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PERIODIC AND QUASI-PERIODIC SOLUTIONS FOR THE COMPLEX SWIFT-HOHENBERG EQUATION*

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Abstract In this paper, we consider the complex Swift-Hohenberg(CSH) equation $\frac{\partial u}{\partial t} = \lambda u - (\alpha + \mathrm{i}\beta)\left(1 + \frac{\partial^2}{\partial x^2}\right)^2 u - (\sigma + \mathrm{i}\rho)|u|^2 u$ subject to periodic boundary conditions. Using an infinite dimensional KAM theorem, we prove that there exist a continuous branch of periodic solutions and a Cantorian branch of quasi-periodic solutions for the above equation.

Keywords Complex Swift-HohenbergC equation, periodic solution, quasi-periodic solution, normal form.

MSC(2010) 37K55, 35Q53.

1. Introduction

Consider the complex Swift-Hohenberg equation (CSH)

$$\frac{\partial u}{\partial t} = \lambda u - (\alpha + \mathrm{i}\beta)\left(1 + \frac{\partial^2}{\partial x^2}\right)^2 u - (\sigma + \mathrm{i}\rho)|u|^2 u, \tag{1.1}$$

where the unknown function $u : \mathbb{T} \times [0, \infty) \to \mathbb{C}$ is a complex-valued function, and $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, $\alpha, \beta, \sigma, \rho$ and λ are all real constants. In particular, α and σ are positive numbers, $\beta \neq 0$. The Swift-Hohenberg equation is a well-known generic model of pattern formation in extended systems [15, 36]. The CSH phenomenologically describes the dynamics of wide-aperture lasers of class A and C [24] close to peak gain. Class B lasers are well described by a combination of a CSH equation and an equation for the population inversion [7, 24]. These order parameter equations are derived from the Maxwell-Bloch equations for a two level, single longitudinal mode laser with flat end reflectors. A CSH equation has been obtained also for semiconductor lasers [29] and for CO_2 lasers with saturable absorber [6]. Besides lasers, the CSH equation has been used as a model for other non-linear optical systems such as optical parametric oscillator [32] and photorefractive materials [35].

As for (1.1) and other related equations, the existence of periodic solutions for travelling wave type, exact soliton solutions, singular solutions, algebro-geometric

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solutions, global attractors have been extensively investigated in many papers, for example [1, 19, 25, 27, 28, 38, 40]. In this paper, we will focus our attention on the periodic and quasi-periodic solutions which are *not* travelling waves.

A powerful tool to deal with quasi-periodic solutions is the KAM theorem constructed by Kolmogorov-Arnold-Moser [2, 20, 31]. By the 1980s, KAM theory has been developed to deal with the existence of quasi-periodic solutions for infinite dimensional Hamiltonian systems defined by partial differential equation (PDE), see [21, 22, 33, 34, 37]. With the rapid development of KAM theory, more and more types of partial differential equations can be dealt with, such as KdV equation, derivatives Schrödinger equation, wave equation and , Schrödinger equation in high dimensional space and so on. In this field of study there are too many references to list here. We give just two survey papers by Kuksin [23] and Bourgain [8] and a classical papers dealing with limiting unbounded perturbation case [26]. The KAM theory provide not only the existence of an invariant torus but also a normal form around it. This would be beneficial to study the dynamics of the PDEs in their neighborhood, for example, the long time stability of the solutions, see [3–5, 9–14, 16–18, 30, 39] and references therein. Consequently, KAM theory becomes a very powerful tool in constructing periodic and quasi-periodic solutions.

The CSH can be considered as a generalization of the nonlinear Schrödinger equation(NLS) with complex coefficients and higher order terms. The first conclusion about this type equation is due to Yuan [10]. Afterward, Cong, Liu and Yuan went on to study the cubic complex Ginzburg-Landau equation, and higher order nonlinear term complex Ginzburg-Landau equation, respectively, see [12, 13]. In order to attain the periodic and quasi-periodic solutions of (1.1) which are not travelling waves, we reduce the infinite dimensional coordinates form of (1.1) into a normal form up to order three in section 2. In section 3, we choose a direction n_0 and two direction n_{01}, n_{02} as tangent direction and others as normal direction, then introduce action-angle variables by coordinate transformation. In section 4, based on an infinite dimension KAM theory in [10], the obtained periodic solutions and quasi-periodic solutions for CSH equation (1.1) are small amplitude. Then, the higher order terms $(\sigma + \mathrm{i}\rho)|u|^2u$ can be regarded as a small perturbation of the linear equation

$$\frac{\partial u}{\partial t} = \lambda u - (\alpha + \mathrm{i}\beta)(1 + \frac{\partial^2}{\partial x^2})^2 u, \tag{1.2}$$

whose linear operator $\lambda - (\alpha + \mathrm{i}\beta)(1 + \frac{\partial^2}{\partial x^2})^2$ possesses eigenvalues

$$\lambda_n = (\lambda - \alpha(1-n^2)^2) - \mathrm{i}\beta(1-n^2)^2, n \in \mathbb{Z}.$$

In the following, we will look for the solution u which is odd in x. We know that

$$u(t,x) = \sum_{n=1}^{\infty} \xi_n e^{t\lambda_n} \sin nx \tag{1.3}$$

is the solution of (1.2) with the initial value

$$u_0(x) = \sum_{n=1}^{\infty} \xi_n \sin nx,$$

where $\xi_n \in \mathbb{R}$. Generally speaking, solution (1.3) is neither periodic nor quasi-periodic in time t, if

$$\Re\lambda_n = (\lambda - \alpha(1-n^2)^2) \neq 0$$

for all $n \in \mathbb{N}$.

Case1. Fix arbitrarily $n = n_0 \in \mathbb{N}$ and assume

$$\lambda = \alpha(1 - n_0^2)^2.$$

Then, the real part of λ_{n_0} vanishes and

$$u_p(t,x) = \xi_{n_0} e^{t\lambda_{n_0}} \sin n_0 x = \xi_{n_0} e^{-\mathrm{i}\beta(1-n_0^2)^2 t} \sin n_0 x$$

is a periodic solution of equation (1.2) for any $\xi_{n_0} \in \mathbb{R}$. We will show that the periodic solution u_p is preserved upon restoring the nonlinearity $|u|^2 u$ at $\lambda = \alpha(1 - n_0^2)^2 + \frac{3}{2\pi}\sigma\xi_{n_0}^2$, provided that $\xi_{n_0} \neq 0$ is sufficiently small. To be more precise, we have the following theorem.

Theorem 1.1. *Assume*

$$\xi_{n_0} = \sqrt{\xi}\epsilon^{4.5}, \xi \in [1,2], 0 < \epsilon \ll 1.$$

Then, when

$$\lambda = \alpha(1 - n_0^2)^2 + \frac{3}{2\pi}\sigma\xi_{n_0}^2,$$

equation (1.1) possesses a time-periodic and normally hyperbolic solution of periodic $2\pi/\omega$ *with positive constant* c *such that**

$$\left|\omega - \left((1 - n_0^2)^2\beta + \frac{3\rho}{2\pi}\xi\epsilon^9\right)\right| \leq c\epsilon^{10}$$

and the solution is odd in the spatial variable x *of the form*

$$u(t,x) = \xi_{n_0} e^{-\mathrm{i}\omega t} \sin n_0 x + \sum_{n \neq n_0} q_n(t) \sin nx$$

with

$$\|q(t)\|^2 = \sum_{n \neq n_0} |n|^{2b_0} e^{2a_0 n} |q_n(t)|^2 \leq c\epsilon^{14},$$

where $a_0 > 0, b_0 > \frac{1}{2}$ *are given constants. Moreover, the solution is analytic in* $(t,x) \in \mathbb{R} \times \mathbb{R}$.

Case2. Fix arbitrarily $n = n_{01}, n_{02} \in \mathbb{N}$, then

$$u_{qp}(t,x) = \xi_{n_{01}} e^{t\lambda_{n_{01}}} \sin n_{01} x + \xi_{n_{02}} e^{t\lambda_{n_{02}}} \sin n_{02} x$$

is still a solution of the linearized equation (1.2) for any $\xi_{n_{01}}, \xi_{n_{02}} \in \mathbb{R}$. Note that

$$\Re\lambda_{n_{01}} = (\lambda - \alpha(1 - n_{01}^2)^2), \Re\lambda_{n_{02}} = (\lambda - \alpha(1 - n_{02}^2)^2)$$

are not zero at the same time. Therefore, the solution $u_{qp}(t,x)$ is neither periodic nor quasi-periodic in time t. However, we can obtain that a quasi-periodic solution will appear in the neighbourhood of u_{qp} when both $\lambda - \alpha(1 - n_{01}^2)^2$ and $\lambda - \alpha(1 - n_{02}^2)^2$ are small enough and the nonlinearity $|u|^2 u$ is restored. More exactly, we have the following theorem.

*In this paper, we will denote by C or c a universal positive constant which might be different in different places.

Theorem 1.2. *Assume $n_{01}, n_{02} \in \mathbb{N}$ are given. For any $(\xi_1, \xi_2) \in [1,2] \times [3,4]$ and $0 < \epsilon \ll 1$. Let*

$$\xi_{n_{01}} = \sqrt{\xi_1}\epsilon^{4.5}, \xi_{n_{02}} = \sqrt{\xi_2}\epsilon^{4.5}.$$

Then, for given $0 < \gamma \ll 1$, there is a Cantorian subset $\Pi_0 \subset [1,2] \times [3,4]$ with Lebesgue measure larger than $1 - \gamma$, such that for any $(\xi_1, \xi_2) \in \Pi_0$, and

$$\alpha = \frac{\sigma\epsilon^9(\xi_1 - \xi_2)}{2\pi(2 - n_{01}^2 - n_{02}^2)(n_{01}^2 - n_{02}^2)},$$
$$\lambda = \frac{\sigma\epsilon^9((1 - n_{02}^2)^2(3\xi_1 + 2\xi_2) - (1 - n_{01}^2)^2(2\xi_1 + 3\xi_2))}{2\pi(2 - n_{01}^2 - n_{02}^2)(n_{01}^2 - n_{02}^2)},$$

equation (1.1) possesses a time quasi-periodic and normally hyperbolic solution with frequency vector $\omega = (\omega_1, \omega_2)$ satisfying

$$\left|\omega_1 - (\beta(1 - n_{01}^2)^2 + \frac{\epsilon^9\rho}{2\pi}(3\xi_1 + 2\xi_2))\right| \le c\epsilon^{10},$$

$$\left|\omega_2 - (\beta(1 - n_{02}^2)^2 + \frac{\epsilon^9\rho}{2\pi}(2\xi_1 + 3\xi_2))\right| \le c\epsilon^{10},$$

and the solution is odd in the spatial variable x of the form

$$u(t,x) = \sum_{j=1,2} \xi_{n_{0j}} e^{-\mathrm{i}\omega_j t} \sin n_{0j}x + \sum_{n \ne n_{01}, n_{02}} q_n(t) \sin nx,$$

with

$$\|q(t)\|^2 = \sum_{n \ne n_{01}, n_{02}} |n|^{2b_0} e^{2a_0 n} |q_n(t)|^2 \le c\epsilon^{14}.$$

Moreover, the solution is analytic in $(t,x) \in \mathbb{R} \times \mathbb{R}$.

It is worth mentioning that we only get 2-dimension KAM torus. Since for fix arbitrarily n_{01}, n_{02}, n_{03}, no matter how to choose the parameter $\xi = (\xi_{n_{01}}, \xi_{n_{02}}, \xi_{n_{03}}) \in \mathbb{R}^3$, it is impossible to make $\lambda - \alpha(1-n_{01}^2)^2$, $\lambda - \alpha(1-n_{02}^2)^2$ and $\lambda - \alpha(1-n_{03}^2)^2$ are small enough at the same time. It results that the solution (1.3) is neither periodic nor quasi-periodic in time t.

2. Normal form

Let $\mathbb{N} = \{1, 2, \cdots\}$. We will see the solutions $u(t,x)$ of equation (1.1) which satisfy

$$u(t,-x) = -u(t,x), (t,x) \in \mathbb{R} \times \mathbb{T}.$$

It is clear to know that $\mu_n = (1 - n^2)^2 (n \in \mathbb{N})$ and $\phi_n(x) = \frac{1}{\sqrt{\pi}} \sin nx (n \in \mathbb{N})$ are respectively the eigenvalues and eigenfunctions of the linear operator $(1 + \partial_{xx})^2$.

Let $\mathcal{H} = \{q = (q_j)_{j \in \mathbb{N}} : q_j \in \mathbb{C}\}$ be the space of complex sequences with

$$\langle q, \tilde{q} \rangle := \sum_{j \in \mathbb{N}} e^{2a_0 j} j^{2b_0} q_j \overline{\tilde{q}_j} < \infty$$

for any $q, \tilde{q} \in \mathcal{H}$. Obviously, $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is a Hilbert space. Let $\|q\|^2 = \langle q, q \rangle$. Denote by $\mathcal{L}(\mathcal{H}, \mathcal{H})$ the set of all linear bounded operators from $\mathcal{H}$ to $\mathcal{H}$.

Let

$$u(t,x) = \sum_{n=1}^{\infty} q_n(t)\phi_n(x),$$

then the equation (1.1) can be written as

$$\dot{q}_n = \lambda_n q_n - (\sigma + \mathrm{i}\rho) \sum_{\pm k \pm l \pm s = n} W_{klsn} q_k \bar{q}_l q_s, \tag{2.1}$$

where

$$\lambda_n = \lambda - \alpha(1-n^2)^2 - \mathrm{i}\beta(1-n^2)^2,$$
$$W_{klsn} = \frac{1}{\pi^2}\int_0^{2\pi} \sin kx \sin lx \sin sx \sin nx dx.$$

Lemma 2.1. *If positive integers n, k, l, s satisfy $n \pm k \pm l \pm s = 0$ and $\{n, l\} \neq \{k, s\}$, then we have*

$$\mu_n - \mu_k + \mu_l - \mu_s = n^4 - k^4 + l^4 - s^4 - 2n^2 + 2k^2 - 2l^2 + 2s^2 \neq 0.$$

Proof. If $n - k - l - s = 0$, we have $n - l = k + s$, thus

$$(n-l)^2 = (k+s)^2, (n-l)^4 = (k+s)^4.$$

It then follows that

$$n^2 - k^2 + l^2 - s^2 = 2(nl + ks),$$

$$\begin{aligned} n^4 - k^4 + l^4 - s^4 &= 4k^3 s + 6k^2 s^2 + 4ks^3 + 4n^3 l - 6n^2 l^2 + 4nl^3 \\ &= ks(4(k+s)^2 - 2ks) + (4(n+l)^2 + 2nl) \\ &= 4(k+s)^2(ks + nl) - 2(k^2 s^2 - n^2 l^2) \\ &= (ks + nl)(4(k+s)^2 - 2(ks - nl)) \\ &= (ks + nl)(k^2 + l^2 + s^2 + n^2 + 2(k+s)^2). \end{aligned}$$

Therefore,

$$\mu_n - \mu_k + \mu_l - \mu_s = (ks + nl)(k^2 + l^2 + s^2 + n^2 + 2(k+s)^2 - 2).$$

Notice that k, s, l, n are positive integer, then $ks + nl \neq 0$ and

$$k^2 + l^2 + s^2 + n^2 + 2(k+s)^2 - 2 \geq k^2 + l^2 + s^2 + n^2 > 0.$$

Consequently,

$$\mu_n - \mu_k + \mu_l - \mu_s \neq 0.$$

By the same method, it is easy to see when $\pm k \pm l \pm s = n$, we have

$$\begin{aligned} |\mu_n - \mu_k + \mu_l - \mu_s| &= |ks \pm nl|(k^2 + l^2 + s^2 + n^2 + 2(k \pm s)^2 - 2) \\ &= |k \pm n||k \pm l|(k^2 + l^2 + s^2 + n^2 + 2(k \pm s)^2 - 2) \\ &> 0. \end{aligned}$$

Above of all,

$$\mu_n - \mu_k + \mu_l - \mu_s \neq 0.$$

The proof is completed. $\square$

Lemma 2.2. *There exists a transformation Ψ which is bounded in a small neighbourhood of the origin in $\mathcal{H}$ and change (2.1) into*

$$\dot{z}_n = \lambda_n z_n - 2(\sigma + \mathrm{i}\rho) \sum_{l\in\mathbb{N}} W_{ln}|z_l|^2 z_n + O(\|z\|^5), \tag{2.2}$$

where

$$W_{ln} = \frac{1}{\pi^2}\int_0^{2\pi} \sin^2 lx \sin^2 nx dx = \begin{cases} \frac{1}{2\pi}, \, l \neq n, \\ \frac{3}{4\pi}, \, l = n. \end{cases}$$

Proof. Define a change Ψ in variables

$$z_n = q_n + \sum_{\pm k \pm l \pm s = n} T_{klsn} q_k \bar{q}_l q_s, n \in \mathbb{N} \tag{2.3}$$

with coefficients

$$T_{klsn} = \begin{cases} \frac{(\sigma+\mathrm{i}\rho)W_{klsn}}{\lambda_k + \bar{\lambda}_l + \lambda_s - \lambda_n}, & n \pm k \pm l \pm s = 0, \{n, l\} \neq \{k, s\}, \\ 0, & \text{otherwise}. \end{cases} \tag{2.4}$$

In view of

$$|\lambda_k + \bar{\lambda}_l + \lambda_s - \lambda_n| \geq |\beta||\mu_k - \mu_l + \mu_s - \mu_n|,$$

then by Lemma 2.1, we know that (2.4) is well defined. Furthermore, the change in variables Ψ is analytic in some neighbourhood of the origin in $\mathcal{H}^p$ and from $\mathcal{H}^p$ into $\mathcal{H}^p$.

It is easy to see that the Ψ is invertible and its inverse is of the form

$$q_n = z_n - \sum_{\pm k \pm l \pm s = n} T_{klsn} z_k \bar{z}_l z_s + O(\|z\|^5), \quad n \geq 1 \tag{2.5}$$

in some neighbourhood of the origin in $\mathcal{H}$.

Differentiating both sides of (2.3) with respect to t, we have

$$\begin{aligned} \dot{z}_n =& \lambda_n q_n - (\sigma + \mathrm{i}\rho) \sum_{\pm k \pm l \pm s = n} W_{klsn} q_k \bar{q}_l q_s \\ &+ \sum_{\pm k \pm l \pm s = n} T_{klsn}(\lambda_k + \bar{\lambda}_l + \lambda_s) q_k \bar{q}_l q_s + O(\|q\|^5). \end{aligned}$$

Substituting (2.5) into the last formula, we have

$$\begin{aligned} \dot{z}_n =& \lambda_n z_n - (\sigma + \mathrm{i}\rho) \sum_{\pm k \pm l \pm s = n} W_{klsn} z_k \bar{z}_l z_s \\ &+ \sum_{\pm k \pm l \pm s = n} T_{klsn}(\lambda_k + \bar{\lambda}_l + \lambda_s - \lambda_n) z_k \bar{z}_l z_s + O(\|z\|^5). \end{aligned}$$

Using (2.4), (2.1) is changed into (2.2). □

3. Action-angle variables

3.1. Fix $n_0 \in \mathbb{N}$

For a given n_0, equation (2.2) can be written as

$$\begin{cases} \dot{z}_{n_0} = \lambda_{n_0} z_{n_0} - \frac{3}{2\pi}(\sigma + \mathrm{i}\rho)|z_{n_0}|^2 z_{n_0} - \frac{1}{\pi}(\sigma + \mathrm{i}\rho) \sum_{l \neq n_0} |z_l|^2 z_{n_0} + O(\|z\|^5), \\ \dot{z}_n = \lambda_n z_n - \frac{1}{\pi}(\sigma + \mathrm{i}\rho)|z_{n_0}|^2 z_n - 2(\sigma + \mathrm{i}\rho) \sum_{l \neq n_0} W_{ln} |z_l|^2 z_n + O(\|z\|^5), \ n \neq n_0. \end{cases} \tag{3.1}$$

Re-scale the space variables as follows:

$$\begin{cases} z_{n_0} = \epsilon^{4.5} y_{n_0}, \\ z_n = \epsilon^6 y_n, \qquad n \neq n_0. \end{cases} \tag{3.2}$$

Then, the equation (3.1) can be rewritten as

$$\begin{cases} \dot{y}_{n_0} = \lambda_{n_0} y_{n_0} - \frac{3}{2\pi}\epsilon^9(\sigma + \mathrm{i}\rho)|y_{n_0}|^2 y_{n_0} + f_0(y), \\ \dot{y}_n = \lambda_n y_n - \frac{1}{\pi}\epsilon^9(\sigma + \mathrm{i}\rho)|y_{n_0}|^2 y_n + f_n(y), \qquad n \neq n_0, \end{cases} \tag{3.3}$$

with $f(y) = (f_n(y), n \in \mathbb{N})$ and $\|f(y)\| \leq C\epsilon^{12}$ in some neighbourhood of $y = 0 \in \mathcal{H}$. Introducing the action-angle variables (I, θ) by

$$y_{n_0} = \sqrt{I}(\cos\theta + \mathrm{i}\sin\theta) = \sqrt{I}e^{\mathrm{i}\theta}, \tag{3.4}$$

and substituting it into (3.3), we have

$$\begin{cases} \dot{I} = 2I\eta + \mathcal{I}(\theta, I, \tilde{y}), \\ \dot{\theta} = \kappa + \Theta_1(\theta, I, \tilde{y}) \\ \dot{y}_n = \lambda_n y_n - \frac{1}{\pi}\epsilon^9(\sigma + \mathrm{i}\rho) I y_n + g_n(\theta, I, \tilde{y}), \ n \neq n_0, \end{cases} \tag{3.5}$$

where $\eta = \lambda - \alpha(1-n_0^2)^2 - \frac{3}{2\pi}\sigma\epsilon^9 I, \kappa = -\beta(1-n_0^2)^2 - \frac{3}{2\pi}\rho\epsilon^9 I, \tilde{y} = (y_n : n \in \mathbb{N}\backslash\{n_0\})$, and $\mathcal{I}, \Theta_1, g_n$ are analytic in the domian $\|(I, \tilde{y})\| \leq c, |\Im\theta| \leq c$ with a given $c > 0$,

$$\|\mathcal{I} \oplus (g_n : n \in \mathbb{N}\backslash\{n_0\})\| \leq C\epsilon^{12}, |\Theta_1| \leq C\epsilon^{12}. \tag{3.6}$$

Let

$$I = \xi + \epsilon r, \xi \in [1, 2], |r| < 1. \tag{3.7}$$

Then,

$$\eta = \eta_0 - \frac{3}{2\pi}\sigma\epsilon^{10} r, \kappa = \kappa_0 \epsilon^9 - \frac{3\rho}{2\pi}\epsilon^{10} r, \tag{3.8}$$

where

$$\eta_0 = \lambda - \alpha(1 - n_0^2)^2 - \frac{3}{2\pi}\sigma\epsilon^9\xi, \kappa_0 = -\frac{3\rho}{2\pi}\xi - \epsilon^{-9}\beta(1 - n_0^2)^2.$$

Assume $\eta_0 = 0$, namely,

$$\lambda = \alpha(1 - n_0^2)^2 + \frac{3}{2\pi}\sigma\epsilon^9\xi. \tag{3.9}$$

Re-scale time by

$$\epsilon^9 t = s. \tag{3.10}$$

Substituting $I = \xi + \epsilon r$ into (3.5), we obtain

$$\begin{cases} \frac{d\theta}{ds} = \kappa_0 - \frac{3}{2\pi}\rho\epsilon r + \epsilon^{-9}\Theta_1, \\ \frac{dr}{ds} = -\frac{3}{\pi}\sigma\xi r - \frac{3}{\pi}\epsilon\sigma r^2 + \epsilon^{-10}\mathcal{I}, \\ \frac{dy_n}{ds} = \Omega_n y_n - \frac{1}{\pi}(\sigma + \mathrm{i}\rho)\epsilon r y_n + \epsilon^{-9} g_n,\ n \neq n_0, \end{cases} \tag{3.11}$$

where

$$\Omega_n = \left(\epsilon^{-9}(\lambda - \alpha(1-n^2)^2) - \frac{\sigma}{2\pi}\xi\right) - \mathrm{i}\left(\epsilon^{-9}\beta(1-n^2)^2 - \frac{\rho}{2\pi}\xi\right). \tag{3.12}$$

Let $\Omega_{n_0} = -\frac{3}{\pi}\sigma\xi$ and $\Omega = \mathrm{diag}(\Omega_n : n \in \mathbb{N})$. By abuse of notation, let $y_{n_0} = r$. Denote by y the infinitely dimensional vector $(y_n : n \in \mathbb{N})$. Then, with the condition of (3.6), equation (3.11) can be written as

$$\begin{cases} \frac{d\theta}{ds} = \kappa_0 + \Theta(\theta, y), \\ \frac{dy}{ds} = \Omega y + \mathcal{Y}(\theta, y), \end{cases} \tag{3.13}$$

where Θ and $\mathcal{Y}$ are analytic in the domain $\|y\| \leq \beta_0$ and $|\Im\theta| \leq \alpha_0$ with some $\alpha_0 > 0, \beta_0 > 0$, and

$$|\Theta| \leq C\epsilon, \|\mathcal{Y}\| \leq C\epsilon. \tag{3.14}$$

3.2. Fix $n_{01}, n_{02} \in \mathbb{N}$

For given n_{01} and n_{02}, re-scale the space variables as follows:

$$\begin{cases} z_{n_{0j}} = \epsilon^{4.5} y_{n_{0j}},\ j = 1, 2, \\ z_n = \epsilon^6 y_n, \qquad n \neq n_{01}, n_{02}. \end{cases}$$

Then, the equation (3.1) can be rewritten as

$$\begin{cases} \dot{y}_{n_{01}} = \lambda_{n_{01}} y_{n_{01}} - \frac{3}{2\pi}\epsilon^9(\sigma + \mathrm{i}\rho)|y_{n_{01}}|^2 y_{n_{01}} - \frac{1}{\pi}\epsilon^9(\sigma + \mathrm{i}\rho)|y_{n_{02}}|^2 y_{n_{01}} + f_{01}(y), \\ \dot{y}_{n_{02}} = \lambda_{n_{02}} y_{n_{02}} - \frac{3}{2\pi}\epsilon^9(\sigma + \mathrm{i}\rho)|y_{n_{02}}|^2 y_{n_{02}} - \frac{1}{\pi}\epsilon^9(\sigma + \mathrm{i}\rho)|y_{n_{01}}|^2 y_{n_{02}} + f_{02}(y), \\ \dot{y}_n = \lambda_n y_n - \frac{1}{\pi}\epsilon^9(\sigma + \mathrm{i}\rho)(|y_{n_{01}}|^2 + |y_{n_{02}}|^2) y_n + f_n(y), \qquad n \neq n_{01}, n_{02} \end{cases} \tag{3.15}$$

with $f(y) = (f_n(y), n \in \mathbb{N})$ and $\|f(y)\| \leq C\epsilon^{12}$ in some neighbourhood of $y = 0 \in \mathcal{H}$. Introduce the action-angle variables (I, θ) by

$$y_{n_{01}} = \sqrt{I_1} e^{\mathrm{i}\theta_1}, y_{n_{02}} = \sqrt{I_2} e^{\mathrm{i}\theta_2}.$$

Then, (3.15) is changed into

$$\begin{cases} \dot{I}_1 = 2\Re\lambda_{n_{01}} - \frac{\sigma}{\pi}\epsilon^9(3I_1+2I_2)I_1 + O(\epsilon^{12}), \\ \dot{I}_2 = 2\Re\lambda_{n_{02}} - \frac{\sigma}{\pi}\epsilon^9(2I_1+3I_2)I_2 + O(\epsilon^{12}), \\ \dot{\theta}_1 = \Im\lambda_{n_{01}} - \frac{3\rho}{2\pi}\epsilon^9 I_1 - \frac{\rho}{\pi}\epsilon^9 I_2 + O(\epsilon^{12}) \\ \dot{\theta}_2 = \Im\lambda_{n_{02}} - \frac{\rho}{\pi}\epsilon^9 I_1 - \frac{3\rho}{2\pi}\epsilon^9 I_2 + O(\epsilon^{12}) \\ \dot{y}_n = \lambda_n y_n - \frac{1}{\pi}\epsilon^9(\sigma+\mathrm{i}\rho)(I_1+I_2)y_n + O(\epsilon^{12}),\ n\neq n_{01}, n_{02}. \end{cases} \tag{3.16}$$

Let

$$I_1 = \xi_1 + \epsilon\rho_1, I_2 = \xi_2 + \epsilon\rho_2, (\xi_1,\xi_2)\in[1,2]\times[3,4], |\rho_1|, |\rho_2| < 1.$$

Assume

$$\alpha = \frac{\sigma\epsilon^9(\xi_1-\xi_2)}{2\pi(2-n_{01}^2-n_{02}^2)(n_{01}^2-n_{02}^2)},$$
$$\lambda = \frac{\sigma\epsilon^9((1-n_{02}^2)^2(3\xi_1+2\xi_2)-(1-n_{01}^2)^2(2\xi_1+3\xi_2))}{2\pi(2-n_{01}^2-n_{02}^2)(n_{01}^2-n_{02}^2)}.$$

Then, the equation (3.16) reads

$$\begin{cases} \frac{d\rho_1}{ds} = -\frac{\sigma(3\rho_1+2\rho_2)}{\pi}\xi_1 + O(\epsilon), \\ \frac{d\rho_2}{ds} = -\frac{\sigma(2\rho_1+3\rho_2)}{\pi}\xi_2 + O(\epsilon), \\ \frac{d\theta_1}{ds} = \kappa_1 + O(\epsilon), \\ \frac{d\theta_2}{ds} = \kappa_2 + O(\epsilon), \\ \frac{dy_n}{ds} = \Omega_n y_n + O(\epsilon), \qquad n\neq n_{01}, n_{02}, \end{cases} \tag{3.17}$$

where

$$-\kappa_1 = \epsilon^{-9}\beta(1-n_{01}^2)^2 + \frac{\rho(3\xi_1+2\xi_2)}{2\pi}, -\kappa_2 = \epsilon^{-9}\beta(1-n_{02}^2)^2 + \frac{\rho(2\xi_1+3\xi_2)}{2\pi},$$

$$\Omega_n = \left[\epsilon^{-9}(\lambda-\alpha(1-n^2)^2) - \frac{1}{\pi}\sigma(\xi_1+\xi_2)\right] - \mathrm{i}\left[\epsilon^{-9}\beta(1-n^2)^2 + \frac{\rho(\xi_1+\xi_2)}{\pi}\right], n\neq n_{01}, n_{02}.$$

Observe that the eigenvalues of the matrix

$$-\begin{pmatrix} \frac{3\sigma\xi_1}{\pi} & \frac{2\sigma\xi_1}{\pi} \\ \frac{2\sigma\xi_2}{\pi} & \frac{3\sigma\xi_2}{\pi} \end{pmatrix}$$

are

$$\Omega_{n_{01}} = -\frac{\sigma\left(3(\xi_1+\xi_2)+\sqrt{9(\xi_1^2+\xi_2^2)-2\xi_1\xi_2}\right)}{2\pi},$$

and

$$\Omega_{n_{02}} = -\frac{\sigma\left(3(\xi_1+\xi_2)-\sqrt{9(\xi_1^2+\xi_2^2)-2\xi_1\xi_2}\right)}{2\pi}.$$

Therefore, there is a change such that

$$\frac{d\rho_1}{ds} = -\frac{\sigma(3\rho_1 + 2\rho_2)}{\pi}\xi_1 + O(\epsilon), \frac{d\rho_2}{ds} = -\frac{\sigma(2\rho_1 + 3\rho_2)}{\pi}\xi_2 + O(\epsilon)$$

are transformed into

$$\frac{d\tilde{\rho}_1}{ds} = \Omega_{n_{01}}\tilde{\rho}_1 + O(\epsilon), \frac{d\tilde{\rho}_2}{ds} = \Omega_{n_{02}}\tilde{\rho}_2 + O(\epsilon).$$

Let $\kappa = (\kappa_1, \kappa_2)$ and $\Omega = \mathrm{diag}(\Omega_n : n \in \mathbb{N})$ and $y = (\tilde{\rho}_1, \tilde{\rho}_2, y_n, n \neq n_{01}, n_{02})$ and $\theta = (\theta_1, \theta_2)$. Then, (3.17) reads

$$\frac{d\theta}{ds} = \kappa + \Theta, \frac{dy}{ds} = \Omega y + \mathcal{Y}, \tag{3.18}$$

where $\Theta = \Theta(\theta, y, \xi)$ and $\mathcal{Y} = \mathcal{Y}(\theta, y, \xi)$ are analytic in the domain $D(c) : |\Im\theta| \leq \alpha_0, \|y\| \leq \beta_0, \xi \in [1,2] \times [3,4]$ with some $\alpha_0 > 0, \beta_0 > 0$, and

$$\sup_{D(c)} |\Theta| \leq C\epsilon, \sup_{D(c)} \|\mathcal{Y}\| \leq C\epsilon.$$

Moreover,

$$\det \frac{\partial \kappa}{\partial \xi} = \frac{5\rho^2}{4\pi^2} \neq 0.$$

4. The proof of main theorem

Because the proof of Theorem 1.2 is similar to that of Theorem 1.1. We only give the proof of Theorem 1.1. Using Taylor's formula, equation (3.13) can be rewritten as follow

$$\begin{cases} \frac{d\theta}{ds} = \kappa_0 + u^0(\theta) + \Xi^0(\theta, y), \\ \frac{dy}{ds} = \Omega y + v^0(\theta) + w^0(\theta)y + \Upsilon^0(\theta, y), \end{cases} \tag{4.1}$$

where

$$u^0 = \Theta(\theta, 0), v^0 = \mathcal{Y}(\theta, 0), w^0 = \partial_y \mathcal{Y}(\theta, 0)$$

and

$$\Xi^0 = \int_0^1 (\partial_y \Theta(\theta, \tau y), y) d\tau, \Upsilon^0 = \int_0^1 \int_0^1 \partial_y^2 \mathcal{Y}(sty)(ty, y) ds dt. \tag{4.2}$$

In our paper, the dimension n of torus $\mathbb{T}^n$ is 1. In order to apply Theorem 5.1, we only need to check that assumptions **(1)-(4)** of Theorem 5.1 hold true.

Let $\Pi_0 = \Pi, \epsilon_0 = \epsilon$. From (3.14) we know

$$\Theta(\theta, y) = O_{\alpha_0, \Pi_0}(\epsilon_0), \mathcal{Y}(\theta, y) = O_{\alpha_0, \Pi_0}(\epsilon_0).$$

Therefore,

$$u^0 = \Theta(\theta, 0) = O_{\alpha_0, \Pi_0}(\epsilon_0^{1/2}), \quad v^0 = \mathcal{Y}(\theta, 0) = O_{\alpha_0, \Pi_0}(\epsilon_0),$$

$$w^0 = \partial_y \mathcal{Y}(\theta, 0) = O_{\alpha_0, \Pi_0}(\epsilon_0^{1/2}).$$

Clearly, u^0, v^0, w^0 are analytic in the domain $\mathcal{U}(\alpha_0)$ and C^1-smooth in $\xi \in \Pi_0$. Namely, assumption **(2)** of Theorem 5.1 holds true.

In view of (4.6),

$$\Xi^0 = \int_0^1 (\partial_y \Theta(\theta, \tau y), y) d\tau = O_{\alpha_0, \beta_0, \Pi_0}(y),$$

$$\Upsilon^0 = \int_0^1 \int_0^1 \partial_y^2 \mathcal{Y}(sty)(ty, y) ds dt = O_{\alpha_0, \beta_0, \Pi_0}(y^2).$$

Then, assumption **(3)** of Theorem 5.1 holds true.

Let $\omega^0 = \kappa_0, \tau_0 = 0$ and $\Lambda^0 = \Omega = \mathrm{diag}(\lambda_j^0, j \in \mathbb{N})$, i.e. $\lambda_j^0 = \Omega_j$, then

$$\inf_{\xi \in \Pi_0} \left| \det \frac{\partial \omega^0}{\partial \xi} \right| = \left| \frac{3\rho}{2\pi} \right| > c = c(1 - \tau_0), \tag{4.3}$$

$$\sup_{\xi \in \Pi_0} |\omega^1(\xi) - \omega^0(\xi)| = \sup_{\xi \in \Pi_0} |\hat{u}^0(0)| \lessdot {}^{\dagger} \epsilon_0^{1/2},$$

where $\hat{u}^0(0)$ is the 0-Fourier coefficient of u^0. Thus,

$$\sup_{\xi \in \Pi_0} |\partial_\xi (\omega^1(\xi) - \omega^0(\xi))| \lessdot \epsilon_0^{1/2}.$$

At the same time,

$$\inf_{\xi \in \Pi_0} |\Re \lambda_j^0| = \inf_{\xi \in \Pi_0} |\epsilon^{-9}(\lambda - \alpha(1 - j^2)^2) - \frac{\sigma}{2\pi}\xi| > c = c(1 - \tau_0).$$

If $i, j \neq n_0$, we have

$$|\Re(\lambda_j^0 - \lambda_i^0)| = |\epsilon^{-9}\alpha((1-j^2)^2 - (1-i^2)^2)| = |\epsilon^{-9}\alpha(2 - i^2 - j^2)(i+j)||i-j| \geq c|i-j|.$$

If $i = n_0$, we have

$$\begin{aligned} |\Re(\lambda_j^0 - \lambda_{n_0}^0)| &= |\epsilon^{-9}(\alpha(1 - n_0^2)^2 + \frac{3}{2\pi}\sigma\epsilon^9\xi - \alpha(1 - j^2)^2) - \frac{\sigma}{2\pi}\xi + \frac{3}{\pi}\sigma\xi| \\ &\geq \frac{\epsilon^{-9}\alpha}{2} |(1 - n_0^2)^2 - (1 - j^2)^2| \geq c|j - n_0| \end{aligned} \tag{4.4}$$

for ϵ small enough. To sum up, we have

$$\inf_{\xi \in \Pi_0} |\Re(\lambda_j^0 - \lambda_i^0)| \geq c|i - j|, \text{for } \forall i, j \in \mathbb{N}.$$

Moreover,

$$\sup_{\xi \in \Pi_0} |\lambda_j^1 - \lambda_j^0| = \sup_{\xi \in \Pi_0} |\hat{w}_{jj}^0(0)| \lessdot \epsilon_0^{1/2},$$

where $\hat{w}_{jj}^0(0)$ is the 0-Fourier coefficient of w_{jj}^0, and w_{ij}^0 is the matrix elements of operator w^0. Consequently,

$$\sup_{\xi \in \Pi_0} |\partial_\xi (\lambda_j^1 - \lambda_j^0)| \lessdot \epsilon_0^{1/2}, \quad \forall j \in \mathbb{N}.$$

Therefore assumption **(1)** of Theorem 5.1 holds true.

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†For two quantities f and g, if there is an absolute constant C such that $|f| \leq |g|C$ (or $|f| \geq |g|C$) with some metric $|\cdot|$, we write $f \lessdot g$ (or $f \gtrdot g$).

By estimate (4.7), we can easily show in a standard proof of KAM theory that there is a subset $\Pi_1 \subset \Pi_0$ with Meas $(\Pi_1) \geq$ Meas $(\Pi_0)(1 - C\gamma_0)$ such that

$$|\langle k, \omega\rangle| \geq \gamma_0 |k|^{-(n+1)}, \quad 0 \neq k \in \mathbb{Z}^n.$$

This finishes the proof of **(4)**.

Because the proof of Theorem 1.2 is similar to that of Theorem 1.1. We only give the proof of Theorem 1.1. Using Taylor's formula, equation (3.13) can be rewritten as follow

$$\begin{cases} \frac{d\theta}{ds} = \kappa_0 + u^0(\theta) + \Xi^0(\theta, y), \\ \frac{dy}{ds} = \Omega y + v^0(\theta) + w^0(\theta)y + \Upsilon^0(\theta, y), \end{cases} \tag{4.5}$$

where

$$u^0 = \Theta(\theta, 0), v^0 = \mathcal{Y}(\theta, 0), w^0 = \partial_y \mathcal{Y}(\theta, 0)$$

and

$$\Xi^0 = \int_0^1 (\partial_y \Theta(\theta, \tau y), y) d\tau, \Upsilon^0 = \int_0^1 \int_0^1 \partial_y^2 \mathcal{Y}(sty)(ty, y) ds dt. \tag{4.6}$$

In our paper, the dimension n of torus $\mathbb{T}^n$ is 1. In order to apply Theorem 5.1, we only need to check that assumptions **(1)-(4)** of Theorem 5.1 hold true.

Let $\Pi_0 = \Pi, \epsilon_0 = \epsilon$. From (3.14) we know

$$\Theta(\theta, y) = O_{\alpha_0, \Pi_0}(\epsilon_0), \mathcal{Y}(\theta, y) = O_{\alpha_0, \Pi_0}(\epsilon_0).$$

Therefore,

$$u^0 = \Theta(\theta, 0) = O_{\alpha_0, \Pi_0}(\epsilon_0^{1/2}),\ v^0 = \mathcal{Y}(\theta, 0) = O_{\alpha_0, \Pi_0}(\epsilon_0),\ w^0 = \partial_y \mathcal{Y}(\theta, 0) = O_{\alpha_0, \Pi_0}(\epsilon_0^{1/2}).$$

Clearly, u^0, v^0, w^0 are analytic in the domain $\mathcal{U}(\alpha_0)$ and C^1-smooth in $\xi \in \Pi_0$. Namely, assumption **(2)** of Theorem 5.1 holds true.

In view of (4.6),

$$\Xi^0 = \int_0^1 (\partial_y \Theta(\theta, \tau y), y) d\tau = O_{\alpha_0, \beta_0, \Pi_0}(y),$$

$$\Upsilon^0 = \int_0^1 \int_0^1 \partial_y^2 \mathcal{Y}(sty)(ty, y) ds dt = O_{\alpha_0, \beta_0, \Pi_0}(y^2).$$

Then, assumption **(3)** of Theorem 5.1 holds true.

Let $\omega^0 = \kappa_0, \tau_0 = 0$ and $\Lambda^0 = \Omega = \mathrm{diag}(\lambda_j^0, j \in \mathbb{N})$, i.e. $\lambda_j^0 = \Omega_j$, then

$$\inf_{\xi \in \Pi_0} \left| \det \frac{\partial \omega^0}{\partial \xi} \right| = \left| \frac{3\rho}{2\pi} \right| > c = c(1 - \tau_0), \tag{4.7}$$

$$\sup_{\xi \in \Pi_0} |\omega^1(\xi) - \omega^0(\xi)| = \sup_{\xi \in \Pi_0} |\hat{u}^0(0)| \lessdot \epsilon_0^{1/2},$$

where $\hat{u}^0(0)$ is the 0-Fourier coefficient of u^0. Thus,

$$\sup_{\xi \in \Pi_0} |\partial_\xi (\omega^1(\xi) - \omega^0(\xi))| \lessdot \epsilon_0^{1/2}.$$

At the same time,

$$\inf_{\xi\in\Pi_0}|\Re\lambda_j^0| = \inf_{\xi\in\Pi_0}|\epsilon^{-9}(\lambda-\alpha(1-j^2)^2) - \frac{\sigma}{2\pi}\xi| > c = c(1-\tau_0).$$

If $i, j \neq n_0$, we have

$$|\Re(\lambda_j^0-\lambda_i^0)| = |\epsilon^{-9}\alpha((1-j^2)^2-(1-i^2)^2)| = |\epsilon^{-9}\alpha(2-i^2-j^2)(i+j)||i-j| \geq c|i-j|.$$

If $i = n_0$, then by $\lambda = \alpha(1-n_0^2)^2 + \frac{3}{2\pi}\sigma\epsilon^9\xi$ we have

$$\begin{aligned}|\Re(\lambda_j^0-\lambda_{n_0}^0)| &= |\epsilon^{-9}(\alpha(1-n_0^2)^2 + \frac{3}{2\pi}\sigma\epsilon^9\xi - \alpha(1-j^2)^2) - \frac{\sigma}{2\pi}\xi + \frac{3}{\pi}\sigma\xi| \\ &\geq \frac{\epsilon^{-9}\alpha}{2}|(1-n_0^2)^2-(1-j^2)^2| \geq c|j-n_0| \qquad (4.8)\end{aligned}$$

for ϵ small enough. To sum up, we have

$$\inf_{\xi\in\Pi_0}|\Re(\lambda_j^0-\lambda_i^0)| \geq c|i-j|, \text{for } \forall i,j\in\mathbb{N}.$$

Moreover,

$$\sup_{\xi\in\Pi_0}|\lambda_j^1-\lambda_j^0| = \sup_{\xi\in\Pi_0}|\hat{w}_{jj}^0(0)| \lessdot \epsilon_0^{1/2},$$

where $\hat{w}_{jj}^0(0)$ is the 0-Fourier coefficient of w_{jj}^0, and w_{ij}^0 is the matrix elements of operator w^0. Consequently,

$$\sup_{\xi\in\Pi_0}|\partial_\xi(\lambda_j^1-\lambda_j^0)| \lessdot \epsilon_0^{1/2}, \quad \forall j\in\mathbb{N}.$$

Therefore assumption **(1)** of Theorem 5.1 holds true.

By estimate (4.7), we can easily show in a standard proof of KAM theory that there is a subset $\Pi_1 \subset \Pi_0$ with Meas $(\Pi_1) \geq$ Meas $(\Pi_0)(1-C\gamma_0)$ such that

$$|\langle k,\omega\rangle| \geq \gamma_0|k|^{-(n+1)}, \quad 0\neq k\in\mathbb{Z}^n.$$

This finishes the proof of **(4)**.

Therefore, Theorem 5.1 can be applied to (4.5). So (4.5) possesses an invariant closed curve which carries a periodic solution. Note that $u^0 = O_{\alpha_0,\Pi_0}(\epsilon)$. Then the solution has the form

$$\begin{cases}\theta(s) = \theta_0 + \kappa_0 + O_1(\epsilon), \\ y(s) = O_2(\epsilon),\end{cases} \qquad (4.9)$$

where both $O_1(\epsilon)\in\mathbb{C}$ and $O_2(\epsilon)\in\mathcal{H}$ are functions of s with the same period $2\pi/\omega^*$. Returning to (2.1) by a series of changes in variables (2.3),(3.2),(3.4),(3.7),(3.10), we conclude that (3.1) possesses a periodic solution with frequency ω:

$$\begin{cases}z_{n_0}(t) = \sqrt{\xi}\epsilon^{4.5}e^{\mathrm{i}\omega t} + O(\epsilon^{14}), \\ z_n(t) = O(\epsilon^{16}), \qquad n\neq n_0,\end{cases} \qquad (4.10)$$

where the frequency ω satisfies

$$\left|\omega - \left((1-n_0^2)^2\beta + \frac{3\rho}{2\pi}\xi\epsilon^9\right)\right| \leq c\epsilon^{10}.$$

By the change of variables (2.5), the periodic solution of (2.1) satisfies

$$\begin{cases} q_{n_0}(t) = \sqrt{\xi}\epsilon^{4.5}e^{-\mathrm{i}\omega t} + O(\epsilon^{14}), \\ q_n(t) = O(\epsilon^{16}), \qquad\qquad n \neq n_0. \end{cases} \tag{4.11}$$

In view of (3.9),(3.11), we see that there are some $n \in \mathbb{N}$ such that $\Re\Omega_n > 0$ and some $n \in \mathbb{N}$ such that $\Re\Omega_n < 0$. Observe that $|\Omega_n - \Lambda_n^*| \ll 1$. From the normal form

$$\begin{aligned} \dot{\theta} &= \omega^\infty + O_{\alpha_0/2}(z), \\ \dot{z} &= \Lambda^* z + O_{\alpha_0/2}(z), \end{aligned} \quad (\theta, z, \xi) \in \mathcal{W}(\alpha_0/2, \beta_0/2) \times \Pi_\infty, \tag{4.12}$$

where $\Lambda^* = \Re \lim_{l\to\infty} \Lambda^l$, we obtain that the solution is normally hyperbolic. The proof of Theorem 1.1 is completed.

5. Appendix: The KAM theorem

The KAM theorem is quoted from [10] by Chung and Yuan. For readers' convenience, we introduce it as follows.

Theorem 5.1. *Suppose the equation* $(Eq)_0$ *:*

$$\begin{aligned} \dot{\theta} &= \omega^0 + u^0(\theta) + \Xi_0(\theta, z), \\ \dot{z} &= \Lambda^0 z + v^0(\theta) + w^0(\theta) z + \Upsilon_0(\theta, z), \end{aligned} \tag{5.1}$$

satisfies assumptions

(1) *the frequencies* ω^0, λ_j^0 *satisfy*

$$\inf_{\xi\in\Pi_0} |\det \frac{\partial \omega^0(\xi)}{\partial \xi}| \geq c(1-\tau_0), \tag{5.2}$$

$$\sup_{\xi\in\Pi_0} |\partial_\xi^k(\omega^1(\xi) - \omega^0(\xi))| \lessdot \epsilon^{\frac{1}{2}}, k = 0, 1, \tag{5.3}$$

$$\inf_{\xi\in\Pi_0} |\Re\lambda_j^0| \geq c(1-\tau_0), \inf_{\xi\in\Pi_0} |\Re(\lambda_j^0 - \lambda_i^0)| \geq c(1-\tau_0)||i| - |j||, \forall i, j \in \mathbb{N}, \tag{5.4}$$

$$\sup_{\xi\in\Pi_0} |\partial_\xi^k(\lambda_j^1 - \lambda_j^0)| \lessdot \epsilon_0^{\frac{1}{2}}, k = 0, 1, j \in \mathbb{N}, \tag{5.5}$$

(2) *the term* u^0, v^0, w^0 *are analytic in the domain* $\mathcal{U}(\alpha_0)$ *and* C^1*-smooth in* $\xi \in \Pi_0$*, and the following estimates hold true:*

$$u^0 = O_{\alpha_0,\Pi_0}(\epsilon^{1/2}), v^0 = O_{\alpha_0,\Pi_0}(\epsilon), w^0 = O_{\alpha_0,\Pi_0}(\epsilon^{1/2}),$$

(3) *the terms* Ξ_0, Υ_0 *are analytic in the domain* $\mathcal{W}(\alpha_0, \beta_0)$ *and* C^1*-smooth in* $\xi \in \Pi_0$*, and the following estimates hold true:*

$$\Xi_0 = O_{\alpha_0,\beta_0,\Pi_0}(z), \Upsilon_0 = O_{\alpha_0,\beta_0,\Pi_0}(z^2),$$

(4) *there is a constant $C > 0$ such that the Lebesgue measure of Π_0 satisfies*

$$\mathrm{Meas}\Pi_0 > 0, \quad \mathrm{Meas}\Pi_1 \geq \mathrm{Meas}\Pi_0(1 - C\gamma_0).$$

Then, for sufficiently small $\epsilon > 0$, equation (5.1) possesses an invariant torus (closed curve if $n = 1$) on which any motion is quasi-periodic (periodic if $n = 1$) with frequency ω^* ($\omega^* = \Re\omega^\infty = \Re\lim_{l\to\infty}\omega^l$) such that $|\omega^* - \omega^0| < \epsilon^{\frac{1}{2}}$, and the torus (curve) can be expressed as

$$\theta(t) = \theta_0 + \omega^* t + u(\omega^* t), z(t) = v(\omega^* t),$$

where $u(\cdot) : \mathbb{T}^n \to \mathbb{C}, v(\cdot) : \mathbb{T}^n \to \mathcal{H}$ are analytic and of period 2π and with the estimates

$$\sup_{\mathbb{T}^n} |u| \leq \epsilon^{\frac{1}{2}}, \sup_{\mathbb{T}^n} |v| \leq \epsilon.$$

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