Journal of Applied Analysis and Computation
Volume 10, Number 2, April 2020, 760–770
Website:http://jaac-online.com/
DOI:10.11948/20190145

THREE RADIAL POSITIVE SOLUTIONS FOR SEMILINEAR ELLIPTIC PROBLEMS IN $\mathbb{R}^N$ *

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Abstract This paper is concerned with the semilinear elliptic problem

$$\begin{cases} -\Delta u = \lambda h(|x|)f(u) & \text{in } \mathbb{R}^N, \\ u(x) > 0 & \text{in } \mathbb{R}^N, \\ u \to 0 & \text{as } |x| \to \infty, \end{cases}$$

where λ is a real parameter and h is a weight function which is positive. We show the existence of three radial positive solutions under suitable conditions on the nonlinearity. Proofs are mainly based on the bifurcation technique.

Keywords Semilinear elliptic problem, radial positive solutions, eigenvalue, bifurcation, connected component.

MSC(2010) 35B20, 35B40, 35J60, 35P05.

1. Introduction

In this paper, we are concerned with the semilinear elliptic problem

$$\begin{cases} -\Delta u = \lambda h(|x|)f(u) & \text{in } \mathbb{R}^N, \\ u(x) > 0 & \text{in } \mathbb{R}^N, \\ u \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{1.1}$$

where $N \geq 3$, λ is a real parameter, and h satisfies

(H1) $h : \mathbb{R}^N \to (0, \infty)$ is continuous and radially symmetric;

(H2) there exist two continuous radially symmetric functions p and P such that

$$0 < p(|x|) \leq h(|x|) \leq P(|x|), \quad x \in \mathbb{R}^N \tag{1.2}$$

and

$$\int_0^{+\infty} r^{N-1} P(r) dr < +\infty. \tag{1.3}$$

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*The authors were supported by National Natural Science Foundation of China (No. 11671322) and The key constructive discipline of Lanzhou City University(LZCU-ZDJSXK-201706).

Edelson & Rumbos [5] have shown that problem

$$\begin{cases} -\Delta u = \lambda h(|x|)u & \text{in } \mathbb{R}^N, \\ u(x) > 0 & \text{in } \mathbb{R}^N, \\ u \to 0 & \text{as } |x| \to \infty \end{cases} \tag{1.4}$$

has a sequence of simple eigenvalue

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots, \quad \lim_{k\to\infty} \lambda_k = +\infty. \tag{1.5}$$

Let ϕ_1 denote the radial positive eigenfunction corresponding to λ_1. Then ϕ_1 satisfies the asymptotic decay law

$$\lim_{r\to+\infty} r^{N-2}\phi_1(r) = c \tag{1.6}$$

for some positive constant c.

Besides the above important results, Edelson & Rumbos [5], Rumbos & Edelson [8] and Edelson & Furi [4] also obtained some interesting results involving existence of positive minimal solution by the Schauder-Tychonoff fixed point theorem or the Rabinowitz global bifurcation theorem [7] for (1.1). Very recently, Dai [2] investigated the spectral structure of (1.4) where the weight function h may change sign, and studied the existence and asymptotic behavior of one-sign and nodal solutions of (1.1) by bifurcation method. However, at most two radial positive solutions of (1.1) were obtained in above mentioned papers.

Therefore, the aim of this paper is to study the global structure of radial positive solutions for problem (1.1) and show that the radial positive solutions set contains a S-shaped connected component, and subsequently, (1.1) possesses at least three radial positive solutions for λ belonging to some open interval.

Denote by X the set

$$\Big\{u \in C(\mathbb{R}^N, \mathbb{R}) : \sup_{x\in\mathbb{R}^N} |u(x)| < \infty\Big\}$$

with the norm

$$||u|| = \sup_{x\in\mathbb{R}^N} |u(x)|, \qquad u \in X.$$

Clearly, X is a Banach space.

Let X_r consist of all radially symmetric functions in X. Let S_1^+ denote the set of positive functions in X_r, and set $S_1^- := -S_1^+$, and $S_1 = S_1^+ \cup S_1^-$. It is clear that S_1^+ and S_1^- are disjoint and open in X_r.

Denote by E the set

$$\Big\{u \in C([0,\infty), \mathbb{R}) : \sup_{r\in[0,\infty)} |u(r)| < \infty\Big\}$$

with the norm

$$||u|| = \sup_{r\in[0,\infty)} |u(r)|, \qquad u \in E.$$

Clearly, E is also a Banach space. Denote P^+ the set of functions in E which are positive in $[0, +\infty)$, $P^- = -P^+$.

The following result is essential in the study of the shape of radial positive solution set of (1.1).

Lemma 1.1. *Let $N \geq 3$. Let s_0 and r_0 be two positive constants. Let $h : [0, \infty) \to (0, \infty)$ and $f : [0, \infty) \to [0, \infty)$ be continuous. Let (λ, u) be the radial positive solution of (1.1) with $||u|| = s_0$. Then there exists a constant $M \in (0, 1)$ depending only on N, r_0 and s_0, such that*

$$\min_{x \in B_{r_0}(0)} u(x) \geq M \sup_{x \in B_{r_0}(0)} u(x) = M||u||,$$

where $B_{r_0}(0) := \{u \in \mathbb{R}^N : |u| < r_0\}$.

Proof. It is as an immediate consequence of Remark 8.16 (b) in [6], and the fact $u(r)$ is decreasing in $[0, \infty)$. □

We also need the following assumptions:

(F0) $f : [0, \infty) \to [0, \infty)$ is a Hölder continuous function with exponent α_1;
(F1) there exist $\alpha > 0$, $f_0 > 0$ and $f_1 > 0$ such that

$$\lim_{s \to 0^+} \frac{f(s) - f_0 s}{s^{1+\alpha}} = -f_1; \tag{1.7}$$

(F2)

$$\lim_{s \to \infty} \frac{f(s)}{s} = 0;$$

(F3) there exists $s_0 > 0$ and $r_0 > 0$, such that

$$\min_{M s_0 \leq s \leq s_0} \frac{f(s)}{s} > \frac{f_0}{\lambda_1 h_0} \eta_1,$$

where M is a constant as in Lemma 1.1, $h_0 = \min\limits_{r \in [0, r_0/2]} h(r)$, and η_1 is the first positive eigenvalue of the problem

$$\begin{cases} -\left(r^{N-1}(v')\right)' = \eta r^{N-1} v, \quad r \in I := [0, r_0/2], \\ v'(0) = v(r_0/2) = 0. \end{cases} \tag{1.8}$$

Notice that the first eigenvalue λ_1 of (1.4) is the minimum of Rayleigh quotient, that is

$$\lambda_1 = \inf \Big\{ \frac{\int_0^{+\infty} r^{N-1} |u'(r)|^2 dr}{\int_0^{+\infty} r^{N-1} h(r) |u(r)|^2 dr} : u \in D^{1,2}([0, \infty)),\ u(r) \not\equiv 0 \Big\},$$

where $D^{1,2}([0, \infty))$ is the completion of $C_c^\infty([0, \infty))$ with respect to the norm

$$||u||_1 = \Big(\int_0^{+\infty} r^{N-1} |u'(r)|^2 dr \Big)^{1/2}.$$

Remark 1.1. It is easy to find that if (F1) holds, then

$$\lim_{s \to 0^+} \frac{f(s)}{s} = f_0. \tag{1.9}$$

Moreover, if (1.9) and (F2) hold, then there exists $f^* > 0$ and $\gamma^* > 0$ such that

$$f(s) \leq f^* s, \qquad s \geq 0 \tag{1.10}$$

and

$$f(s) \geq \gamma^* s, \qquad 0 \leq s \leq s_0. \tag{1.11}$$

As we mentioned above, to show the existence of three radial positive solutions of (1.1), we shall employ a bifurcation technique. Indeed, under (F1) we have an unbounded connected component which is bifurcating from $(\lambda_1/f_0, 0)$. Conditions (H1), (H2), (F0) and (F1) push the direction of bifurcation to the right near $u = 0$. Conditions (F3) and (F2) will guarantee that the connected component grows to the right from the initial point $(\lambda_1/f_0, 0)$, to the left at some point and to the right near $\lambda = \infty$.

Using the similar idea to show the existence of three positive solutions of one-dimensional p-Laplacian problem and arguing the shape of bifurcation in [9], we have the following

Theorem 1.1. *Let (H1), (H2), (F0), (F1), (F2) and (F3) hold. Then there exist* $\lambda_* \in (0, \lambda_1/f_0)$ *and* $\lambda^* \in (\lambda_1/f_0, \infty)$ *such that*

(i) (1.1) has at least one radial positive solution if $\lambda = \lambda_*$*;*

(ii) (1.1) has at least two radial positive solutions if $\lambda_* < \lambda \leq \lambda_1/f_0$*;*

(iii) (1.1) has at least three radial positive solutions if $\lambda_1/f_0 < \lambda < \lambda^*$*;*

(iv) (1.1) has at least two radial positive solutions if $\lambda = \lambda^*$*;*

(v) (1.1) has at least one radial positive solution if $\lambda > \lambda^*$*.*

2. Preliminaries

To find a radially symmetric solution of problem (1.1) is equivalent to find a solution of the following boundary value problem

$$\begin{cases} -\left(r^{N-1}u'\right)' = \lambda r^{N-1}h(r)f(u), \qquad r \in (0, +\infty), \\ u'(0) = \lim\limits_{r\to+\infty} u(r) = 0. \end{cases} \tag{2.1}$$

We first establish a global bifurcation result for problem (2.1) with $f(s) = f_0 s + \xi(s)$, i.e.,

$$\begin{cases} -\left(r^{N-1}u'\right)' = \lambda r^{N-1}h(r)[f_0 u + \xi(u)], \qquad r \in (0, +\infty), \\ u'(0) = \lim\limits_{r\to+\infty} u(r) = 0, \end{cases} \tag{2.2}$$

where $\xi : \mathbb{R}^+ \to \mathbb{R}$ with $\mathbb{R}^+ := [0, +\infty)$ satisfies $\lim\limits_{s\to 0^+} \xi(s)/s = 0$ and the following sublinear growth restriction

$$|\xi(s)| \leq C(1 + |s|)$$

for some constant $C \in (0, +\infty)$.

Let ω_N be the volume of the unit ball of $\mathbb{R}^N$. Let $c_N = \frac{1}{N(N-2)\omega_N}$, and let

$$\Gamma_N(x - y) = c_N|x - y|^{2-N}.$$

Let us define an integral operator $T : X_r \to X_r$

$$T(u) = \int_{\mathbb{R}^N} \Gamma_N(x-y)h(y)u(y)dy, \qquad u \in X_r.$$

Then by an argument similar to that of [8], we may show that u is a radially symmetric positive $C^{2+\alpha}$ solution of problem (2.2) if and only if u is a solution of the operator equation

$$u = \lambda T(f_0 u) + \lambda T(\xi(u)), \qquad u \in X_r. \tag{2.3}$$

Similarly to Proposition 1 in [8], we also can show that $T : X_r \to X_r$ is a linear completely continuous operator.

Define

$$\mathcal{S} := \overline{\{(\lambda, u) : u = \lambda T(f(u)) \text{ and } u \not\equiv 0\}}^{\mathbb{R}\times X_r}. \tag{2.4}$$

Applying the Rabinowitz global bifurcation theorem [7] to operator equation (2.3), one has that there exists a component $\mathcal{C}_1$ of $\mathcal{S}$ which contains $(\lambda_1/f_0, 0)$ and either is unbounded or passes through $(\mu/f_0, 0)$, where μ is another characteristic value of T. Furthermore, using the Dancer unilateral global bifurcation theorem [3], one gets that there are two distinct continua, $\mathcal{C}_1^+$ and $\mathcal{C}_1^-$, consisting of the bifurcation branch $\mathcal{C}_1$ emanating from $(\lambda_1/f_0, 0)$, which satisfy either $\mathcal{C}_1^+$ and $\mathcal{C}_1^-$ are both unbounded or $\mathcal{C}_1^+ \cap \mathcal{C}_1^- \neq \{(\mu/f_0, 0)\}$.

By the same method to prove [3, Theorem 1.3], with obvious changes, we may prove that

$$\mathcal{C}_1^\nu \subset \big(\{(\lambda_1/f_0, 0)\} \cup (\mathbb{R} \times P^\nu)\big), \qquad \nu \in \{+, -\}. \tag{2.5}$$

Thus, we get

Lemma 2.1. *Let (H1), (H2), (F0), (F1) and (F2) hold. Then $(\lambda_1/f_0, 0)$ is the only bifurcation point of the positive solutions of (2.2). Moreover, $\mathcal{C}_1^+$ and $\mathcal{C}_1^-$ are both unbounded in $[0, \infty) \times X_r$.*

By the same method to prove Lemma 2.3 in [9] with obvious changes, we may get

Lemma 2.2. *Let (H1), (H2), (F0) hold. Assume that (F1) and (F2) hold. Let $\{(\eta_j, u_j)\}$ be a sequence of positive solutions to (2.2) which satisfies $||u_j|| \to 0$ and $\eta_j \to \lambda_1/f_0$. Let $\phi_1(r)$ be the radial eigenfunction of (1.4) corresponding to λ_1, which satisfies $||\phi_1|| = 1$. Then there exists a subsequence of $\{u_j\}$, again denoted by $\{u_j\}$, such that $u_j/||u_j||$ converges uniformly to ϕ_1 on $[0, \infty)$.*

Lemma 2.3. *Assume that (H1), (H2), (F0), (F1) and (F2) hold. Let $\mathcal{C}_1^+$ be as in Lemma 2.1. Then there exists $\delta > 0$ such that $(\lambda, u) \in \mathcal{C}_1^+$ and $|\lambda - \lambda_1/f_0| + ||u|| \leq \delta$ imply $\lambda > \lambda_1/f_0$.*

Proof. Assume on the contrary that there exists a sequence $\{(\eta_j, u_j)\}$ such that $(\eta_j, u_j) \in \mathcal{C}_1^+$, $\eta_j \to \lambda_1/f_0$, $||u_j|| \to 0$ and $\eta_j \leq \lambda_1/f_0$. By Lemma 2.2, there exists a subsequence of $\{u_j\}$, again denoted by $\{u_j\}$, such that $u_j/||u_j||$ converges uniformly to ϕ_1 on $[0, \infty)$, where $\phi_1(r) > 0$ is the first eigenfunction of (1.4) which satisfies $||\phi_1|| = 1$. Multiplying the equation of (2.1) with $(\lambda, u) = (\eta_j, u_j)$ by u_j and integrating it over $[0, +\infty)$, we obtain

$$\eta_j \int_0^\infty r^{N-1}h(r)f(u_j(r))u_j(r)dr = \int_0^\infty r^{N-1}(u_j'(r))^2 dr - \lim_{t\to\infty} t^{N-1}u_j'(t)u_j(t).$$

From $\eta_j \to (\lambda_1/f_0) + \delta$ and (1.10) and $||u_j|| \to 0$ as $j \to \infty$, it follows that

$$-t^{N-1}u_j'(t) = \eta_j \int_0^t s^{N-1}h(s)f(u_j(s))ds \le \eta_j f^* \int_0^t s^{N-1}h(s)ds||u_j|| \le C_2$$

for some constant $C_2 > 0$. Combining this with the fact $\lim_{t\to\infty} u_j(t) = 0$ and using the fact $u_j(t)$ is decreasing in t, it deduces that

$$\lim_{t\to\infty} t^{N-1}u_j'(t)u_j(t) = 0,$$

and subsequently

$$\eta_j \int_0^\infty r^{N-1}h(r)f(u_j(r))u_j(r)dr = \int_0^\infty r^{N-1}(u_j'(r))^2dr.$$

Since λ_1 is the minimum of Rayleigh quotient, we get

$$\eta_j \int_0^\infty r^{N-1}h(r)f(u_j(r))u_j(r)dr \ge \lambda_1 \int_0^\infty r^{N-1}h(r)(u_j(r))^2dr.$$

It is easy to see that

$$\int_0^\infty r^{N-1}h(r)\frac{f(u_j(r))-f_0|u_j(r)|}{|u_j(r)|^{1+\alpha}}\Big|\frac{u_j(r)}{||u_j||}\Big|^{2+\alpha}dr \ge \frac{\lambda_1 - f_0\eta_j}{\eta_j||u_j||^\alpha}\int_0^\infty r^{N-1}h(r)\Big|\frac{u_j(r)}{||u_j||}\Big|^2dr.$$

Lebesgue's dominated convergence theorem and (F1) imply that

$$\int_0^\infty r^{N-1}h(r)\frac{f(u_j(r))-f_0|u_j(r)|}{|u_j(r)|^{1+\alpha}}\Big|\frac{u_j(r)}{||u_j||}\Big|^{2+\alpha}dr \to -f_1\int_0^\infty r^{N-1}h(r)|\phi_1(r)|^{2+\alpha}dr < 0$$

and

$$\int_0^\infty r^{N-1}h(r)\Big|\frac{u_j(r)}{||u_j||}\Big|^2dr \to \int_0^\infty r^{N-1}h(r)|\phi_1(r)|^2dr > 0.$$

This contradicts $\eta_j \le \lambda_1/f_0$. □

3. Direction turn of bifurcation

In this section, we show that there is a direction turn of the bifurcation under (F3) condition.

Lemma 3.1. *Let (H1), (H2) and (F0) hold. Let $u \in C^1[0,\infty)$ be the positive solution of (2.1) with $||u|| = s_0$ for some $s_0 > 0$. Then*

$$M||u|| \le u(r) \le ||u||, \qquad r \in I = [0, r_0/2] \tag{3.1}$$

for some $M \in (0,1)$ depending only on N, r_0 and s_0.

Proof. From the fact $u(r)$ is decreasing on $[0,\infty)$, we have that

$$||u|| = u(0) = \max_{r\in[0,\rho]} u(r) = \max_{r\in[0,\infty)} u(r)$$

for any $\rho \in (0,\infty)$. From Lemma 1.1,

$$\min_{r\in[0,r_0/2]} u \ge M \sup_{r\in[0,r_0/2]} u = M||u||, \tag{3.2}$$

where $M \in (0, 1)$ is as in Lemma 1.1. □

Lemma 3.2. *Assume that (H1), (H2), (F0) and (F3) hold. Let u be a positive solution of (2.1) with $||u|| = s_0$. Then*

$$\lambda < \lambda_1/f_0.$$

Proof. Let u be a positive solution of (2.1). By Lemma 3.1,

$$Ms_0 \le u(r) \le s_0, \qquad r \in I.$$

Now we assume $\lambda \ge \lambda_1/f_0$. Then by (F3),

$$\lambda \frac{f(u(r))}{u(r)} > \frac{\lambda_1}{f_0} \frac{f_0 \eta_1}{\lambda_1 h_0}, \qquad r \in I, \tag{3.3}$$

where η_1 is the first positive eigenvalue of the problem

$$\begin{cases} -\left(r^{N-1}v'(r)\right)' = \eta r^{N-1} v(r), \qquad r \in I, \\ v'(0) = 0 = v(r_0/2). \end{cases} \tag{3.4}$$

Obviously, (3.3) implies

$$\lambda h(r) \frac{f(u(r))}{u(r)} > \eta_1 + \epsilon_1 \tag{3.5}$$

for some $\epsilon_1 \in (0, \infty)$.

Let w_1 be the corresponding eigenfunction of η_1. Then

$$w_1(r) > 0, \qquad r \in I.$$

Note that u is a positive solution of

$$\begin{cases} -\left(r^{N-1}u'(r)\right)' = \lambda r^{N-1} h(r) \dfrac{f(u(r))}{u(r)} u(r), \quad r \in (0, r_0/2), \\ u'(0) = 0, \quad u(r_0/2) > 0. \end{cases} \tag{3.6}$$

From (3.5) and the fact $\eta_1 < \eta_1 + \epsilon_1$, we have from Sturm comparison theorem that u has a zero in $[0, r_0/2)$. However, this is a contradiction. □

4. Second turn and proof of Theorem 1.1

In this section, we shall give a block for a parameter and a priori estimate and finally a proof of Theorem 1.1.

Lemma 4.1. *Assume that (H1), (H2), (F0), (F1) and (F2) hold. Let J be a compact interval in $(0, \infty)$. Then for all $\lambda \in J$, there exists $M_J > 0$ such that all possible positive solutions u of (2.1) satisfy $||u|| \le M_J$.*

Proof. Assume on the contrary that there exists a sequence $\{(\beta_j, u_j)\}$ of positive solutions of (2.1) such that

$$\beta_j \in J; \qquad u_j > 0 \text{ in } [0, \infty) \text{ and } ||u_j|| \to \infty \quad \text{as } j \to \infty.$$

By the same method to prove the claim in the proof of Lemma 4.1, we may deduce that

$$\lim_{j\to\infty}\frac{f(u_j(r))}{||u_j||}=0 \qquad \text{uniformly for } r\in[0,\infty). \tag{4.1}$$

Let $v_j := u_j/||u_j||$. Then $||v_j|| = 1$ and

$$\begin{cases} -\left(r^{N-1}v_j'\right)'=\beta_j r^{N-1}h(r)\dfrac{f(u_j)}{u_j}v_j, & r\in(0,\infty),\\ v_j(r)>0 & \text{in } [0,\infty),\\ v_j(r)\to 0 & \text{as } r\to\infty. \end{cases}$$

By the standard method, we may show that after taking a subsequence and relabeling if necessary, $v_j \to v_*$ in X_r. Moreover, v^* is a solution of

$$\begin{cases} -\left(r^{N-1}v_*'\right)'=r^{N-1}h(r)\,0\,v_*, & r\in(0,\infty),\\ v_*(r)\ge 0 & \text{in } [0,\infty),\\ v_*(r)\to 0 & \text{as } r\to\infty. \end{cases}$$

Letting $b \to +\infty$, it follows from (4.1) that

$$v_*(r)\equiv 0, \qquad r\in[0,\infty).$$

However, this contradicts the fact $||v_*|| = \lim_{j\to\infty}||v_j|| = 1$. □

Lemma 4.2. *Assume that (H1), (H2), (F0), (F1) and (F2) hold. Let (λ, u) be a positive solution of (1.1) with $||u|| \le s_0$. Then there exists $C_1 > 0$ independent of u such that $\lambda \underline{f}(||u||) < C_1$, where*

$$\underline{f}(s):=\min_{Ms\le t\le s} f(t)/t.$$

Proof. Let $h_* := \min_{x\in B_{r_0}(0)} h(x)$. Then

$$\begin{aligned} u(x)&=\lambda\int_{\mathbb{R}^N}\Gamma_N(x-y)h(y)f(u(y))dy\\ &\ge\lambda\int_{B_{r_0}(0)}\Gamma_N(x-y)h(y)f(u(y))dy\\ &\ge\lambda\int_{B_{r_0}(0)}\Gamma_N(x-y)h(y)\frac{f(u(y))}{u(y)}u(y)dy\\ &\ge\lambda\int_{B_{r_0}(0)}\Gamma_N(x-y)h_*\underline{f}(||u||)\,M||u||dy, \end{aligned}$$

which implies that $\lambda\underline{f}(||u||) < C_1$ for some $C_1 > 0$. □

Proof of Theorem 1.1. Let $\mathcal{C}_1^+$ be as in Lemma 2.1. By Lemma 2.3, $\mathcal{C}_1^+$ is bifurcating from $(\lambda_1/f_0, 0)$ and goes rightward.

We claim that there exists a sequence $\{(\beta_j, u_j)\} \subset \mathcal{C}_1^+$ satisfying

$$\beta_j\to+\infty, \qquad ||u_j||_\infty > s_0. \tag{4.2}$$

Assume on the contrary that there exists $\beta^* > 0$, such that

$$||u|| \le s_0 \quad \text{for all } (\lambda, u) \in \mathcal{C}_1^+ \text{ with } \lambda > \beta^*. \tag{4.3}$$

Then it follows from Lemma 4.2 that

$$\lambda \underline{f}(||u||) < C_1 \quad \forall \text{ for all } (\lambda, u) \in \mathcal{C}_1^+ \text{ with } \lambda > \beta^*. \tag{4.4}$$

Notice that $0 \le ||u|| \le s_0$ implies $\underline{f}(||u||) \ge \delta_0 > 0$. However this contradicts (4.4). Therefore, (4.1) holds.

Thus, there exists $(\beta_0, u_0) \in \mathcal{C}_1^+$ such that $||u_0|| = s_0$. Lemma 3.2 implies that $\beta_0 < \lambda_1/f_0$. By Lemmas 2.3, 3.2 and 4.1, $\mathcal{C}_1^+$ passes through some points $(\lambda_1/f_0, v_1)$ and $(\lambda_1/f_0, v_2)$ with $||v_1|| < s_0 < ||v_2||$. By Lemmas 2.3, 3.2 and 4.1 again, there exist $\bar{\lambda}$ and $\underline{\lambda}$ which satisfy $0 < \underline{\lambda} < \lambda_1/f_0 < \bar{\lambda}$ and both (i) and (ii):

(i) if $\lambda \in (\lambda_1/f_0, \bar{\lambda}]$, then there exists u and v such that (λ, u), $(\lambda, v) \in \mathcal{C}_1^+$ and $||u|| < ||v|| < s_0$;

(ii) if $\lambda \in (\underline{\lambda}, \lambda_1/f_0]$, then there exists u and v such that (λ, u), $(\lambda, v) \in \mathcal{C}_1^+$ and $||u|| < s_0 < ||v||$.

Define $\lambda^* = \sup\{\bar{\lambda} : \bar{\lambda} \text{ satisfies (i)}\}$ and $\lambda_* = \inf\{\underline{\lambda} : \underline{\lambda} \text{ satisfies (ii)}\}$. Then by the standard argument, (1.1) has a positive solution at $\lambda = \lambda_*$ and $\lambda = \lambda^*$, respectively. Since $\mathcal{C}_1^+$ passes through $(\lambda_1/f_0, v_2)$ and (β_j, u_j), Lemma 2.3 shows that, for each $\lambda > \lambda_1/f_0$, there exists w such that $(\lambda, w) \in \mathcal{C}_1^+$ and $||w|| > s_0$. This completes the proof. □

5. Estimation of M

In this section, we give an example in which $M = \frac{1}{2}$.

Let

$$G(t,s) = \begin{cases} \dfrac{t^{2-N}}{N-2}, & 0 \le s \le t < \infty, \\ \dfrac{s^{2-N}}{N-2}, & 0 \le t \le s < \infty \end{cases} \tag{5.1}$$

is the Green's function of the problem

$$\begin{cases} -\left(r^{N-1}u'\right)' = \lambda r^{N-1} h(r) f(u(r)), & r \in (0, +\infty), \\ u'(0) = \lim\limits_{r \to +\infty} u(r) = 0. \end{cases} \tag{5.2}$$

Then, for the radial solution u, (1.1) is equivalent (5.2), and (5.2) can be rewritten as

$$u(t) = \lambda \int_0^\infty G(t,s) s^{N-1} h(s) f(u(s)) ds. \tag{5.3}$$

From (5.3), it is easy to deduce

$$u'(t) = -\lambda t^{1-N} \int_0^t s^{N-1} h(s) f(u(s)) ds.$$

Moreover

$$0 \ge u'(t) \ge -\lambda M_0 t, \tag{5.4}$$

where

$$M_0 := \frac{1}{N}||h||_\infty \max\{f(x) : x \in [0, s_0]\}. \tag{5.5}$$

Let

$$\underline{f}(s) := \inf\left\{\frac{f(r)}{r} : r \in (0, s]\right\}.$$

Then from

$$\begin{aligned} u(1) &= \lambda \int_0^\infty G(1,s)s^{N-1}h(s)f(u(s))ds \\ &\geq \lambda \int_{1/2}^1 G(1,s)s^{N-1}h(s)f(u(s))ds \\ &\geq \lambda \int_{1/2}^1 G(1,s)s^{N-1}h(s)\underline{f}(u(s))ds \\ &\geq \lambda \int_{1/2}^1 G(1,s)s^{N-1}h(s)\underline{f}(s_0)u(1)ds \end{aligned}$$

it follows that

$$\lambda \leq \frac{1}{\int_{1/2}^1 G(1,s)s^{N-1}h(s)\underline{f}(s_0)ds} := b_0.$$

This together with (5.4) imply that

$$u(t) \geq ||u||_\infty - b_0M_0t \geq s_0 - b_0M_0, \qquad t \in [0, \min\{1, \frac{s_0}{b_0M_0}\}].$$

Let $r_0 := \frac{1}{2}\min\{1, \frac{s_0}{b_0M_0}\}$. Then

$$\min_{r\in[0,r_0]} u(t) \geq \frac{1}{2} \max_{r\in[0,r_0]} u(t).$$

Acknowledgements

The authors are very grateful to the anonymous referees for their valuable suggestions.

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