n \geq 0$ $(n \in \mathbf{N} = \{1, 2, \cdots\})$, such that

$$0 < \int_0^\infty x^{p[1-(\frac{\sigma}{p}+\frac{\sigma_1}{q})]-1}f^p(x)dx < \infty, 0 < \sum_{n=1}^\infty (n-\xi)^{q[1-(\frac{\sigma}{p}+\frac{\sigma_1}{q})]-1}a_n^q.$$

Lemma 2.1. *Define the following weight functions:*

$$\omega_\sigma(\sigma_1, n) := (n-\xi)^\sigma \int_0^\infty \frac{x^{\sigma_1-1}}{1+[x(n-\xi)]^\lambda}dx \ (n \in \mathbf{N}), \tag{2.1}$$

$$\varpi_{\sigma_1}(\sigma, x) := x^{\sigma_1}\sum_{n=1}^\infty \frac{(n-\xi)^{\sigma-1}}{1+[x(n-\xi)]^\lambda} \ (x \in \mathbf{R}_+). \tag{2.2}$$

We have the following equality and inequalities:

$$\omega_\sigma(\sigma_1, n) = \frac{\pi}{\lambda\sin(\pi\sigma_1/\lambda)}(n-\xi)^{\sigma-\sigma_1} \ (n \in \mathbf{N}), \tag{2.3}$$

$$\left[\frac{\pi}{\lambda\sin(\pi\sigma/\lambda)} - \frac{[x(1-\xi)]^\sigma}{\sigma}\right]x^{\sigma_1-\sigma} < \varpi_{\sigma_1}(\sigma, x) < \frac{\pi}{\lambda\sin(\pi\sigma/\lambda)}x^{\sigma_1-\sigma} \ (x \in \mathbf{R}_+). \tag{2.4}$$

Proof. Setting $u = x^\lambda(n-\xi)^\lambda$, we find

$$\begin{aligned}\omega_\sigma(\sigma_1, n) &= (n-\xi)^\sigma \frac{1}{\lambda}\int_0^\infty \frac{1}{1+u}\frac{u^{(\sigma_1-1)/\lambda}}{(n-\xi)^{\sigma_1-1}}\frac{u^{(1/\lambda)-1}}{n-\xi}du \\ &= (n-\xi)^{\sigma-\sigma_1}\frac{1}{\lambda}\int_0^\infty \frac{u^{(\sigma_1/\lambda)-1}}{1+u}du \\ &= \frac{\pi}{\lambda\sin(\pi\sigma_1/\lambda)}(n-\xi)^{\sigma-\sigma_1},\end{aligned}$$

and then (2.3) follows.

In view of the fact that $\frac{u^{\sigma-1}}{1+u^\lambda} > 0$,

$$\begin{aligned}\frac{d}{du}\frac{u^{\sigma-1}}{1+u^\lambda} &= \frac{(\sigma-1)u^{\sigma-2}}{1+u^\lambda} - \frac{\lambda u^{\sigma+\lambda-2}}{(1+u^\lambda)^2} < 0, \\ \frac{d^2}{du^2}\frac{u^{\sigma-1}}{1+u^\lambda} &= \frac{(\sigma-1)(\sigma-2)u^{\sigma-3}}{1+u^\lambda} - \frac{(\sigma-1)\lambda u^{\sigma+\lambda-3}}{(1+u^\lambda)^2} \\ &\quad - \frac{(\sigma+\lambda-2)\lambda u^{\sigma+\lambda-3}}{(1+u^\lambda)^2} + \frac{\lambda^2 u^{\sigma+2\lambda-3}}{(1+u^\lambda)^3} > 0,\end{aligned}$$

by Hemite-Hadamard's inequality (cf. [6]), we find

$$\begin{aligned}\varpi_{\sigma_1}(\sigma, x) &< x^{\sigma_1}\int_{\frac{1}{2}}^\infty \frac{(t-\xi)^{\sigma-1}}{1+[x(t-\xi)]^\lambda}dt \\ &\le x^{\sigma_1-\sigma}\frac{1}{\lambda}\int_0^\infty \frac{u^{(\sigma/\lambda)-1}}{1+u}du = \frac{\pi}{\lambda\sin(\pi\sigma/\lambda)}x^{\sigma_1-\sigma}.\end{aligned}$$

In view of the decreasingness property, we obtain

$$\begin{aligned}\varpi_{\sigma_1}(\sigma, x) &> x^{\sigma_1}\int_1^\infty \frac{(t-\xi)^{\sigma-1}}{1+[x(t-\xi)]^\lambda}dt \\ &= x^{\sigma_1-\sigma}\frac{1}{\lambda}\int_{[x(1-\xi)]^\lambda}^\infty \frac{u^{(\sigma/\lambda)-1}}{1+u}du \\ &\ge x^{\sigma_1-\sigma}\frac{1}{\lambda}\left[\frac{\pi}{\lambda\sin(\pi\sigma/\lambda)} - \int_0^{[x(1-\xi)]^\lambda} u^{(\sigma/\lambda)-1}du\right] \\ &= \left[\frac{\pi}{\lambda\sin(\pi\sigma/\lambda)} - \frac{[x(1-\xi)]^\sigma}{\sigma}\right]x^{\sigma_1-\sigma}.\end{aligned}$$

Hence, (2.4) follows.

The lemma is proved. □

Lemma 2.2. *Setting* $k_\lambda(\eta) := \frac{\pi}{\lambda\sin(\pi\eta/\lambda)}(\eta = \sigma, \sigma_1)$, *we have the following inequality:*

$$\begin{aligned}I &:= \int_0^\infty \sum_{n=1}^\infty \frac{a_n f(x)}{1+[x(n-\xi)]^\lambda}dx = \sum_{n=1}^\infty \int_0^\infty \frac{a_n f(x)}{1+[x(n-\xi)]^\lambda}dx \\ &< k_\lambda^{\frac{1}{p}}(\sigma)k_\lambda^{\frac{1}{q}}(\sigma_1)\left\{\int_0^\infty x^{p[1-(\frac{\sigma}{p}+\frac{\sigma_1}{q})]-1}f^p(x)dx\right\}^{\frac{1}{p}}\end{aligned}$$

$$\times \left\{ \sum_{n=1}^{\infty} (n-\xi)^{q[1-(\frac{\sigma}{p}+\frac{\sigma_1}{q})]-1} a_n^q \right\}^{\frac{1}{q}}. \tag{2.5}$$

Proof. By Hölder's inequality (cf. [6]), we have

$$\begin{aligned} I &= \int_0^\infty \sum_{n=1}^\infty \frac{1}{1+[x(n-\xi)]^\lambda} \left[\frac{x^{(1-\sigma_1)/q} f(x)}{(n-\xi)^{(1-\sigma)/p}} \right] \left[\frac{(n-\xi)^{(1-\sigma)/p}}{x^{(1-\sigma_1)/q}} a_n \right] dx \\ &\leq \left\{ \int_0^\infty \sum_{n=1}^\infty \frac{1}{1+[x(n-\xi)]^\lambda} \frac{x^{(1-\sigma_1)p/q} f^p(x)}{(n-\xi)^{1-\sigma}} dx \right\}^{\frac{1}{p}} \\ &\quad \times \left\{ \sum_{n=1}^\infty \int_0^\infty \frac{1}{1+[x(n-\xi)]^\lambda} \frac{(n-\xi)^{(1-\sigma)q/p}}{x^{1-\sigma_1}} a_n^q dx \right\}^{\frac{1}{q}} \\ &= \left[\int_0^\infty \varpi_{\sigma_1}(\sigma,x) x^{p(1-\sigma_1)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\sum_{n=1}^\infty \omega_\sigma(\sigma_1,n)(n-\xi)^{q(1-\sigma)-1} a_n^q \right]^{\frac{1}{q}}. \end{aligned}$$

Then by (2.3) and (2.4), we have (2.5).

The lemma is proved. □

By (2.5), for $\sigma_1 = \sigma$, we find $0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty, 0 < \sum_{n=1}^\infty (n-\xi)^{q(1-\sigma)-1} a_n^q < \infty$, and

$$\begin{aligned} &\int_0^\infty \sum_{n=1}^\infty \frac{a_n f(x)}{1+[x(n-\xi)]^\lambda} dx \\ &< k_\lambda(\sigma) \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\sum_{n=1}^\infty (n-\xi)^{q(1-\sigma)-1} a_n^q \right]^{\frac{1}{q}}. \end{aligned} \tag{2.6}$$

Lemma 2.3. *The constant factor* $k_\lambda(\sigma) = \frac{\pi}{\lambda \sin(\pi\sigma/\lambda)}$ *in (2.6) is the best possible.*

Proof. For $0 < \varepsilon < q\sigma$, we set

$$\widetilde{a}_n := (n-\xi)^{\sigma-\frac{\varepsilon}{q}-1} \ (n \in \mathbf{N}), \widetilde{f}(x) := \begin{cases} x^{\sigma+\frac{\varepsilon}{p}-1}, & o < x \leq 1, \\ 0, & x > 1. \end{cases}$$

If there exists a constant $M \leq k_\lambda(\sigma)$, such that (2.6) is valid when replacing $k_\lambda(\sigma)$ by M, then for $a_n = \widetilde{a}_n$, $f = \widetilde{f}$, we have

$$\begin{aligned} \widetilde{I} &:= \int_0^\infty \sum_{n=1}^\infty \frac{\widetilde{a}_n \widetilde{f}(x)}{1+[x(n-\xi)]^\lambda} dx \\ &< M \left[\int_0^\infty x^{p(1-\sigma)-1} \widetilde{f}^p(x) dx \right]^{\frac{1}{p}} \left[\sum_{n=1}^\infty (n-\xi)^{q(1-\sigma)-1} \widetilde{a}_n^q \right]^{\frac{1}{q}}. \end{aligned}$$

We obtain

$$\widetilde{I} < M\left[\int_0^1 x^{p(1-\sigma)-1}x^{p(\sigma+\frac{\varepsilon}{p}-1)}dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}(n-\xi)^{q(1-\sigma)-1}(n-\xi)^{q(\sigma-\frac{\varepsilon}{q}-1)}\right]^{\frac{1}{q}}$$
$$= M\left(\int_0^1 x^{\varepsilon-1}dx\right)^{\frac{1}{p}}\left[(1-\xi)^{-\varepsilon-1}+\sum_{n=2}^{\infty}(n-\xi)^{-\varepsilon-1}\right]^{\frac{1}{q}}$$
$$< M\left(\int_0^1 x^{\varepsilon-1}dx\right)^{\frac{1}{p}}\left[(1-\xi)^{-\varepsilon-1}+\int_1^{\infty}(t-\xi)^{-\varepsilon-1}dt\right]^{\frac{1}{q}}$$
$$= \frac{M}{\varepsilon}\left[\varepsilon(1-\xi)^{-\varepsilon-1}+(1-\xi)^{-\varepsilon}\right]^{\frac{1}{q}}.$$

In view of (2.4) (for $\sigma_1 = \sigma$), we find

$$\widetilde{I} = \int_0^1 x^{\varepsilon-1}\left\{x^{(\sigma-\frac{\varepsilon}{q})}\sum_{n=1}^{\infty}\frac{(n-\xi)^{(\sigma-\frac{\varepsilon}{q})-1}}{1+[x(n-\xi)]^{\lambda}}\right\}dx$$
$$> \int_0^1 x^{\varepsilon-1}\left\{k_{\lambda}(\sigma-\frac{\varepsilon}{q})-\frac{[x(1-\xi)]^{\sigma-\frac{\varepsilon}{q}}}{\sigma-\frac{\varepsilon}{q}}\right\}dx$$
$$= \frac{1}{\varepsilon}\left[k_{\lambda}(\sigma-\frac{\varepsilon}{q})-\frac{\varepsilon(1-\xi)^{\sigma-\frac{\varepsilon}{q}}}{\sigma-\frac{\varepsilon}{q}}\int_0^1 x^{\sigma+\frac{\varepsilon}{p}-1}dx\right]$$
$$= \frac{1}{\varepsilon}\left[k_{\lambda}(\sigma-\frac{\varepsilon}{q})-\frac{\varepsilon(1-\xi)^{\sigma-\frac{\varepsilon}{q}}}{(\sigma-\frac{\varepsilon}{q})(\sigma+\frac{\varepsilon}{p})}\right].$$

Then we have

$$k_{\lambda}(\sigma-\frac{\varepsilon}{q})-\frac{\varepsilon(1-\xi)^{\sigma-\frac{\varepsilon}{q}}}{(\sigma-\frac{\varepsilon}{q})(\sigma+\frac{\varepsilon}{p})}$$
$$< \varepsilon\widetilde{I} < M\left[\varepsilon(1-\xi)^{-\varepsilon-1}+(1-\xi)^{-\varepsilon}\right]^{\frac{1}{q}}.$$

For $\varepsilon \to 0^+$, in view of the continuous property of the sine function, we find $k_{\lambda}(\sigma) \leq M$. Hence, $M = k_{\lambda}(\sigma)$ is the best possible constant factor of (2.6).

The lemma is proved. □

Note. Setting $\widetilde{\sigma} = \frac{\sigma}{p} + \frac{\sigma_1}{q}(\sigma, \sigma_1 \in (0,\lambda) \subset (0,1))$, we may rewrite (2.5) as follows:

$$I < k_{\lambda}^{\frac{1}{p}}(\sigma)k_{\lambda}^{\frac{1}{q}}(\sigma_1)\left[\int_0^{\infty} x^{p(1-\widetilde{\sigma})-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}(n-\xi)^{q(1-\widetilde{\sigma})-1}\right]^{\frac{1}{q}}. \tag{2.7}$$

By Hölder's inequality (cf. [6]), the parameter $\widetilde{\sigma}$ in (2.7) also satisfies

$$0 < k_{\lambda}(\widetilde{\sigma}) = k_{\lambda}(\frac{\sigma}{p}+\frac{\sigma_1}{q}) = \int_0^{\infty}\frac{1}{1+u^{\lambda}}(u^{\frac{\sigma-1}{p}})(u^{\frac{\sigma_1-1}{q}})du$$
$$\leq \left(\int_0^{\infty}\frac{1}{1+u^{\lambda}}u^{\sigma-1}du\right)^{\frac{1}{p}}\left(\int_0^{\infty}\frac{1}{1+u^{\lambda}}u^{\sigma_1-1}du\right)^{\frac{1}{q}}$$
$$= k_{\lambda}^{\frac{1}{p}}(\sigma)k_{\lambda}^{\frac{1}{q}}(\sigma_1) < \infty, \tag{2.8}$$

and for $0 < \widetilde{\sigma} < \frac{\lambda}{p} + \frac{\lambda}{q} = \lambda$, it follows that

$$k_\lambda(\widetilde{\sigma}) - \frac{[x(1-\xi)]^{\widetilde{\sigma}}}{\widetilde{\sigma}} < x^{\widetilde{\sigma}} \sum_{n=1}^{\infty} \frac{(n-\xi)^{\widetilde{\sigma}-1}}{1+[x(n-\xi)]^\lambda} < k_\lambda(\widetilde{\sigma}).$$

Lemma 2.4. *If the constant factor $k_\lambda^{\frac{1}{p}}(\sigma)k_\lambda^{\frac{1}{q}}(\sigma_1)$ in (2.7) is the best possible, then we have $\sigma_1 = \sigma$.*

Proof. If the constant factor $k_\lambda^{\frac{1}{p}}(\sigma)k_\lambda^{\frac{1}{q}}(\sigma_1)$ in (2.7) is the best possible, then by (2.6), the unique best possible constant factor must be $k_\lambda(\widetilde{\sigma})(\in \mathbf{R}_+)$, namely, $k_\lambda(\widetilde{\sigma}) = k_\lambda^{\frac{1}{p}}(\sigma)k_\lambda^{\frac{1}{q}}(\sigma_1)$. We observe that (2.8) keeps the form of equality if and only if there exist constants A and B, such that they are not all zero and $Au^{\sigma-1} = Bu^{\sigma_1-1}$ a.e. in $\mathbf{R}_+$ (cf. [6]). Assuming that $A \neq 0$, it follows that $u^{\sigma-\sigma_1} = B/A$ a.e. in $\mathbf{R}_+$, and then $\sigma - \sigma_1 = 0$, namely, $\sigma_1 = \sigma$.

The lemma is proved. □

3. Main results and some corollaries

Theorem 3.1. *Inequality (2.5) is equivalent to the following inequalities:*

$$J_1 := \left\{ \sum_{n=1}^{\infty} (n-\xi)^{p(\frac{\sigma}{p}+\frac{\sigma_1}{q})-1} \left[\int_0^\infty \frac{f(x)}{1+[x(n-\xi)]^\lambda} dx \right]^p \right\}^{\frac{1}{p}}$$
$$< k_\lambda^{\frac{1}{p}}(\sigma)k_\lambda^{\frac{1}{q}}(\sigma_1) \left\{ \int_0^\infty x^{p[1-(\frac{\sigma}{p}+\frac{\sigma_1}{q})]-1} f^p(x) dx \right\}^{\frac{1}{p}}, \tag{3.1}$$

$$J_2 := \left\{ \int_0^\infty x^{q(\frac{\sigma}{p}+\frac{\sigma_1}{q})-1} \left[\sum_{n=1}^{\infty} \frac{a_n}{1+[x(n-\xi)]^\lambda} \right]^q dx \right\}^{\frac{1}{q}}$$
$$< k_\lambda^{\frac{1}{p}}(\sigma)k_\lambda^{\frac{1}{q}}(\sigma_1) \left\{ \sum_{n=1}^{\infty} (n-\xi)^{q[1-(\frac{\sigma}{p}+\frac{\sigma_1}{q})]-1} a_n^q \right\}^{\frac{1}{q}}. \tag{3.2}$$

If the constant factor in (2.5) is the best possible, then, so is the constant factor in (3.1) and (3.2).

Proof. Suppose that (3.1) ((3.2)) is valid. By Hölder's inequality (cf. [6]), we have

$$I = \sum_{n=1}^{\infty} \left[(n-\xi)^{\frac{-1}{p}+(\frac{\sigma}{p}+\frac{\sigma_1}{q})} \int_0^\infty \frac{f(x)dx}{1+[x(n-\xi)]^\lambda} \right] \left[(n-\xi)^{\frac{1}{p}-(\frac{\sigma}{p}+\frac{\sigma_1}{q})} a_n \right]$$
$$\leq J_1 \left\{ \sum_{n=1}^{\infty} (n-\xi)^{q[1-(\frac{\sigma}{p}+\frac{\sigma_1}{q})]-1} a_n^q \right\}^{\frac{1}{q}}, \tag{3.3}$$

$$I = \int_0^\infty \left[x^{\frac{1}{q}-(\frac{\sigma}{p}+\frac{\sigma_1}{q})} f(x) \right] \left[x^{\frac{-1}{q}+(\frac{\sigma}{p}+\frac{\sigma_1}{q})} \sum_{n=1}^{\infty} \frac{a_n}{1+[x(n-\xi)]^\lambda} \right] dx$$
$$\leq \left\{ \int_0^\infty x^{p[1-(\frac{\sigma}{p}+\frac{\sigma_1}{q})]-1} f^p(x) dx \right\}^{\frac{1}{p}} J_2. \tag{3.4}$$

Then by (3.1) ((3.2)), we have (2.5). On the other hand, assuming that (2.5) is valid, we set

$$a_n := (n-\xi)^{p(\frac{\sigma}{p}+\frac{\sigma_1}{q})-1}\left[\int_0^\infty \frac{f(x)}{1+[x(n-\xi)]^\lambda}dx\right]^{p-1} \quad (n\in\mathbf{N}).$$

If $J_1 = 0$, then (3.1) is naturally valid; if $J_1 = \infty$, then it is impossible to make (3.1) valid. Suppose that $0 < J_1 < \infty$. By (2.5) we have

$$\begin{aligned}&\sum_{n=1}^\infty (n-\xi)^{q[1-(\frac{\sigma}{p}+\frac{\sigma_1}{q})]-1}a_n^q\\ &= J_1^p = I < k_\lambda^{\frac{1}{p}}(\sigma)k_\lambda^{\frac{1}{q}}(\sigma_1)\left\{\int_0^\infty x^{p[1-(\frac{\sigma}{p}+\frac{\sigma_1}{q})]-1}f^p(x)dx\right\}^{\frac{1}{p}}\\ &\quad\times\left\{\sum_{n=1}^\infty (n-\xi)^{q[1-(\frac{\sigma}{p}+\frac{\sigma_1}{q})]-1}a_n^q\right\}^{\frac{1}{q}},\end{aligned}$$

$$\begin{aligned}&\left\{\sum_{n=1}^\infty (n-\xi)^{q[1-(\frac{\sigma}{p}+\frac{\sigma_1}{q})]-1}a_n^q\right\}^{\frac{1}{p}}\\ &= J_1 < k_\lambda^{\frac{1}{p}}(\sigma)k_\lambda^{\frac{1}{q}}(\sigma_1)\left\{\int_0^\infty x^{p[1-(\frac{\sigma}{p}+\frac{\sigma_1}{q})]-1}f^p(x)dx\right\}^{\frac{1}{p}},\end{aligned}$$

namely, (3.1) follows.

In the same way, assuming that (2.5) is valid, we set

$$f(x) := x^{q(\frac{\sigma}{p}+\frac{\sigma_1}{q})-1}\left[\sum_{n=1}^\infty \frac{a_n}{1+[x(n-\xi)]^\lambda}\right]^{q-1} \quad (x\in\mathbf{R}_+).$$

If $J_2 = 0$, then (3.2) is naturally valid; if $J_2 = \infty$, then it is impossible to make (3.2) valid. Suppose that $0 < J_2 < \infty$. By (2.5), we have

$$\begin{aligned}&\int_0^\infty x^{p[1-(\frac{\sigma}{p}+\frac{\sigma_1}{q})]-1}f^p(x)dx\\ &= J_2^q = I < k_\lambda^{\frac{1}{p}}(\sigma)k_\lambda^{\frac{1}{q}}(\sigma_1)\left\{\int_0^\infty x^{p[1-(\frac{\sigma}{p}+\frac{\sigma_1}{q})]-1}f^p(x)dx\right\}^{\frac{1}{p}}\\ &\quad\times\left\{\sum_{n=1}^\infty (n-\xi)^{q[1-(\frac{\sigma}{p}+\frac{\sigma_1}{q})]-1}a_n^q\right\}^{\frac{1}{q}},\end{aligned}$$

$$\begin{aligned}&\left\{\int_0^\infty x^{p[1-(\frac{\sigma}{p}+\frac{\sigma_1}{q})]-1}f^p(x)dx\right\}^{\frac{1}{q}}\\ &= J_2 < k_\lambda^{\frac{1}{p}}(\sigma)k_\lambda^{\frac{1}{q}}(\sigma_1)\left\{\sum_{n=1}^\infty (n-\xi)^{q[1-(\frac{\sigma}{p}+\frac{\sigma_1}{q})]-1}a_n^q\right\}^{\frac{1}{q}},\end{aligned}$$

namely, (3.2) follows. Hence, inequalities (2.5), (3.1) and (3.2) are equivalent.

If the constant factor in (2.5) is the best possible, then so is constant factor in (3.1) ((3.2)). Otherwise, by (3.3) ((3.4)), we would reach a contradiction that the constant factor in (2.5) is not the best possible.

The theorem is proved. □

Theorem 3.2. *The following statements (i), (ii), (iii) and (iv) are equivalent:*

(i) $k_\lambda^{\frac{1}{p}}(\sigma)k_\lambda^{\frac{1}{q}}(\sigma_1)$ *is independent of* p, q*;*

(ii) $k_\lambda^{\frac{1}{p}}(\sigma)k_\lambda^{\frac{1}{q}}(\sigma_1)$ *is expressible as a single integral;*

(iii) $k_\lambda^{\frac{1}{p}}(\sigma)k_\lambda^{\frac{1}{q}}(\sigma_1)$ *in (2.5) is the best possible constant;*

(iv) $\sigma_1 = \sigma$.

If the statement (iv) follows, then we have the following equivalent inequalities with the best possible constant factor $\frac{\pi}{\lambda\sin(\pi\sigma/\lambda)}$*:*

$$\int_0^\infty \sum_{n=1}^\infty \frac{a_n f(x)}{1+[x(n-\xi)]^\lambda}dx < \frac{\pi}{\lambda\sin(\pi\sigma/\lambda)}\left[\int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\sum_{n=1}^\infty (n-\xi)^{q(1-\sigma)-1}a_n^q\right]^{\frac{1}{q}}, \quad (3.5)$$

$$\left\{\sum_{n=1}^\infty (n-\xi)^{p\sigma-1}\left[\int_0^\infty \frac{f(x)}{1+[x(n-\xi)]^\lambda}dx\right]^p\right\}^{\frac{1}{p}} < \frac{\pi}{\lambda\sin(\pi\sigma/\lambda)}\left[\int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}, \quad (3.6)$$

$$\left\{\int_0^\infty x^{q\sigma-1}\left[\sum_{n=1}^\infty \frac{a_n}{1+[x(n-\xi)]^\lambda}\right]^q dx\right\}^{\frac{1}{q}} < \frac{\pi}{\lambda\sin(\pi\sigma/\lambda)}\left[\sum_{n=1}^\infty (n-\xi)^{q(1-\sigma)-1}a_n^q\right]^{\frac{1}{q}}. \quad (3.7)$$

Proof. $(i) => (ii)$. By (i) we have

$$k_\lambda^{\frac{1}{p}}(\sigma)k_\lambda^{\frac{1}{q}}(\sigma_1) = \lim_{p\to 1^+} k_\lambda^{\frac{1}{p}}(\sigma)k_\lambda^{\frac{1}{q}}(\sigma_1) = k_\lambda(\sigma),$$

namely, $k_\lambda^{\frac{1}{p}}(\sigma)k_\lambda^{\frac{1}{q}}(\sigma_1)$ is expressible as a single integral.

$(ii) => (iv)$. In (2.8), if $k_\lambda^{\frac{1}{p}}(\sigma)k_\lambda^{\frac{1}{q}}(\sigma_1)$ is expressible as a single integral $k_\lambda(\frac{\sigma}{p}+\frac{\sigma_1}{q})$, then (2.8) keeps the form of equality. In view of the proof of Lemma 4, we have $\sigma_1 = \sigma$.

$(iv) => (i)$. If $\sigma_1 = \sigma$, then $k_\lambda^{\frac{1}{p}}(\sigma)k_\lambda^{\frac{1}{q}}(\sigma_1) = k_\lambda(\sigma)$, which is independent of p, q. Hence, we have $(i) <=> (ii) <=> (iv)$.

$(iii) => (iv)$. By Lemma 4, we have $\sigma_1 = \sigma$.

$(iv) => (iii)$. By Lemma 3, $k_\lambda^{\frac{1}{p}}(\sigma)k_\lambda^{\frac{1}{q}}(\sigma_1) = k_\lambda(\sigma)$ in (2.5) (for $\sigma_1 = \sigma$) is the best possible constant. Therefore, we have $(iii) <=> (iv)$.

Hence, the statements (i), (ii), (iii) and (iv) are equivalent.

The theorem is proved. □

Replacing x by $\frac{1}{x}$and then $x^{\lambda-2}f(\frac{1}{x})$ by $f(x)$ in Theorem 3.1 and Theorem 3.2, setting $\sigma_1 = \lambda - \mu$, we have

Corollary 3.1. *The following inequalities with the homogeneous kernel are equivalent:*

$$\int_0^\infty \sum_{n=1}^\infty \frac{a_n f(x)}{x^\lambda + (n-\xi)^\lambda} dx < k_\lambda^{\frac{1}{p}}(\sigma) k_\lambda^{\frac{1}{q}}(\lambda-\mu) \left\{ \int_0^\infty x^{p[1-(\frac{\lambda-\sigma}{p}+\frac{\mu}{q})]-1} f^p(x) dx \right\}^{\frac{1}{p}} \times \left\{ \sum_{n=1}^\infty (n-\xi)^{q[1-(\frac{\sigma}{p}+\frac{\lambda-\mu}{q})]-1} a_n^q \right\}^{\frac{1}{q}}, \tag{3.8}$$

$$\left\{ \sum_{n=1}^\infty (n-\xi)^{p(\frac{\sigma}{p}+\frac{\lambda-\mu}{q})-1} \left[\int_0^\infty \frac{f(x)}{x^\lambda + (n-\xi)^\lambda} dx \right]^p \right\}^{\frac{1}{p}} < k_\lambda^{\frac{1}{p}}(\sigma) k_\lambda^{\frac{1}{q}}(\lambda-\mu) \left\{ \int_0^\infty x^{p[1-(\frac{\lambda-\sigma}{p}+\frac{\mu}{q})]-1} f^p(x) dx \right\}^{\frac{1}{p}}, \tag{3.9}$$

$$\left\{ \int_0^\infty x^{q(\frac{\lambda-\sigma}{p}+\frac{\sigma_1}{q})-1} \left[\sum_{n=1}^\infty \frac{a_n}{x^\lambda + (n-\xi)^\lambda} \right]^q dx \right\}^{\frac{1}{q}} < k_\lambda^{\frac{1}{p}}(\sigma) k_\lambda^{\frac{1}{q}}(\lambda-\mu) \left\{ \sum_{n=1}^\infty (n-\xi)^{q[1-(\frac{\sigma}{p}+\frac{\lambda-\mu}{q})]-1} a_n^q \right\}^{\frac{1}{q}}. \tag{3.10}$$

If the constant factor in (3.8) is the best possible, then so is the constant factor in (3.9) and (3.10).

Corollary 3.2. *The following statements (I), (II), (III) and (IV) are equivalent:*

(I) $k_\lambda^{\frac{1}{p}}(\sigma) k_\lambda^{\frac{1}{q}}(\lambda-\mu)$ *is independent of* p, q;

(II) $k_\lambda^{\frac{1}{p}}(\sigma) k_\lambda^{\frac{1}{q}}(\lambda-\mu)$ *is expressible as a single integral;*

(III) $k_\lambda^{\frac{1}{p}}(\sigma) k_\lambda^{\frac{1}{q}}(\lambda-\mu)$ *in (3.8) is the best possible constant;*

(IV) $\mu + \sigma = \lambda$.

If the statement (IV) follows, then we have the following equivalent inequalities with the best possible constant factor:

$$\int_0^\infty \sum_{n=1}^\infty \frac{a_n f(x)}{x^\lambda + (n-\xi)^\lambda} dx < \frac{\pi}{\lambda \sin(\pi\sigma/\lambda)} \left[\int_0^\infty x^{p(1-\mu)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\sum_{n=1}^\infty (n-\xi)^{q(1-\sigma)-1} a_n^q \right]^{\frac{1}{q}}, \tag{3.11}$$

$$\left\{ \sum_{n=1}^\infty (n-\xi)^{p\sigma-1} \left[\int_0^\infty \frac{f(x)}{x^\lambda + (n-\xi)^\lambda} dx \right]^p \right\}^{\frac{1}{p}} < \frac{\pi}{\lambda \sin(\pi\sigma/\lambda)} \left[\int_0^\infty x^{p(1-\mu)-1} f^p(x) dx \right]^{\frac{1}{p}}, \tag{3.12}$$

$$\left\{\int_0^\infty x^{q\mu-1}\left[\sum_{n=1}^\infty \frac{a_n}{x^\lambda+(n-\xi)^\lambda}\right]^q dx\right\}^{\frac{1}{q}}$$
$$< \frac{\pi}{\lambda\sin(\pi\sigma/\lambda)}\left[\sum_{n=1}^\infty (n-\xi)^{q(1-\sigma)-1}a_n^q\right]^{\frac{1}{q}}. \tag{3.13}$$

4. Operator expressions and a remark

(1) We set functions: $\varphi(x) := x^{p[1-(\frac{\sigma}{p}+\frac{\sigma_1}{q})]-1}, \psi(n) := (n-\xi)^{q[1-(\frac{\sigma}{p}+\frac{\sigma_1}{q})]-1}$, wherefrom,

$$\varphi^{1-q}(x) = x^{q(\frac{\sigma}{p}+\frac{\sigma_1}{q})-1}, \psi^{1-p}(n) = (n-\xi)^{p(\frac{\sigma}{p}+\frac{\sigma_1}{q})-1}\ (x\in\mathbf{R}_+, n\in\mathbf{N}).$$

Define the following real normed spaces:

$$L_{p,\varphi}(\mathbf{R}_+) := \left\{f; f=f(x), x\in\mathbf{R}_+, ||f||_{p,\varphi} := \left(\int_0^\infty \varphi(x)|f(x)|^p dx\right)^{\frac{1}{p}} < \infty\right\},$$
$$L_{q,\varphi^{1-q}}(\mathbf{R}_+) := \left\{h; h=h(x), x\in\mathbf{R}_+, ||h||_{q,\varphi^{1-q}} = (\int_0^\infty \varphi^{1-q}(x)|h(x)|^q dx)^{\frac{1}{q}} < \infty\right\},$$
$$l_{q,\psi} := \left\{a; a=\{a_n\}_{n=1}^\infty, ||a||_{q,\psi} = \left(\sum_{n=1}^\infty \psi(n)|a_n|^q\right)^{\frac{1}{q}} < \infty\right\},$$
$$l_{p,\psi^{1-p}} := \left\{b; b=\{b_n\}_{n=1}^\infty, ||b||_{p,\psi^{1-p}} = \left(\sum_{n=1}^\infty \psi^{1-p}(n)|b_n|^p\right)^{\frac{1}{p}} < \infty\right\}.$$

Assuming that $f\in L_{p,\varphi}(\mathbf{R}_+)$, setting $b=\{b_n\}_{n=1}^\infty, b_n := \int_0^\infty \frac{f(x)}{1+[x(n-\xi)]^\lambda}dx, n\in\mathbf{N}$, we can rewrite (3.1) as

$$||b||_{p,\psi^{1-p}} < k_\lambda^{\frac{1}{p}}(\sigma)k_\lambda^{\frac{1}{q}}(\sigma_1)||f||_{p,\varphi} < \infty,$$

namely, $b\in l_{p,\psi^{1-p}}$.

Definition 4.1. Define a half-discrete Hilbert's operator with the nonhomogeneous kernel $T_1 : L_{p,\varphi}(\mathbf{R}_+) \to l_{p,\psi^{1-p}}$ as follows: For any $f\in L_{p,\varphi}(\mathbf{R}_+)$, there exists a unique representation $T_1 f = b \in l_{p,\psi^{1-p}}$. Define the formal inner product of $T_1 f$ and $a\in l_{q,\psi}$, and the norm of T_1 as follows:

$$(T_1 f, a) := \sum_{n=1}^\infty\left\{\int_0^\infty \frac{f(x)}{1+[x(n-\xi)]^\lambda}dx\right\}a_n,$$
$$||T_1|| := \sup_{f(\neq\theta)\in L_{p,\varphi}(\mathbf{R}_+)} \frac{||T_1 f||_{p,\psi^{1-p}}}{||f||_{p,\varphi}}.$$

Assuming that $a\in l_{q,\psi}$, setting $h=h(x), h(x) := \sum_{n=1}^\infty \frac{a_n}{1+[x(n-\xi)]^\lambda}, x\in\mathbf{R}_+$, we can rewrite (3.2) as

$$||h||_{q,\varphi^{1-q}} < k_\lambda^{\frac{1}{p}}(\sigma)k_\lambda^{\frac{1}{q}}(\sigma_1)||a||_{q,\psi} < \infty,$$

namely, $h\in L_{q,\varphi^{1-q}}(\mathbf{R}_+)$.

Definition 4.2. Define a half-discrete Hilbert's operator with the nonhomogeneous kernel $T_2 : l_{q,\psi} \to L_{q,\varphi^{1-q}}(\mathbf{R}_+)$ as follows: For any $a \in l_{q,\psi}$, there exists a unique representation $T_2 a = h \in L_{q,\varphi^{1-q}}$. Define the formal inner product of $f \in L_{p,\varphi}(\mathbf{R}_+)$ and $T_2 a$, and the norm of T_2 as follows:

$$(f, T_2 a) := \int_0^\infty \left\{ \sum_{n=1}^\infty \frac{a_n}{1 + [x(n-\xi)]^\lambda} \right\} f(x) dx,$$

$$||T_2|| := \sup_{a(\neq\theta) \in l_{q,\psi}} \frac{||T_2 a||_{q,\varphi^{1-q}}}{||a||_{q,\psi}}.$$

By Theorem 3.1 and Theorem 3.2, we have

Theorem 4.1. *If* $f = f(x)(\geq 0) \in L_{p,\varphi}(\mathbf{R}_+), a = \{a_n\}_{n=1}^\infty (\geq 0) \in l_{q,\psi}, ||f||_{p,\varphi}, ||a||_{q,\psi} > 0$, *then we have the following equivalent inequalities:*

$$(T_1 f, a) = (f, T_2 a) < k_\lambda^{\frac{1}{p}}(\sigma) k_\lambda^{\frac{1}{q}}(\sigma_1) ||f||_{p,\varphi} ||a||_{q,\psi}, \tag{4.1}$$

$$||T_1 f||_{p,\psi^{1-p}} < k_\lambda^{\frac{1}{p}}(\sigma) k_\lambda^{\frac{1}{q}}(\sigma_1) ||f||_{p,\varphi}, \tag{4.2}$$

$$||T_2 a||_{q,\varphi^{1-q}} < k_\lambda^{\frac{1}{p}}(\sigma) k_\lambda^{\frac{1}{q}}(\sigma_1) ||a||_{q,\psi}. \tag{4.3}$$

Moreover, if and only if $\sigma_1 = \sigma$, the constant factor $k_\lambda^{\frac{1}{p}}(\sigma) k_\lambda^{\frac{1}{q}}(\sigma_1)$ in the above inequalities is the best possible, namely,

$$||T_1|| = ||T_2|| = k_\lambda(\sigma) = \frac{\pi}{\lambda \sin(\pi\sigma/\lambda)}.$$

(2) We set functions: $\Phi(x) := x^{p[1-(\frac{\lambda-\sigma}{p}+\frac{\sigma}{q})]-1}, \Psi(n) := (n-\xi)^{q[1-(\frac{\sigma}{p}+\frac{\lambda-\mu}{q})]-1}$, wherefrom,

$$\Phi^{1-q}(x) = x^{q(\frac{\lambda-\sigma}{p}+\frac{\sigma}{q})-1}, \Psi^{1-p}(n) = (n-\xi)^{p(\frac{\sigma}{p}+\frac{\lambda-\mu}{q})-1} \ (x \in \mathbf{R}_+, n \in \mathbf{N}).$$

Define the following real normed spaces:

$$L_{p,\Phi}(\mathbf{R}_+) := \left\{ f; f = f(x), x \in \mathbf{R}_+, ||f||_{p,\Phi} := \left(\int_0^\infty \Phi(x) |f(x)|^p dx \right)^{\frac{1}{p}} < \infty \right\},$$

$$L_{q,\Phi^{1-q}}(\mathbf{R}_+) := \left\{ h; h = h(x), x \in \mathbf{R}_+, ||h||_{q,\Phi^{1-q}} = (\int_0^\infty \Phi^{1-q}(x) |h(x)|^q dx)^{\frac{1}{q}} < \infty \right\},$$

$$l_{q,\Psi} := \left\{ a; a = \{a_n\}_{n=1}^\infty, ||a||_{q,\Psi} = \left(\sum_{n=1}^\infty \Psi(n) |a_n|^q \right)^{\frac{1}{q}} < \infty \right\},$$

$$l_{p,\Psi^{1-p}} := \left\{ b; b = \{b_n\}_{n=1}^\infty, ||b||_{p,\Psi^{1-p}} = \left(\sum_{n=1}^\infty \Psi^{1-p}(n) |b_n|^p \right)^{\frac{1}{p}} < \infty \right\}.$$

Assuming that $f \in L_{p,\Phi}(\mathbf{R}_+)$, setting $b = \{b_n\}_{n=1}^\infty, b_n := \int_0^\infty \frac{f(x)}{x^\lambda + (n-\xi)^\lambda} dx, n \in \mathbf{N}$, we can rewrite (3.9) as

$$||b||_{p,\Psi^{1-p}} < k_\lambda^{\frac{1}{p}}(\sigma) k_\lambda^{\frac{1}{q}}(\lambda - \mu) ||f||_{p,\Phi} < \infty,$$

namely, $b \in l_{p,\Psi^{1-p}}$.

Definition 4.3. Define a half-discrete Hilbert's operator with the homogeneous kernel $T_3 : L_{p,\Phi}(\mathbf{R}_+) \to l_{p,\Psi^{1-p}}$ as follows: For any $f \in L_{p,\Phi}(\mathbf{R}_+)$, there exists a unique representation $T_3 f = b \in l_{p,\Psi^{1-p}}$. Define the formal inner product of $T_3 f$ and $a \in l_{q,\Psi}$, and the norm of T_3 as follows:

$$(T_3 f, a) := \sum_{n=1}^{\infty} \left[\int_0^{\infty} \frac{f(x)}{x^{\lambda} + (n-\xi)^{\lambda}} dx \right] a_n,$$

$$||T_3|| := \sup_{f(\neq\theta) \in L_{p,\Phi}(\mathbf{R}_+)} \frac{||T_1 f||_{p,\Psi^{1-p}}}{||f||_{p,\Phi}}.$$

Assuming that $a \in l_{q,\Psi}$, setting $h = h(x), h(x) := \sum_{n=1}^{\infty} \frac{a_n}{x^{\lambda}+(n-\xi)^{\lambda}}, x \in \mathbf{R}_+$, we can rewrite (3.10) as

$$||h||_{q,\Phi^{1-q}} < k_{\lambda}^{\frac{1}{p}}(\sigma) k_{\lambda}^{\frac{1}{q}}(\lambda-\mu) ||a||_{q,\Psi} < \infty,$$

namely, $h \in L_{q,\Phi^{1-q}}(\mathbf{R}_+)$.

Definition 4.4. Define a half-discrete Hilbert's operator with the homogeneous kernel $T_4 : l_{q,\Psi} \to L_{q,\Phi^{1-q}}(\mathbf{R}_+)$ as follows: For any $a \in l_{q,\Psi}$, there exists a unique representation $T_4 a = h \in L_{q,\Phi^{1-q}}$. Define the formal inner product of $f \in L_{p,\Phi}(\mathbf{R}_+)$ and $T_4 a$, and the norm of T_4 as follows:

$$(f, T_4 a) := \int_0^{\infty} \left[\sum_{n=1}^{\infty} \frac{a_n}{x^{\lambda} + (n-\xi)^{\lambda}} \right] f(x) dx,$$

$$||T_4|| := \sup_{a(\neq\theta) \in l_{q,\Psi}} \frac{||T_4 a||_{q,\Phi^{1-q}}}{||a||_{q,\Psi}}.$$

By Corollary 3.1 and Corollary 3.2, we have

Corollary 4.1. *If* $f = f(x)(\geq 0) \in L_{p,\Phi}(\mathbf{R}_+), a = \{a_n\}_{n=1}^{\infty}(\geq 0) \in l_{q,\Psi}, ||f||_{p,\Phi}, ||a||_{q,\Psi} > 0$, *then we have the following equivalent inequalities:*

$$(T_3 f, a) = (f, T_4 a) < k_{\lambda}^{\frac{1}{p}}(\sigma) k_{\lambda}^{\frac{1}{q}}(\lambda-\mu) ||f||_{p,\Phi} ||a||_{q,\Psi}, \tag{4.4}$$

$$||T_3 f||_{p,\Psi^{1-p}} < k_{\lambda}^{\frac{1}{p}}(\sigma) k_{\lambda}^{\frac{1}{q}}(\lambda-\mu) ||f||_{p,\Phi}, \tag{4.5}$$

$$||T_4 a||_{q,\varphi^{1-q}} < k_{\lambda}^{\frac{1}{p}}(\sigma) k_{\lambda}^{\frac{1}{q}}(\lambda-\mu) ||a||_{q,\Psi}. \tag{4.6}$$

Moreover, if and only if $\mu + \sigma = \lambda$, the constant factor $k_{\lambda}^{\frac{1}{p}}(\sigma) k_{\lambda}^{\frac{1}{q}}(\lambda-\mu)$ in the above inequalities is the best possible, namely,

$$||T_3|| = ||T_4|| = k_{\lambda}(\sigma) = \frac{\pi}{\lambda \sin(\pi\sigma/\lambda)}.$$

Remark 4.1. (i) For $\sigma = \frac{1}{p}(< \lambda)$ in (3.5), (3.6) and (3.7), we have the following equivalent inequalities with the nonhomogeneous kernel and the best possible constant factor $\frac{\pi}{\lambda \sin(\pi/p\lambda)}$:

$$\int_0^{\infty} \sum_{n=1}^{\infty} \frac{a_n f(x)}{1 + [x(n-\xi)]^{\lambda}} dx$$
$$< \frac{\pi}{\lambda \sin(\pi/p\lambda)} \left(\int_0^{\infty} x^{p-2} f^p(x) dx \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} a_n^q \right)^{\frac{1}{q}}, \tag{4.7}$$

$$\left\{\sum_{n=1}^{\infty}\left[\int_0^{\infty}\frac{f(x)}{1+[x(n-\xi)]^{\lambda}}dx\right]^p\right\}^{\frac{1}{p}}<\frac{\pi}{\lambda\sin(\pi/p\lambda)}\left(\int_0^{\infty}x^{p-2}f^p(x)dx\right)^{\frac{1}{p}}, \quad (4.8)$$

$$\left\{\int_0^{\infty}x^{q-2}\left[\sum_{n=1}^{\infty}\frac{a_n}{1+[x(n-\xi)]^{\lambda}}\right]^q dx\right\}^{\frac{1}{q}}<\frac{\pi}{\lambda\sin(\pi/p\lambda)}\left(\sum_{n=1}^{\infty}a_n^q\right)^{\frac{1}{q}}. \quad (4.9)$$

(ii) For $\sigma=\frac{1}{q}(<\lambda)$ in (3.5), (3.6) and (3.7), we have the following equivalent inequalities with the best possible constant factor $\frac{\pi}{\lambda\sin(\pi/q\lambda)}$:

$$\int_0^{\infty}\sum_{n=1}^{\infty}\frac{a_nf(x)}{1+[x(n-\xi)]^{\lambda}}dx$$
$$<\frac{\pi}{\lambda\sin(\pi/q\lambda)}\left(\int_0^{\infty}f^p(x)dx\right)^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}(n-\xi)^{q-2}a_n^q\right]^{\frac{1}{q}}, \quad (4.10)$$

$$\left\{\sum_{n=1}^{\infty}(n-\xi)^{p-2}\left[\int_0^{\infty}\frac{f(x)}{1+[x(n-\xi)]^{\lambda}}dx\right]^p\right\}^{\frac{1}{p}}<\frac{\pi}{\lambda\sin(\pi/q\lambda)}\left(\int_0^{\infty}f^p(x)dx\right)^{\frac{1}{p}}, \quad (4.11)$$

$$\left\{\int_0^{\infty}\left[\sum_{n=1}^{\infty}\frac{a_n}{1+[x(n-\xi)]^{\lambda}}\right]^q dx\right\}^{\frac{1}{q}}<\frac{\pi}{\lambda\sin(\pi/q\lambda)}\left[\sum_{n=1}^{\infty}(n-\xi)^{q-2}a_n^q\right]^{\frac{1}{q}}. \quad (4.12)$$

(iii) For $\lambda=1,\mu=\frac{1}{q},\sigma=\frac{1}{p}$ in (3.11), (3.12) and (3.13), we have the following equivalent inequalities with the homogeneous kernel and the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$:

$$\int_0^{\infty}\sum_{n=1}^{\infty}\frac{a_nf(x)}{x+n-\xi}dx<\frac{\pi}{\sin(\pi/p)}\left(\int_0^{\infty}f^p(x)dx\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}a_n^q\right)^{\frac{1}{q}}, \quad (4.13)$$

$$\left[\sum_{n=1}^{\infty}\left(\int_0^{\infty}\frac{f(x)}{x+n-\xi}dx\right)^p\right]^{\frac{1}{p}}<\frac{\pi}{\sin(\pi/p)}\left(\int_0^{\infty}f^p(x)dx\right)^{\frac{1}{p}}, \quad (4.14)$$

$$\left[\int_0^{\infty}\left(\sum_{n=1}^{\infty}\frac{a_n}{x+n-\xi}\right)^q dx\right]^{\frac{1}{q}}<\frac{\pi}{\sin(\pi/p)}\left(\sum_{n=1}^{\infty}a_n^q\right)^{\frac{1}{q}}. \quad (4.15)$$

(iv) For $\lambda=1,\mu=\frac{1}{p},\sigma=\frac{1}{q}$ in (3.11), (3.12) and (3.13), we have the following equivalent inequalities with the homogeneous kernel and the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$:

$$\int_0^{\infty}\sum_{n=1}^{\infty}\frac{a_nf(x)}{x+n-\xi}dx<\frac{\pi}{\sin(\pi/p)}\left(\int_0^{\infty}x^{p-2}f^p(x)dx\right)^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}(n-\xi)^{q-2}a_n^q\right]^{\frac{1}{q}}, \quad (4.16)$$

$$\left[\sum_{n=1}^{\infty}(n-\xi)^{p-2}\left(\int_0^{\infty}\frac{f(x)}{x+n-\xi}dx\right)^p\right]^{\frac{1}{p}}<\frac{\pi}{\sin(\pi/p)}\left(\int_0^{\infty}x^{p-2}f^p(x)dx\right)^{\frac{1}{p}}, \quad (4.17)$$

$$\left[\int_0^{\infty}x^{q-2}\left(\sum_{n=1}^{\infty}\frac{a_n}{x+n-\xi}\right)^q dx\right]^{\frac{1}{q}}<\frac{\pi}{\sin(\pi/p)}\left[\sum_{n=1}^{\infty}(n-\xi)^{q-2}a_n^q\right]^{\frac{1}{q}}. \quad (4.18)$$

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