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ANALYSIS OF A PREDATOR-PREY MODEL WITH CROWLEY-MARTIN AND MODIFIED LESLIE-GOWER SCHEMES WITH STOCHASTIC PERTURBATION

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Abstract In this paper, we study a nonautonomous predator-prey model with Crowley-Martin and modified Leslie-Gower schemes with stochastic perturbation. The existence of a global positive solution and stochastically ultimate boundedness are obtained. Sufficient conditions are established for extinction, persistence in the mean, and stochastic permanence of the system. Finally, simulations are carried out to verify our results.

Keywords Stochastic, Crowley-Martin functional response, modified Leslie-Gower functional response, extinction, persistence.

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1. Introduction

As we all know, functional response is one of the most important factors in the dynamic relationship between predators and their preys [4,5,8,11,25]. The classical types of functional response involve Holling types [9], Beddington-DeAngelis type [3], ratio-dependence type [1], etc. In 1975, Crowley and Martin [20] proposed the Crowley-Martin functional response, which considers the interaction between predator and prey. Even assuming the prey density is enough large, the catches still decline with the increase of predator density. In addition, whether or not a predator search for prey, there always exists interference between predators. Hence it's more in line with the natural biological phenomenon.

Meanwhile, considering the fact that reduction in a predator population has a reciprocal relationship with per capita availability of its preferred food, Leslie [13] introduced a predator-prey model where the carrying capacity of the predator's environment is proportional to the number of prey. For this predator-prey model, Leslie and Gower [14], Pielou [23] had discussed the predator dynamics which can be written as follows

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ry\left(1 - \frac{y}{\alpha x}\right),$$

in which the growth of the predator population is of logistic form if letting αx be the carrying capacity. Here, the denominator αx measures the carrying capacity set by the environmental resources and is proportional to prey abundance, $\alpha > 0$

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is the conversion factor of prey into predator. The term $\frac{y}{\alpha x}$ is named the Leslie-Gower term, which measures the loss of the predator population due to the rarity of its favorite food. In real world, when the prey population is severely scarce, the predator y can search for other food, but its growth will be limited because it is the fact that its most favorite food, the prey x is not enough. So, it's vital to add a positive constant to the denominator, which is generally called modified Leslie-Gower functional response, then the above equation becomes

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ry\left(1 - \frac{y}{\alpha x + d}\right).$$

In recent years, the predator-prey models with modified Leslie-Gower functional response have received great attention and have been studied extensively [6, 11, 17, 24]. However, so far as our knowledge is concerned, no results related to predator-prey models with Crowley-Martin and modified Leslie-Gower schemes have been reported. Motivated by these, in this paper, we will concentrate on the nonautonomous predator-prey model with Crowley-Martin and modified Leslie-Gower schemes:

$$\begin{cases} \dot{x}(t) = x(t)\left[a(t) - b(t)x(t) - \dfrac{c(t)y(t)}{(1+\alpha(t)x(t))(1+\beta(t)y(t))}\right], \\ \dot{y}(t) = y(t)\left[r(t) - h(t)y(t) - \dfrac{f(t)y(t)}{x(t)+m(t)}\right] \end{cases}$$

with initial value $x(0) = x_0 > 0$, $y(0) = y_0 > 0$, where $x(t)$ and $y(t)$ stand for the population density of prey and predator at time t respectively; $a(t)$ and $r(t)$ are the growth rates of the prey and predator, respectively; $b(t)$ and $h(t)$ represent density-dependent coefficients of $x(t)$ and $y(t)$; $c(t)$ is the capturing rate of predator; $f(t)$ is the maximum value of the per capita reduction rate of $y(t)$ due to $x(t)$; $m(t)$ measures the extent to which the environment provides protection to predator $y(t)$; $\alpha(t)$ and $\beta(t)$ represent the effects of handling time and magnitude of interference among predators. Furthermore, $c(t) \geq 0$, $f(t) \geq 0$ and $a(t)$, $b(t)$, $r(t)$, $h(t)$, $m(t)$, $\alpha(t)$, $\beta(t)$ are all continuous and bounded above and below by positive constants on $\mathbb{R}_+ = (0, +\infty)$.

In realistic environment, population systems are often affected by noise, and hence stochastic differential equation models play important roles in various branches of applied sciences including biology and population dynamics, as they provide some additional degree of realism compared to their deterministic counterpart. Moreover, due to continuous fluctuations in the environment, parameters involved in models are not absolute constants, but they always fluctuate around some average value. As a result, the population density never attains a fixed value with the advancement of time but rather exhibits continuous oscillation around some average values. Based on these factors, stochastic population models have gotten more and more attention [2, 16, 19, 26]. In this paper, we add stochastic perturbations in this way:

$$a(t) \to a(t) + \sigma_1(t)\dot{B}_1(t), \qquad r(t) \to r(t) + \sigma_2(t)\dot{B}_2(t).$$

Then the stochastic model takes the following form:

$$\begin{cases} \mathrm{d}x(\mathrm{t}) = x(t)\left[a(t) - b(t)x(t) - \dfrac{c(t)y(t)}{(1+\alpha(t)x(t))(1+\beta(t)y(t))}\right]\mathrm{d}t \\ \qquad + \sigma_1(t)x(t)\mathrm{d}B_1(\mathrm{t}), \\ \mathrm{d}y(\mathrm{t}) = y(t)\left[r(t) - h(t)y(t) - \dfrac{f(t)y(t)}{x(t)+m(t)}\right]\mathrm{d}t + \sigma_2(\mathrm{t})y(\mathrm{t})\mathrm{d}B_2(\mathrm{t}), \end{cases} \tag{1.1}$$

which is studied in this paper.

Throughout this paper, unless otherwise specified, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Let $\sigma_1(t)$ and $\sigma_2(t)$ stand for the intensities of the white noises, $B_1(t)$ and $B_2(t)$ denote the independent standard Brownian motions defined on this probability space. We denote $\mathbb{R}^2_+ = \{X(t) = (x(t), y(t))|x(t) > 0, y(t) > 0\}$ and $|X(t)| = (x(t)^2 + y(t)^2)^{\frac{1}{2}}$.

The paper is designed as follows. In Section 2, we make some preliminaries and give some important results of stochastic equations. In Section 3, we use comparison theorem of stochastic equations and Itô formula to obtain the global existence of a unique positive solution of the system (1.1). Moreover, by using Chebyshev inequality, we obtain the stochastically ultimate boundedness of the system. In Sections 4 and 5, we investigate the long time behavior of the system (1.1). The conditions of extinction, persistence in the mean, and the stochastic permanence are established. Finally, simulations are carried out to support our results.

2. Preliminaries

For convenience, we do some definitions and notations:

$$f^u = \sup_{t\geq 0} f(t), \quad f^\ell = \inf_{t\geq 0} f(t), \quad \langle f(t)\rangle = \frac{1}{t}\int_0^t f(s)\mathrm{d}s,$$
$$f^* = \limsup_{t\to+\infty} f(t), \quad f_* = \liminf_{t\to+\infty} f(t).$$

Definition 2.1 ([15]). Let $X(t) = (x(t), y(t))$ be the solution of system (1.1), if for any $\varepsilon \in (0,1)$, there exists a constant $\chi > 0$ such that

$$\limsup_{t\to+\infty} \mathbb{P}(|X(t)| > \chi) < \varepsilon,$$

then we say system (1.1) is stochastically ultimately bounded.

Definition 2.2 ([15]). Let $X(t) = (x(t), y(t))$ be the solution of system (1.1), if for any $\varepsilon \in (0,1)$, there are constants $\delta > 0,\ \chi > 0$ such that

$$\liminf_{t\to+\infty} \mathbb{P}(|X(t)| \geq \delta) \geq 1-\varepsilon \quad \text{and} \quad \liminf_{t\to+\infty} \mathbb{P}(|X(t)| \leq \chi) \geq 1-\varepsilon,$$

then we say system (1.1) is stochastically permanent.

Definition 2.3 ([27]). In the predator-prey system (1.1), we say the prey $x(t)$ is

(1) Extinction, if $lim_{t\to+\infty} x(t) = 0$, $a.s.$

(2) Non-persistence in the mean, if $\langle x(t)\rangle^* = 0$.

(3) Weak-persistence in the mean, if $\langle x(t)\rangle^* > 0$.

(4) Strong persistence in the mean, if $\langle x(t)\rangle_* > 0$.

Lemma 2.1 ([10], Stochastic comparison theorem). *Considering the two one-dimensional stochastic differential equations:*

$$\mathrm{d}x_1(t) = f_1(x_1(t), t)\mathrm{d}t + g(x_1(t), t)\mathrm{d}B_1(t), \quad x_1(0) = \tilde{x}_1 \in \mathbb{R},$$

$$\mathrm{d}x_2(t) = f_2(x_2(t), t)\mathrm{d}t + g(x_2(t), t)\mathrm{d}B_1(t), \quad x_2(0) = \tilde{x}_2 \in \mathbb{R}.$$

If the solutions of the two equations exist and at least one is unique, and satisfy:

$$\tilde{x}_1 = \tilde{x}_2, \quad f_1(x, t) \le f_2(x, t), \quad \forall t \ge 0, \ x \in \mathbb{R},$$

then

$$x_1(t) \le x_2(t) \ a.s.$$

Lemma 2.2 ([22], Existence and uniqueness of the local solutions). *Consider the $d-$dimensional stochastic differential equation:*

$$\mathrm{d}x(t) = f(x(t), t)\mathrm{d}t + g(x(t), t)\mathrm{d}B(t).$$

Assume that for every integer $n \ge 1$, there exists a positive constant K_n such that for all $t \in [t_0, T]$ and for all x, $y \in \mathbb{R}^d$ with $|x| \vee |y| \le n$,

$$|f(x, t) - f(y, t)|^2 \vee |g(x, t) - g(y, t)|^2 \le K_n|x - y|^2,$$

then there exists a unique solution $x(t)$ to the above equation and the solution belongs to $\mathcal{M}^2([t_0, T]; \mathbb{R}^d)$.

Lemma 2.3 ([21], Strong law of large numbers). *Let $M = \{M_t\}_{t\ge 0}$ be a real-valued continuous local martingale vanishing at $t = 0$. Then*

$$\lim_{t\to+\infty} \langle M, M\rangle_t = \infty \ a.s. \ \Rightarrow \ \lim_{t\to+\infty} \frac{M_t}{\langle M, M\rangle_t} = 0 \, a.s.$$

and also

$$\limsup_{t\to+\infty} \frac{\langle M, M\rangle_t}{t} < \infty \ a.s. \ \Rightarrow \ \lim_{t\to+\infty} \frac{M_t}{t} = 0 \, a.s.$$

Lemma 2.4 ([21], *Itô* formula). *Let $x(t)$ be a $d-$dimensional Itô process on $t \ge 0$ with the stochastic differential*

$$\mathrm{d}x(t) = f(t)\mathrm{d}t + g(t)\mathrm{d}B(t),$$

where $f \in \mathcal{L}^1(\mathbb{R}_+; \mathbb{R}^d)$ and $g \in \mathcal{L}^2(\mathbb{R}_+; \mathbb{R}^{d\times m})$. Let $V \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R})$. Then V(x(t),t) is again an Itô process with the stochastic differential given by

$$\begin{aligned}\mathrm{d}V(x(t), t) = {} & [V_t(x(t), t) + V_x(x(t), t)f(t) + \frac{1}{2}trace(g^T(t)V_{xx}(x(t), t)g(t))]\mathrm{d}t \\ & + V_x(x(t), t)g(t)\mathrm{d}B(t) \, a.s.\end{aligned}$$

Lemma 2.5 ([21], Hölder inequality). *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $X \in L^p$, $Y \in L^q$, then*

$$|\mathbb{E}(X^T Y)| \leq (\mathbb{E}|X|^p)^{\frac{1}{p}} (\mathbb{E}|Y|^q)^{\frac{1}{q}}.$$

Lemma 2.6 ([21], Chebyshev inequality). *If $p > 0$, $c > 0$, $X \in L^p$, then*

$$c^{-p}\mathbb{E}|X(t)|^p \geq \mathbb{P}(|X(t)| \geq c).$$

Lemma 2.7 ([21], Burkholder-Davis-Gundy inequality). *Let $g \in \mathcal{L}^2(\mathbb{R}_+; \mathbb{R}^{n\times m})$. Define for $t \geq 0$,*

$$x(t) = \int_0^t g(s)\mathrm{d}B(s) \quad and \quad A(t) = \int_0^t |g(s)|^2 \mathrm{d}s.$$

Then for every $p > 0$, there exist universal positive constants c_p, C_p, which are only dependent on p, such that

$$c_p \mathbb{E}|A(t)|^{\frac{p}{2}} \leq \mathbb{E}(\sup_{0\leq s\leq t} |x(s)|^p) \leq C_p \mathbb{E}|A(t)|^{\frac{p}{2}}$$

for all $t > 0$. In particular, one may take

$$\begin{aligned}
&c_p = \left(\frac{p}{2}\right)^p, && C_p = \left(\frac{32}{p}\right)^{\frac{p}{2}}, && \text{if } 0 < p < 2;\\
&c_p = 1, && C_p = 4, && \text{if } p = 2;\\
&c_p = (2p)^{-\frac{p}{2}}, && C_p = \left[\frac{p^{p+1}}{2(p-1)^{p-1}}\right]^{\frac{p}{2}}, && \text{if } p > 2.
\end{aligned}$$

Lemma 2.8 ([21], Borel-Cantelli's lemma). *(i) If $\{A_k\} \subset \mathcal{F}$ and $\sum_{k=1}^{\infty} \mathbb{P}(A_k) < \infty$, then*

$$\mathbb{P}(\limsup_{k\to\infty} A_k) = 0.$$

That is, there exists a set $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ and an integer-valued random variable k_0 such that for every $\omega \in \Omega_0$ we have $\omega \notin A_k$ whenever $k \geq k_0(\omega)$.

(ii) If the sequence $\{A_k\} \subset \mathcal{F}$ is independent and $\sum_{k=1}^{\infty} \mathbb{P}(A_k) = \infty$, then

$$\mathbb{P}(\limsup_{k\to\infty} A_k) = 1.$$

That is, there exists a set $\Omega_\theta \in \mathcal{F}$ with $\mathbb{P}(\Omega_\theta) = 1$ such that for every $\omega \in \Omega_\theta$, there exists a sub-sequence $\{A_{k_i}\}$ such that the ω belongs to every A_{k_i}.

Lemma 2.9 ([19]). *Suppose $x(t) \in C[\Omega\times\mathbb{R}_+, \mathbb{R}_+^0]$, where $\mathbb{R}_+^0 := \{a|a > 0, a \in \mathbb{R}\}$.*

(i) If there are positive constants λ_0, T and $\lambda \geq 0$ such that

$$\ln x(t) \leq \lambda t - \lambda_0 \int_0^t x(t)\mathrm{d}s + \sum_{i=1}^{n} \beta_i B_i(t)$$

for $t \geq T$, where β_i is a constant, $1 \leq i \leq n$, then $\langle x(t)\rangle^* \leq \frac{\lambda}{\lambda_0}$ a.s.

(ii) If there are positive constants λ_0, T and $\lambda \geq 0$ such that

$$\ln x(t) \geq \lambda t - \lambda_0 \int_0^t x(t)\mathrm{d}s + \sum_{i=1}^{n} \beta_i B_i(t)$$

for $t \geq T$, where β_i is a constant, $1 \leq i \leq n$, then $\langle x(t)\rangle_* \geq \frac{\lambda}{\lambda_0}$ a.s.

Lemma 2.10 ([12]). *For the logistic equation:*

$$\mathrm{d}x(t) = x(t)\left(a(t) - b(t)x(t)\right)\mathrm{d}t + \alpha(t)x(t)\mathrm{d}B(t)$$

with initial value $x(0) = x_0$. Assume $a(t)$, $b(t)$, $\alpha(t)$ are continuous bounded functions on $\mathbb{R}_+ = [0, +\infty)$ and $a(t) > 0$, $b(t) > 0$, then the equation has an explicit solution of the form:

$$x(t) = \frac{\mathrm{e}^{\int_0^t \left[a(s) - \frac{1}{2}\alpha^2(s)\right]\mathrm{d}s + \int_0^t \alpha(s)\mathrm{d}B(s)}}{\frac{1}{x_0} + \int_0^t b(s)\mathrm{e}^{\int_0^s [a(\tau) - \frac{1}{2}\alpha^2(\tau)]\mathrm{d}\tau + \int_0^s \alpha(\tau)\mathrm{d}B(\tau)}\mathrm{d}s}.$$

3. Existence, uniqueness and stochastically ultimate boundedness

In this section, taking into account the biological meanings, we are only interested in the positive solutions. Here, by making the change of variables and comparison theorem for stochastic equations, we show the existence and uniqueness of the positive solution. Moreover, by using Chebyshev inequality, we obtain the stochastically ultimate boundedness of the system (1.1).

Here is our first result.

Lemma 3.1. *There is a unique positive local solution $(x(t), y(t))$ for $t \in [0, \tau_\varrho)$ to system (1.1) a.s. for the initial value $x_0 > 0$, $y_0 > 0$.*

Proof. Consider the equation

$$\begin{cases} \mathrm{d}u(\mathrm{t}) = \left[a(t) - \dfrac{\sigma_1^2(t)}{2} - b(t)e^{u(t)} - \dfrac{c(t)e^{v(t)}}{(1+\alpha(t)e^{u(t)})(1+\beta(t)e^{v(t)})}\right]\mathrm{d}t \\ \qquad\qquad + \sigma_1(t)\mathrm{d}B_1(\mathrm{t}), \\ \mathrm{d}v(\mathrm{t}) = \left[r(t) - \dfrac{\sigma_2^2(t)}{2} - h(t)e^{v(t)} - \dfrac{f(t)e^{v(t)}}{e^{u(t)} + m(t)}\right]\mathrm{d}t + \sigma_2(\mathrm{t})\mathrm{d}B_2(\mathrm{t}) \end{cases} \tag{3.1}$$

on $t \geq 0$ with initial value $u(0) = \ln x_0$, $v(0) = \ln y_0$. Obviously, the coefficients of system (3.1) satisfy the local Lipschitz condition, then there is unique local solution $(u(t), v(t))$ on $t \in [0, \tau_\varrho)$, where τ_ϱ is the explosion time. By Itô's formula, it is easy to see $x(t) = e^{u(t)}$, $y(t) = e^{v(t)}$ is the unique positive local solution to system (1.1) with initial value $x_0 > 0$, $y_0 > 0$. $\square$

Noting that Lemma 3.1 only tells us the unique positive solution is local. Next, we will show a more important result, that is, the unique positive solution is global. It suffices to prove $\tau_\varrho = \infty$. Since the solution is positive, we have

$$\mathrm{d}x(t) \leq x(t)\left(a(t) - b(t)x(t)\right)\mathrm{d}t + \sigma_1(t)x(t)\mathrm{d}B_1(t).$$

Let $\Phi(t)$ be the unique solution of equation

$$\begin{cases} \mathrm{d}\Phi(t) = \Phi(t)\left(a(t) - b(t)\Phi(t)\right)\mathrm{d}t + \sigma_1(t)\Phi(t)\mathrm{d}B_1(t), \\ \Phi(0) = x_0. \end{cases}$$

Then, by Lemma 2.10,

$$\Phi(t) = \frac{e^{\int_0^t \left[a(s) - \frac{\sigma_1^2(s)}{2}\right]\mathrm{d}s + \int_0^t \sigma_1(s)\mathrm{d}B_1(s)}}{\frac{1}{x_0} + \int_0^t b(s) e^{\int_0^s \left[a(\tau) - \frac{\sigma_1^2(\tau)}{2}\right]\mathrm{d}\tau + \int_0^s \sigma_1(\tau)\mathrm{d}B_1(\tau)}\mathrm{d}s}.$$

According to the comparison theorem of stochastic equations, we know

$$x(t) \le \Phi(t),\ t \in [0, \tau_\varrho),\ a.s. \tag{3.2}$$

From the first equation of system (1.1), we have

$$\mathrm{d}x(t) \ge x(t)\left[a(t) - \frac{c(t)}{\beta(t)} - b(t)x(t)\right]\mathrm{d}t + \sigma_1(t)x(t)\mathrm{d}B_1(t).$$

By the comparison theorem of stochastic equations, we get

$$x(t) \ge \phi(t),\ t \in [0, \tau_\varrho),\ a.s.$$

where $\phi(t)$ is the unique solution of equation

$$\begin{cases} \mathrm{d}\phi(t) = \phi(t)\left[a(t) - \dfrac{c(t)}{\beta(t)} - b(t)\phi(t)\right]\mathrm{d}t + \sigma_1(t)\phi(t)\mathrm{d}B_1(t), \\ \phi(0) = x_0. \end{cases}$$

It follows from the second equation of system (1.1) that

$$\mathrm{d}y(t) \ge y(t)\left[r(t) - \frac{m(t)h(t) + f(t)}{m(t)}y(t)\right]\mathrm{d}t + \sigma_2(t)y(t)\mathrm{d}B_2(t).$$

Then by the comparison theorem of stochastic equations, we derive

$$y(t) \ge \psi(t),\ t \in [0, \tau_\varrho),\ a.s.$$

where $\psi(t)$ is the unique solution of equation

$$\begin{cases} \mathrm{d}\psi(t) = \psi(t)\left[r(t) - \dfrac{m(t)h(t) + f(t)}{m(t)}\psi(t)\right]\mathrm{d}t + \sigma_2(t)\psi(t)\mathrm{d}B_2(t), \\ \psi(0) = y_0. \end{cases} \tag{3.3}$$

By virtue of (3.2) and the second equation of system (1.1), we get

$$\mathrm{d}y(t) \le y(t)\left[r(t) - \frac{f(t)y(t)}{m(t) + \Phi(t)}\right]\mathrm{d}t + \sigma_2(t)y(t)\mathrm{d}B_2(t).$$

Therefore, by the comparison theorem of stochastic equations, we also have

$$y(t) \le \Psi(t),\ t \in [0, \tau_\varrho),\ a.s.$$

Here, $\Psi(t)$ is the unique solution of equation

$$\begin{cases} \mathrm{d}\Psi(t) = \Psi(t)\left[r(t) - \dfrac{f(t)}{m(t)+\Phi(t)}\Psi(t)\right]\mathrm{d}t + \sigma_2(t)\Psi(t)\mathrm{d}B_2(t), \\ \Psi(0) = y_0. \end{cases}$$

Noting that $\phi(t), \psi(t), \Phi(t), \Psi(t)$ are existence on $t \geq 0$. That is to say $x(t)$, $y(t)$ are existence on $t \geq 0$. Hence $\tau_\varrho = \infty$. Therefore, we get the following theorem

Theorem 3.1. *There is a unique positive solution $(x(t), y(t))$ of system (1.1) a.s. for any initial value $x_0 > 0$, $y_0 > 0$. Moreover, there exist $\phi(t)$, $\psi(t)$, $\Phi(t)$, $\Psi(t)$ defined as above such that*

$$\phi(t) \leq x(t) \leq \Phi(t), \quad \psi(t) \leq y(t) \leq \Psi(t), t \geq 0, \quad a.s.$$

Theorem 3.2. *The solution of system (1.1) is stochastically ultimately bounded for any initial value $X_0 = (x_0, y_0) \in \mathbb{R}^2_+$.*

Proof. Now we need to show that for any $\varepsilon \in (0,1)$, there exists a positive constant $\delta = \delta(\varepsilon)$ such that for any initial value $X_0 = (x_0, y_0) \in \mathbb{R}^2_+$, the solution $X(t)$ of system (1.1) satisfies

$$\limsup_{t\to+\infty} \mathbb{P}\{|X(t)| > \delta\} < \varepsilon.$$

Let $V_1(x(t)) = x^p(t)$, $V_2(y(t)) = y^p(t)$ for $(x(t), y(t)) \in \mathbb{R}^2_+$ and $p > 0$. Then, we obtain

$$\begin{aligned} \mathrm{d}(x^p(t)) &= px^{p-1}(t)\mathrm{d}x(t) + \frac{p(p-1)}{2}x^{p-2}(t)(\mathrm{d}x(t))^2 \\ &= px^{p-1}(t)x(t)\left[a(t) - b(t)x(t) - \frac{c(t)y(t)}{(1+\alpha(t)x(t))(1+\beta(t)y(t))}\right]\mathrm{d}t \\ &\quad + \frac{p(p-1)}{2}x^p(t)\sigma_1^2(t)\mathrm{d}t + px^{p-1}(t)x(t)\sigma_1(t)\mathrm{d}B_1(t) \\ &= px^p(t)\left[a(t) + \frac{p-1}{2}\sigma_1^2(t) - \frac{c(t)y(t)}{(1+\alpha(t)x(t))(1+\beta(t)y(t))}\right]\mathrm{d}t \\ &\quad - pb(t)x^{p+1}(t)\mathrm{d}t + px^p(t)\sigma_1(t)\mathrm{d}B_1(t) \\ &\leq px^p(t)\left[a(t) + \frac{p}{2}\sigma_1^2(t)\right]\mathrm{d}t - pb(t)x^{p+1}(t)\mathrm{d}t + px^p(t)\sigma_1(t)\mathrm{d}B_1(t) \\ &\leq px^p(t)\left[a^u + \frac{p}{2}(\sigma_1^u)^2\right]\mathrm{d}t - pb^\ell x^{p+1}(t)\mathrm{d}t + px^p(t)\sigma_1(t)\mathrm{d}B_1(t). \end{aligned}$$

Integrating it from 0 to t and taking expectation, we have

$$\mathbb{E}\left(x^p(t)\right) \leq x_0^p + \int_0^t p\left(a^u + \frac{p}{2}(\sigma_1^u)^2\right)\mathbb{E}\left(x^p(s)\right)\mathrm{d}s - \int_0^t pb^\ell\mathbb{E}\left(x^{p+1}(s)\right)\mathrm{d}s.$$

Thus,

$$\begin{aligned} \frac{\mathrm{d}\mathbb{E}(x^p(t))}{\mathrm{d}t} &\leq p\left[a^u + \frac{p}{2}(\sigma_1^u)^2\right]\mathbb{E}\left(x^p(t)\right) - pb^\ell\mathbb{E}\left(x^{p+1}(t)\right) \\ &\leq p\left[a^u + \frac{p}{2}(\sigma_1^u)^2\right]\mathbb{E}\left(x^p(t)\right) - pb^\ell[\mathbb{E}\left(x^p(t)\right)]^{1+\frac{1}{p}}. \end{aligned}$$

By using the stochastic comparison theorem, we know

$$\mathbb{E}(x^p(t)) \leq \left[\frac{a^u + \frac{p}{2}(\sigma_1^u)^2}{b^\ell}\right]^p := K_1(p). \tag{3.4}$$

In the same way, we obtain

$$\begin{aligned}
\mathrm{d}(y^p(t)) &= py^{p-1}(t)\mathrm{d}y(t) + \frac{p(p-1)}{2}y^{p-2}(t)(\mathrm{d}y(t))^2 \\
&= py^{p-1}(t)y(t)\left[r(t) - h(t)y(t) - \frac{f(t)y(t)}{m(t)+x(t)}\right]\mathrm{d}t + \frac{p(p-1)}{2}y^p(t)\sigma_2^2(t)\mathrm{d}t \\
&\quad + py^{p-1}(t)y(t)\sigma_2(t)\mathrm{d}B_2(t) \\
&= py^p(t)\left[r(t) + \frac{p-1}{2}\sigma_2^2(t) - \frac{f(t)y(t)}{m(t)+x(t))}\right]\mathrm{d}t - ph(t)y^{p+1}(t)\mathrm{d}t \\
&\quad + py^p(t)\sigma_2(t)\mathrm{d}B_2(t) \\
&\leq py^p(t)\left[r(t) + \frac{p}{2}\sigma_2^2(t)\right]\mathrm{d}t - ph(t)y^{p+1}(t)\mathrm{d}t + py^p(t)\sigma_2(t)\mathrm{d}B_2(t) \\
&\leq py^p(t)\left[r^u + \frac{p}{2}(\sigma_2^u)^2\right]\mathrm{d}t - ph^\ell y^{p+1}(t)\mathrm{d}t + py^p(t)\sigma_2(t)\mathrm{d}B_2(t).
\end{aligned}$$

Integrating it from 0 to t and taking expectation, we attain

$$\mathbb{E}\left(y^p(t)\right) \leq y_0^p + \int_0^t p\left[r^u + \frac{p}{2}(\sigma_2^u)^2\right]\mathbb{E}\left(y^p(s)\right)\mathrm{d}s - \int_0^t ph^\ell\mathbb{E}\left(y^{p+1}(s)\right)\mathrm{d}s.$$

Therefore,

$$\begin{aligned}
\frac{\mathrm{d}\mathbb{E}\left(y^p(t)\right)}{\mathrm{d}t} &\leq p\left[r^u + \frac{p}{2}(\sigma_2^u)^2\right]\mathbb{E}\left(y^p(t)\right) - ph^\ell\mathbb{E}\left(y^{p+1}(t)\right) \\
&\leq p\left[r^u + \frac{p}{2}(\sigma_2^u)^2\right]\mathbb{E}(y^p(t)) - ph^\ell[\mathbb{E}(y^p(t))]^{1+\frac{1}{p}}.
\end{aligned}$$

By employing the stochastic comparison theorem, we derive

$$\mathbb{E}(y^p(t)) \leq \left[\frac{r^u + \frac{p}{2}(\sigma_2^u)^2}{h^\ell}\right]^p := K_2(p). \tag{3.5}$$

Since

$$\begin{aligned}
\left(\sqrt{x^2(t)+y^2(t)}\right)^p &\leq \left(\sqrt{2[x^2(t)\vee y^2(t)]}\right)^p \\
&\leq 2^{\frac{p}{2}}\left(\sqrt{x^2(t)}^p + \sqrt{y^2(t)}^p\right) \\
&= 2^{\frac{p}{2}}\left(x^p(t) + y^p(t)\right),
\end{aligned}$$

then

$$|X(t)|^p \leq 2^{\frac{p}{2}}(x^p(t) + y^p(t)), \qquad X(t) = (x(t), y(t)).$$

Consequently,

$$\mathbb{E}|X(t)|^p \leq 2^{\frac{p}{2}}[\mathbb{E}(x^p(t)) + \mathbb{E}(y^p(t))] \leq 2^{\frac{p}{2}}\left(K_1(p) + K_2(p)\right) := K(p).$$

By the Chebyshev inequality, the proof is completed. $\square$

4. Persistence and extinction

To prove the persistence and extinction of the system, we first state an important result which can be used in the sequel.

Lemma 4.1. *If $(x(t), y(t))$ is the solution of system (1.1), then*

$$\limsup_{t\to+\infty} \frac{\ln x(t)}{t} \le 0, \quad \limsup_{t\to+\infty} \frac{\ln y(t)}{t} \le 0.$$

Proof. Define

$$V(X(t)) = V(x(t), y(t)) = x(t) + y(t).$$

By Itô's formula, we get

$$\begin{aligned} \mathrm{d}V(X(t)) &= \mathrm{d}x(t) + \mathrm{d}y(t) \le a(t)x(t) + r(t)y(t) + \sigma_1(t)x(t)\mathrm{d}B_1(t) + \sigma_2(t)y(t)\mathrm{d}B_2(t) \\ &\le \max\{a^u, r^u\}(x(t) + y(t)) + \sigma_1(t)x(t)\mathrm{d}B_1(t) + \sigma_2(t)y(t)\mathrm{d}B_2(t). \end{aligned}$$

Integrating it from t to r and taking expectation, yields

$$\begin{aligned} \mathbb{E}\left[\sup_{t\le r\le t+1} V(X(r))\right] &\le \mathbb{E}V\left(X(t)\right) + \max\{a^u, r^u\}\int_t^{t+1} \mathbb{E}V(X(s))\mathrm{d}s \\ &\quad + \mathbb{E}\left[\sup_{t\le r\le t+1}\int_t^r (\sigma_1(s)x(s)\mathrm{d}B_1(s) + \sigma_2(s)y(s)\mathrm{d}B_2(s))\right]. \end{aligned} \tag{4.1}$$

By (3.4) and (3.5), we obtain

$$\begin{aligned} \limsup_{t\to+\infty} \mathbb{E}\left(V(X(t))\right) &= \limsup_{t\to+\infty} \mathbb{E}(x(t) + y(t)) \\ &\le \limsup_{t\to+\infty} \mathbb{E}(x(t)) + \limsup_{t\to+\infty} \mathbb{E}(y(t)) \\ &\le K_1(1) + K_2(1) := 2^{-\frac{1}{2}}K(1), \end{aligned} \tag{4.2}$$

and

$$\limsup_{t\to+\infty} \mathbb{E}\int_t^{t+1} |X(s)|^2\mathrm{d}s \le K(2).$$

By the well-known Burkholder-Davis-Gundy inequality (Lemma 2.7) and Hölder inequality (Lemma 2.5), we attain

$$\begin{aligned} &\mathbb{E}\left[\sup_{t\le r\le t+1}\int_t^r (\sigma_1(s)x(s)\mathrm{d}B_1(s) + \sigma_2(s)y(s)\mathrm{d}B_2(s))\right] \\ &\le \max\{\sigma_1(s), \sigma_2(s)\}\mathbb{E}\left[\sup_{t\le r\le t+1}\int_t^{t+1} [x(s)\mathrm{d}B_1(s) + y(s)\mathrm{d}B_2(s)]\right] \\ &\le \max\{\sigma_1^u, \sigma_2^u\}\mathbb{E}\left[\sup_{t\le r\le t+1}\int_t^{t+1} X(s)\mathrm{d}B(s)\right] \\ &\le 3\max\{\sigma_1^u, \sigma_2^u\}\mathbb{E}\left[\int_t^{t+1} X^2(s)\mathrm{d}s\right]^{\frac{1}{2}} \\ &\le 3\max\{\sigma_1^u, \sigma_2^u\}\left[\mathbb{E}\int_t^{t+1} X^2(s)\mathrm{d}s\right]^{\frac{1}{2}} \\ &\le 3\max\{\sigma_1^u, \sigma_2^u\}\left(K(2)\right)^{\frac{1}{2}}. \end{aligned}$$

Therefore,

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\leq r\leq t+1} V(X(r))\right]\\&\leq \mathbb{E}V(X(t))+\max\{a^u,r^u\}\int_t^{t+1}\mathbb{E}V(X(s))\mathrm{d}s+3\max\{\sigma_1^u,\sigma_2^u\}\left(K(2)\right)^{\frac{1}{2}},\end{aligned}$$

which combing with (4.2) leads to

$$\begin{aligned}&\limsup_{t\to+\infty}\mathbb{E}\left[\sup_{t\leq r\leq t+1} V(X(r))\right]\\&\leq (1+\max\{a^u,r^u\})K(1)2^{-\frac{1}{2}}+3\max\{\sigma_1^u,\sigma_2^u\}(K(2))^{\frac{1}{2}}.\end{aligned}$$

Since $|X(t)|\leq x(t)+y(t)$, then

$$\limsup_{t\to+\infty}\mathbb{E}\left[\sup_{t\leq r\leq t+1}|X(t)|\right]\leq (1+\max\{a^u,r^u\})K(1)2^{-\frac{1}{2}}+3\max\{\sigma_1^u,\sigma_2^u\}(K(2))^{\frac{1}{2}}. \tag{4.3}$$

From (4.3), there exists a positive constant $\bar{K}$ such that

$$\mathbb{E}\left[\sup_{k\leq t\leq k+1}|X(t)|\right]\leq \bar{K},\quad k=1,2,...$$

Let $\varepsilon>0$ be arbitrary. Then according to the Chebyshev inequality, we have

$$\mathbb{P}\left\{\sup_{k\leq t\leq k+1}|X(t)|>k^{1+\varepsilon}\right\}\leq \frac{\bar{K}}{k^{1+\varepsilon}},\quad k=1,2,...$$

Using the Borel-Cantelli lemma, we know that for almost all $\omega\in\Omega$,

$$\sup_{k\leq t\leq k+1}|X(t)|\leq k^{1+\varepsilon} \tag{4.4}$$

holds for all but finitely many k. Hence, there exists a $k_0(\omega)$, for almost all $\omega\in\Omega$, (4.4) holds whenever $k\geq k_0$. Consequently, for almost all $\omega\in\Omega$, if $k\geq k_0$ and $k\leq t\leq k+1$,

$$\frac{\ln|X(t)|}{\ln t}\leq\frac{(1+\varepsilon)\ln k}{\ln k}=1+\varepsilon.$$

Therefore

$$\limsup_{t\to+\infty}\frac{\ln|X(t)|}{\ln t}\leq 1+\varepsilon.$$

Letting $\varepsilon\to 0$, we get

$$\limsup_{t\to+\infty}\frac{\ln|X(t)|}{\ln t}\leq 1.$$

Consequently,

$$\limsup_{t\to+\infty}\frac{\ln x(t)}{\ln t}\leq 1,\qquad \limsup_{t\to+\infty}\frac{\ln y(t)}{\ln t}\leq 1.$$

Thus,

$$\limsup_{t\to+\infty}\frac{\ln x(t)}{t}=\limsup_{t\to+\infty}\frac{\ln x(t)}{\ln t}\times\limsup_{t\to+\infty}\frac{\ln t}{t}\leq\limsup_{t\to+\infty}\frac{\ln t}{t}=0.$$

In the same way, we get
$$\limsup_{t\to+\infty}\frac{\ln y(t)}{t}\le 0.$$
This completes the proof. □

Theorem 4.1. *For the prey population $x(t)$ in system (1.1), the following conclusions hold*

(1) If $\langle a(t)-\frac{\sigma_1^2(t)}{2}\rangle^*<0$, then $\lim\limits_{t\to+\infty}x(t)=0$.

(2) If $\langle a(t)-\frac{\sigma_1^2(t)}{2}\rangle^*=0$, then $\langle x(t)\rangle^*=0$.

(3) If $\langle a(t)-\frac{\sigma_1^2(t)}{2}\rangle^*>0$, then $\langle x(t)\rangle^*\le M_x:=\frac{\langle a(t)-\frac{\sigma_1^2(t)}{2}\rangle^*}{b^\ell}$.

(4) If $\langle a(t)-\frac{\sigma_1^2(t)}{2}\rangle^*>0$, $\langle r(t)-\frac{\sigma_2^2(t)}{2}\rangle^*<(h^\ell+\frac{f^\ell}{m^u})\langle\psi(t)\rangle^*$, then $\langle x(t)\rangle^*>0$. Here, $\psi(t)$ is the solution of equation (3.3).

(5) If $\langle a(t)-\frac{\sigma_1^2(t)}{2}\rangle_*-\langle\frac{c(t)}{\beta(t)}\rangle^*>0$, then $\langle x(t)\rangle_*>0$.

Proof. (1) For the system (1.1), by using Itô's formula, we have
$$\begin{cases}\mathrm{d}\ln x(t)=\Big(a(t)-b(t)x(t)-\dfrac{c(t)y(t)}{(1+\alpha(t)x(t))(1+\beta(t)y(t))}-\dfrac{\sigma_1^2(t)}{2}\Big)\mathrm{d}t+\sigma_1(t)\mathrm{d}B_1(t),\\ \mathrm{d}\ln y(t)=\Big(r(t)-h(t)y(t)-\dfrac{f(t)y(t)}{m(t)+x(t)}-\dfrac{\sigma_2^2(t)}{2}\Big)\mathrm{d}t+\sigma_2(t)\mathrm{d}B_2(t).\end{cases}\tag{4.5}$$
Integrating the first equation of (4.5), we have
$$\begin{aligned}\frac{\ln x(t)-\ln x_0}{t}&=\Big\langle a(t)-\frac{\sigma_1^2(t)}{2}\Big\rangle-\langle b(t)x(t)\rangle\\&\quad-\Big\langle\frac{c(t)y(t)}{(1+\alpha(t)x(t))(1+\beta(t)y(t))}\Big\rangle+\frac{1}{t}\int_0^t\sigma_1(t)\mathrm{d}B_1(t).\end{aligned}\tag{4.6}$$
Let
$$M_1(t)=\int_0^t\sigma_1(s)\mathrm{d}B_1(s),\quad M_2(t)=\int_0^t\sigma_2(s)\mathrm{d}B_2(s).$$
Then, $M_i(t)\,(i=1,2)$ are local martingales, and the quadratic variations satisfy
$$\langle M_1,M_1\rangle_t=\int_0^t\sigma_1^2(s)\mathrm{d}s\le(\sigma_1^u)^2t$$
and
$$\langle M_2,M_2\rangle_t=\int_0^t\sigma_2^2(s)\mathrm{d}s\le(\sigma_2^u)^2t.$$
According to the strong law of large numbers for martingales, we have
$$\limsup_{t\to+\infty}\frac{M_i(t)}{t}=0\ \ a.s.\tag{4.7}$$
For (4.6), it follows from (4.7) and the property of the superior limit
$$\Big(\frac{\ln x(t)-\ln x_0}{t}\Big)^*\le\Big\langle a(t)-\frac{\sigma_1^2(t)}{2}\Big\rangle^*<0.$$

Then

$$\lim_{t\to\infty} x(t) = 0.$$

(2) By virtue of the superior limit and (4.7), we can show that for any $\varepsilon > 0$, there exists $T > 0$ such that

$$\left\langle a(t) - \frac{\sigma_1^2(t)}{2}\right\rangle < \left\langle a(t) - \frac{\sigma_1^2(t)}{2}\right\rangle^* + \frac{\varepsilon}{2}, \quad \text{and} \quad \frac{M_1(t)}{t} < \frac{\varepsilon}{2}, \quad \text{for all} \quad t > T.$$

It follows from the first equation of (4.5), we achieve

$$\frac{\ln x(t) - \ln x_0}{t} \le \left\langle a(t) - \frac{\sigma_1^2(t)}{2}\right\rangle^* + \varepsilon - b^\ell \langle x(t)\rangle = \varepsilon - b^\ell \langle x(t)\rangle.$$

According to Lemma 2.9, we get

$$\langle x(t)\rangle^* \le \frac{\varepsilon}{b^\ell}.$$

By the arbitrariness of ε, the desired conclusion is obtained.

(3) From the first equation of (4.5), we have

$$\mathrm{d}\ln x(t) \le \left(a(t) - b(t)x(t) - \frac{\sigma_1^2(t)}{2}\right)\mathrm{d}t + \sigma_1(t)\mathrm{d}B_1(t).$$

Thus

$$\frac{\ln x(t) - \ln x_0}{t} \le \left\langle a(t) - \frac{\sigma_1^2(t)}{2}\right\rangle - b^\ell\langle x(t)\rangle + \frac{M_1(t)}{t}.$$

By virtue of the superior limit and (4.7), for any given positive number $\varepsilon > 0$, there exists $T_1 > 0$ satisfying

$$\langle a(t) - \frac{\sigma_1^2(t)}{2}\rangle < \left\langle a(t) - \frac{\sigma_1^2(t)}{2}\right\rangle^* + \frac{\varepsilon}{2}, \quad \text{and} \quad \frac{M_1(t)}{t} < \frac{\varepsilon}{2}, \quad \text{for all} \quad t > T_1.$$

Therefore

$$\frac{\ln x(t) - \ln x_0}{t} \le \left\langle a(t) - \frac{\sigma_1^2(t)}{2}\right\rangle^* - b^\ell\langle x(t)\rangle + \varepsilon.$$

According to Lemma 2.9 and the arbitrariness of ε, we get

$$\langle x(t)\rangle^* \le \frac{\langle a(t) - \frac{\sigma_1^2(t)}{2}\rangle^*}{b^\ell} := M_x.$$

(4) Here, we note $\psi(t)$ is the solution of (3.3). According to the comparison theorem of stochastic equations, we know

$$y(t) \ge \psi(t), \ t \in [0, \tau_\varrho), \quad a.s.$$

By (4.7) and Lemma 4.1, we have

$$\begin{aligned} & b^u\langle x(t)\rangle^* + c^u\langle y(t)\rangle^* \\ \ge & \left(\frac{\ln x(t)}{t}\right)^* + \langle b(t)x(t)\rangle^* + \left\langle \frac{c(t)y(t)}{(1+\alpha(t)x(t))(1+\beta(t)y(t))}\right\rangle^* - \left\langle \frac{M_1(t)}{t}\right\rangle_* \\ \ge & \left\langle a(t) - \frac{\sigma_1^2(t)}{2}\right\rangle^* > 0. \end{aligned} \tag{4.8}$$

There must exist $\langle x(t)\rangle^* > 0$ a.s. If not, for arbitrary $\nu \in \{\langle x(t,\nu)\rangle^* = 0\}$. By (4.8) we have $\langle y(t,\nu)\rangle^* > 0$. Meanwhile, from the second equation of system (4.5), we get

$$\begin{aligned}\Big(\frac{\ln y(t) - \ln y_0}{t}\Big)^* &= \Big\langle r(t) - \frac{\sigma_2^2(t)}{2}\Big\rangle^* - \langle h(t)y(t)\rangle^* - \Big\langle \frac{f(t)y(t)}{x(t)+m(t)}\Big\rangle^* \\ &\le \Big\langle r(t) - \frac{\sigma_2^2(t)}{2}\Big\rangle^* - h^\ell\langle y(t)\rangle^* - \frac{f^\ell}{m^u}\langle y(t)\rangle^* \\ &= \Big\langle r(t) - \frac{\sigma_2^2(t)}{2}\Big\rangle^* - \Big(h^\ell + \frac{f^\ell}{m^u}\Big)\langle y(t)\rangle^* \\ &\le \Big\langle r(t) - \frac{\sigma_2^2(t)}{2}\Big\rangle^* - \Big(h^\ell + \frac{f^\ell}{m^u}\Big)\langle \psi(t)\rangle^* \\ &< 0.\end{aligned}$$

Then, $\lim\limits_{t\to\infty} y(t,\nu) = 0$, which contradicts with $\langle y(t,\nu)\rangle^* > 0$. The proof is completed.

(5) By the condition $\langle a(t) - \frac{\sigma_1^2(t)}{2}\rangle_* - \langle \frac{c(t)}{\beta(t)}\rangle^* > 0$, there exists a sufficiently small $\varepsilon > 0$ such that

$$\Big\langle a(t) - \frac{\sigma_1^2(t)}{2}\Big\rangle_* - \Big\langle \frac{c(t)}{\beta(t)}\Big\rangle^* - \varepsilon > 0.$$

By (4.7), for this $\varepsilon > 0$, there exists $T_2 > 0$, such that for all $t > T_2$,

$$\Big\langle a(t) - \frac{\sigma_1^2(t)}{2}\Big\rangle > \Big\langle a(t) - \frac{\sigma_1^2(t)}{2}\Big\rangle_* - \frac{\varepsilon}{3}, \quad \Big\langle \frac{c(t)}{\beta(t)}\Big\rangle < \Big\langle \frac{c(t)}{\beta(t)}\Big\rangle^* + \frac{\varepsilon}{3}, \quad \frac{M_1(t)}{t} > -\frac{\varepsilon}{3}.$$

Then,

$$\begin{aligned}\frac{\ln x(t) - \ln x_0}{t} &\ge \Big\langle a(t) - \frac{\sigma_1^2(t)}{2}\Big\rangle - b^u\langle x(t)\rangle - \Big\langle \frac{c(t)}{\beta(t)}\Big\rangle + \frac{M_1(t)}{t} \\ &\ge \Big\langle a(t) - \frac{\sigma_1^2(t)}{2}\Big\rangle_* - \frac{\varepsilon}{3} - \Big\langle \frac{c(t)}{\beta(t)}\Big\rangle^* - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} - b^u\langle x(t)\rangle \\ &= \Big\langle a(t) - \frac{\sigma_1^2(t)}{2}\Big\rangle_* - \Big\langle \frac{c(t)}{\beta(t)}\Big\rangle^* - \varepsilon - b^u\langle x(t)\rangle.\end{aligned}$$

According to Lemma 2.9 and the arbitrariness of ε, we have

$$\langle x(t)\rangle_* \ge \frac{\langle a(t) - \frac{\sigma_1^2(t)}{2}\rangle_* - \langle \frac{c(t)}{\beta(t)}\rangle^*}{b^u} > 0.$$

□

Theorem 4.2. *For the predator population $y(t)$ in system (1.1), the following conclusions hold:*

(1) If $\langle r(t) - \frac{\sigma_2^2(t)}{2}\rangle^* < 0$, then $\lim\limits_{t\to+\infty} y(t) = 0$.

(2) If $\langle r(t) - \frac{\sigma_2^2(t)}{2}\rangle^* = 0$, then $\langle y(t)\rangle^* = 0$.

(3) If $\langle r(t) - \frac{\sigma_2^2(t)}{2}\rangle^* > 0$, then $\langle y(t)\rangle^* \le M_y := \frac{\langle r(t) - \frac{\sigma_2^2(t)}{2}\rangle^*}{h^\ell}$.

(4) If $\langle r(t)-\frac{\sigma_2^2(t)}{2}\rangle^*-\langle\frac{f(t)\tilde{y}(t)}{m(t)+\tilde{x}(t)}\rangle^*>0$*, then* $\langle y(t)\rangle^*>0$*. Here* $(\tilde{x}(t),\tilde{y}(t))$ *is the solution of the following equation*

$$\begin{cases} \mathrm{d}x(t)=x(t)\left(a(t)-b(t)x(t)\right)\mathrm{d}t+\sigma_1(t)x(t)\mathrm{d}B_1(t), \\ \mathrm{d}y(t)=y(t)\left(r(t)-h(t)y(t)\right)\mathrm{d}t+\sigma_2(t)y(t)\mathrm{d}B_2(t). \end{cases} \tag{4.9}$$

(5) If $\langle r(t)-\frac{\sigma_2^2(t)}{2}\rangle_*>0$*, then* $\langle y(t)\rangle_*>0$*.*

Proof. (1) For the second equation of (4.5), we integrate it from 0 to t and get

$$\begin{aligned} \frac{\ln y(t)-\ln y_0}{t} &= \left\langle r(t)-\frac{\sigma_2^2(t)}{2}\right\rangle-\langle h(t)y(t)\rangle \\ &\quad -\left\langle\frac{f(t)y(t)}{m(t)+x(t)}\right\rangle+\frac{1}{t}\int_0^t\sigma_2(t)\mathrm{d}B_2(t), \end{aligned} \tag{4.10}$$

which combing with (4.7) and the property of the superior limit produces

$$\left(\frac{\ln y(t)-\ln y_0}{t}\right)^*\le\left\langle r(t)-\frac{\sigma_2^2(t)}{2}\right\rangle^*<0.$$

Then,

$$\lim_{t\to\infty}y(t)=0.$$

(2) By virtue of the superior limit and (4.7), we can show that for any $\varepsilon>0$, there exists $T>0$, such that for all $t>T$

$$\left\langle r(t)-\frac{\sigma_2^2(t)}{2}\right\rangle<\left\langle r(t)-\frac{\sigma_2^2(t)}{2}\right\rangle^*+\frac{\varepsilon}{2},\quad\text{and}\quad\frac{M_2(t)}{t}<\frac{\varepsilon}{2}.$$

From (4.10), we know

$$\frac{\ln y(t)-\ln y_0}{t}\le\left\langle r(t)-\frac{\sigma_2^2(t)}{2}\right\rangle^*+\varepsilon-h^\ell\langle y(t)\rangle=\varepsilon-h^\ell\langle y(t)\rangle.$$

According to Lemma 2.9, we get

$$\langle y(t)\rangle^*\le\frac{\varepsilon}{h^\ell}.$$

By the arbitrariness of ε, the desired conclusion is obtained.
(3) From the second equation of (4.5), we achieve

$$\mathrm{d}\ln y(t)\le\left(r(t)-h(t)y(t)-\frac{\sigma_2^2(t)}{2}\right)\mathrm{d}t+\sigma_2(t)\mathrm{d}B_2(t).$$

Thereby,

$$\frac{\ln y(t)-\ln y_0}{t}\le\left\langle r(t)-\frac{\sigma_2^2(t)}{2}\right\rangle-h^\ell\langle y(t)\rangle+\frac{M_2(t)}{t}.$$

By exploiting the superior limit and (4.7), for any given positive number $\varepsilon>0$, there exists $T_1>0$ satisfying, for all $t>T_1$,

$$\left\langle r(t)-\frac{\sigma_2^2(t)}{2}\right\rangle<\left\langle r(t)-\frac{\sigma_2^2(t)}{2}\right\rangle^*+\frac{\varepsilon}{2},\quad\text{and}\quad\frac{M_2(t)}{t}<\frac{\varepsilon}{2}.$$

Therefore,

$$\frac{\ln y(t)-\ln y_0}{t}\le\Big\langle r(t)-\frac{\sigma_2^2(t)}{2}\Big\rangle^*+\varepsilon-h^\ell\langle y(t)\rangle.$$

According to Lemma 2.9 and the arbitrariness of ε, we have

$$\langle y(t)\rangle^*\le\frac{\langle r(t)-\frac{\sigma_2^2(t)}{2}\rangle^*}{h^\ell}:=M_y.$$

(4) Here, we want to prove that $\langle y(t)\rangle^*>0$ a.s. If not, for arbitrary $\varepsilon_1>0$, there exists a solution $(x(t),y(t))$ with initial value $(x_0,y_0)\in\mathbb{R}_+^2$ such that $P\{\langle y(t)\rangle^*<\varepsilon_1\}>0$. Let ε_1 be sufficiently small such that

$$\Big\langle r(t)-\frac{\sigma_2^2(t)}{2}\Big\rangle^*-\Big\langle\frac{f(t)\tilde{y}(t)}{m(t)+\tilde{x}(t)}\Big\rangle^*>\left[h^u+\frac{f^ub^u\langle r(t)-\frac{\sigma_2^2(t)}{2}\rangle^*}{(m^\ell)^2b^\ell h^\ell}\right]\varepsilon_1.$$

Here $(\tilde{x}(t),\tilde{y}(t))$ is the solution of (4.9) and $(x(t),y(t))$ is the solution of (1.1). Then, by the comparison theorem, we have

$$x(t)\le\tilde{x}(t),\quad y(t)\le\tilde{y}(t).$$

Since

$$\begin{aligned}\frac{\ln y(t)-\ln y_0}{t}&=\Big\langle r(t)-\frac{\sigma_2^2(t)}{2}\Big\rangle-\langle h(t)y(t)\rangle-\Big\langle\frac{f(t)y(t)}{m(t)+x(t)}\Big\rangle+\frac{M_2(t)}{t}\\&=\Big\langle r(t)-\frac{\sigma_2^2(t)}{2}\Big\rangle-\langle h(t)y(t)\rangle-\Big\langle\frac{f(t)\tilde{y}(t)}{m(t)+\tilde{x}(t)}\Big\rangle+\frac{M_2(t)}{t}\\&\quad+\Big\langle\frac{f(t)\tilde{y}(t)}{m(t)+\tilde{x}(t)}-\frac{f(t)y(t)}{m(t)+x(t)}\Big\rangle.\end{aligned}$$

Then

$$\begin{aligned}\left(\frac{\ln y(t)-\ln y_0}{t}\right)^*&=\Big\langle r(t)-\frac{\sigma_2^2(t)}{2}\Big\rangle^*-\langle h(t)y(t)\rangle^*-\Big\langle\frac{f(t)y(t)}{m(t)+x(t)}\Big\rangle^*\\&=\Big\langle r(t)-\frac{\sigma_2^2(t)}{2}\Big\rangle^*-\langle h(t)y(t)\rangle^*-\Big\langle\frac{f(t)\tilde{y}(t)}{m(t)+\tilde{x}(t)}\Big\rangle^*\\&\quad+\Big\langle\frac{f(t)\tilde{y}(t)}{m(t)+\tilde{x}(t)}-\frac{f(t)y(t)}{m(t)+x(t)}\Big\rangle^*.\end{aligned}\tag{4.11}$$

In addition,

$$\begin{aligned}&\Big\langle\frac{f(t)\tilde{y}(t)}{m(t)+\tilde{x}(t)}-\frac{f(t)y(t)}{m(t)+x(t)}\Big\rangle^*\\&=\Big\langle\frac{f(t)m(t)(\tilde{y}(t)-y(t))+f(t)y(t)(x(t)-\tilde{x}(t))+f(t)x(t)(\tilde{y}(t)-y(t))}{(m(t)+\tilde{x}(t))((m(t)+x(t))}\Big\rangle^*\\&\ge\Big\langle\frac{f(t)y(t)(x(t)-\tilde{x}(t))}{(m(t)+\tilde{x}(t))((m(t)+x(t))}\Big\rangle^*\\&\ge-\frac{f^u}{(m^\ell)^2}\langle y(t)(\tilde{x}(t)-x(t))\rangle^*\\&\ge-\frac{f^u}{(m^\ell)^2}\langle y(t)\rangle^*\langle\tilde{x}(t)-x(t)\rangle^*.\end{aligned}$$

By the above result, we know $\langle y(t)\rangle^* \le M_y$. Then we have

$$\Big\langle \frac{f(t)\tilde{y}(t)}{m(t)+\tilde{x}(t)} - \frac{f(t)y(t)}{m(t)+x(t)} \Big\rangle^* \ge -\frac{f^u}{(m^\ell)^2} M_y \langle \tilde{x}(t)-x(t)\rangle^*. \tag{4.12}$$

Combining with (4.11), we acquire

$$\begin{aligned}\Big(\frac{\ln y(t)}{t}\Big)^* \ge \Big\langle r(t) - \frac{\sigma_2^2(t)}{2}\Big\rangle^* - h^u \langle y(t)\rangle^* - \Big\langle \frac{f(t)\tilde{y}(t)}{m(t)+\tilde{x}(t)}\Big\rangle^* \\ - \frac{f^u}{(m^\ell)^2} M_y \langle \tilde{x}(t)-x(t)\rangle^*.\end{aligned} \tag{4.13}$$

Define the Lyapunov function $V_3(t) = \ln \tilde{x}(t) - \ln x(t)$, which is a positive function on $\mathbb{R}_+$. Thus,

$$\begin{aligned} \mathrm{D}^+ V_3(t) &= \Big(\frac{\mathrm{d}\tilde{x}(t)}{\tilde{x}(t)} - \frac{(\mathrm{d}\tilde{x}(t))^2}{2\tilde{x}^2(t)}\Big) - \Big(\frac{\mathrm{d}x(t)}{x(t)} - \frac{(\mathrm{d}x(t))^2}{2x^2(t)}\Big) \\ &= \Big[-b(t)\tilde{x}(t) + b(t)x(t) + \frac{c(t)y(t)}{(1+\alpha(t)x(t))(1+\beta(t)y(t))}\Big]\mathrm{d}t \\ &\le \big[-b^\ell(\tilde{x}(t)-x(t)) + c^u y(t)\big]\mathrm{d}t. \end{aligned} \tag{4.14}$$

Integrating (4.14), we get

$$\frac{V_3(t)-V_3(0)}{t} \le c^u\langle y(t)\rangle - b^\ell \langle \tilde{x}(t)-x(t)\rangle.$$

That is

$$\langle \tilde{x}(t)-x(t)\rangle \le \frac{c^u}{b^\ell}\langle y(t)\rangle,$$

which indicates that

$$\langle \tilde{x}(t)-x(t)\rangle^* \le \frac{c^u}{b^\ell}\langle y(t)\rangle^*.$$

Combining with (4.13), we have

$$\begin{aligned} \Big(\frac{\ln y(t)}{t}\Big)^* &\ge \Big\langle r(t) - \frac{\sigma_2^2(t)}{2}\Big\rangle^* - h^u\langle y(t)\rangle^* - \Big\langle \frac{f(t)\tilde{y}(t)}{m(t)+\tilde{x}(t)}\Big\rangle^* - \frac{f^u c^u}{(m^\ell)^2 b^\ell} M_y \langle y(t)\rangle^* \\ &= \Big\langle r(t) - \frac{\sigma_2^2(t)}{2}\Big\rangle^* - \Big\langle \frac{f(t)\tilde{y}(t)}{m(t)+\tilde{x}(t)}\Big\rangle^* - \Big[h^u + \frac{f^u c^u \langle r(t) - \frac{\sigma_2^2(t)}{2}\rangle^*}{(m^\ell)^2 b^\ell h^\ell}\Big]\langle y(t)\rangle^* \\ &\ge \Big\langle r(t) - \frac{\sigma_2^2(t)}{2}\Big\rangle^* - \Big\langle \frac{f(t)\tilde{y}(t)}{m(t)+\tilde{x}(t)}\Big\rangle^* - \Big[h^u + \frac{f^u c^u \langle r(t) - \frac{\sigma_2^2(t)}{2}\rangle^*}{(m^\ell)^2 b^\ell h^\ell}\Big]\varepsilon_1 \\ &> 0, \end{aligned} \tag{4.15}$$

which contradicts with Lemma 4.1 and thus we complete the proof.

(5) By the condition $\langle r(t) - \frac{\sigma_2^2(t)}{2}\rangle_* > 0$, there exists a sufficiently small $\varepsilon > 0$ such that

$$\Big\langle r(t) - \frac{\sigma_2^2(t)}{2}\Big\rangle_* - \varepsilon > 0.$$

By the second equation of (4.5), we attain

$$\frac{\ln y(t) - \ln y_0}{t} \ge \Big\langle r(t) - \frac{\sigma_2^2(t)}{2}\Big\rangle - h^u\langle y(t)\rangle - \frac{f^u}{m^\ell}\langle y(t)\rangle + \frac{M_2(t)}{t}.$$

For any given positive number $\varepsilon > 0$, there exists $T_2 > 0$ satisfying, for all $t > T_2$

$$\left\langle r(t) - \frac{\sigma_2^2(t)}{2}\right\rangle > \left\langle r(t) - \frac{\sigma_2^2(t)}{2}\right\rangle_* - \frac{\varepsilon}{2}, \quad \text{and} \quad \frac{M_2(t)}{t} > -\frac{\varepsilon}{2}.$$

Therefore,

$$\frac{\ln y(t) - \ln y_0}{t} \geq \left\langle r(t) - \frac{\sigma_2^2(t)}{2}\right\rangle_* - \varepsilon - \left(h^u + \frac{f^u}{m^\ell}\right)\langle y(t)\rangle.$$

According to Lemma 2.9 and the arbitrariness of ε, we have

$$\langle y(t)\rangle_* \geq \frac{\langle r(t) - \frac{\sigma_2^2(t)}{2}\rangle_*}{h^u + \frac{f^u}{m^\ell}} > 0.$$

□

Remark 4.1. According to the proof of Theorem 4.1 and Theorem 4.2, we can know that if $\langle a(t) - \frac{\sigma_1^2}{2}\rangle^* > 0$ and $\langle r(t) - \frac{\sigma_2^2}{2}\rangle^* < 0$, then although the prey population survives, the predators die out because of the too large diffusion coefficients σ_2^2.

5. Stochastic permanence

Theorem 5.1. *Suppose that* $\min\{a^\ell - \frac{c^u}{\beta^\ell}, r^\ell\} - \frac{1}{2}\max\{(\sigma_1^u)^2, (\sigma_2^u)^2\} > 0$, *then the system (1.1) is stochastically permanent.*

Proof. The proof is motivated by Li and Mao [15] and Liu and Wang [18]. The whole proof is divided into two parts.

In the first part, we prove that for arbitraty $\varepsilon > 0$, there exists a constant $\delta > 0$ such that $\mathbb{P}\{|X(t)| \geq \delta\} \geq 1 - \varepsilon$. Above all, we claim that for any initial value $X(0) = (x(0), y(0)) \in \mathbb{R}_2^+$, the solution $X(t) = (x(t), y(t))$ satisfies

$$\limsup_{t\to+\infty} \mathbb{E}\left(\frac{1}{|X(t)|^\theta}\right) \leq H.$$

Here, $\theta < 2$ is an arbitrary positive constant satisfying

$$\min\left\{a^\ell - \frac{c^u}{\beta^\ell}, r^\ell\right\} - \frac{1}{2}(\theta + 1)\max\{(\sigma_1^u)^2, (\sigma_2^u)^2\} > 0. \tag{5.1}$$

Define $V_4(x(t), y(t)) = x(t) + y(t)$, then

$$\begin{aligned}
\mathrm{d}V_4(x(t), y(t)) &= \mathrm{d}x(t) + \mathrm{d}y(t) \\
&= x(t)\left[a(t) - b(t)x(t) - \frac{c(t)y(t)}{(1 + \alpha(t)x(t))(1 + \beta(t)y(t))}\right]\mathrm{d}t \\
&\quad + y(t)\left[r(t) - h(t)y(t) - \frac{f(t)y(t)}{m(t) + x(t)}\right]\mathrm{d}t \\
&\quad + \sigma_1(t)x(t)\mathrm{d}B_1(t) + \sigma_2(t)y(t)\mathrm{d}B_2(t).
\end{aligned}$$

Let $U(t) = \frac{1}{V_4(x(t),y(t))}$. By Itô's formula, we obtain

$$\begin{aligned}
\mathrm{d}U(t) &= -U^2(t)\mathrm{d}V_4 + U^3(t)(\mathrm{d}V_4)^2\\
&= -U^2(t)\left[x(t)\left(a(t) - b(t)x(t) - \frac{c(t)y(t)}{(1+\alpha(t)x(t))(1+\beta(t)y(t))}\right)\right.\\
&\quad \left. + y(t)\left(r(t) - h(t)y(t) - \frac{f(t)y(t)}{m(t)+x(t)}\right)\right]\mathrm{d}t + U^3(t)(\sigma_1^2(t)x^2(t)\\
&\quad + \sigma_2^2(t)y^2(t))\mathrm{d}t - U^2(t)\sigma_1(t)x(t)\mathrm{d}B_1(t) - U^2(t)\sigma_2(t)y(t)\mathrm{d}B_2(t)\\
&= LU(t)\mathrm{d}t - U^2(t)\sigma_1(t)x(t)\mathrm{d}B_1(t) - U^2(t)\sigma_2(t)y(t)\mathrm{d}B_2(t).
\end{aligned}$$

Here,

$$\begin{aligned}
LU(t) &= -U^2(t)\left[x(t)\left(a(t) - b(t)x(t) - \frac{c(t)y(t)}{(1+\alpha(t)x(t))(1+\beta(t)y(t))}\right)\right.\\
&\quad \left. + y(t)\left(r(t) - h(t)y(t) - \frac{f(t)y(t)}{m(t)+x(t)}\right)\right]\\
&\quad + U^3(t)(\sigma_1^2(t)x^2(t) + \sigma_2^2(t)y^2(t)).
\end{aligned}$$

Noting that the positive constant $\theta < 2$ obeying (5.1), so we can choose a suitable $p > 0$ such that it satisfies the following inequality

$$\theta\min\left\{a^\ell - \frac{c^u}{\beta^\ell}, r^\ell\right\} - \frac{1}{2}\theta(\theta+1)\max\{(\sigma_1^u)^2, (\sigma_2^u)^2\} - p > 0. \tag{5.2}$$

Denote $W(t) = e^{pt}(1+U(t))^\theta$, then

$$\begin{aligned}
\mathrm{d}W(t) &= pe^{pt}(1+U(t))^\theta\mathrm{d}t + e^{pt}\theta(1+U(t))^{\theta-1}\mathrm{d}U(t)\\
&\quad + \frac{1}{2}e^{pt}\theta(\theta-1)(1+U(t))^{\theta-2}(\mathrm{d}U(t))^2\\
&= LW(t)\mathrm{d}t - e^{pt}\theta(1+U(t))^{\theta-1}U^2(t)\sigma_1(t)x(t)\mathrm{d}B_1(t)\\
&\quad - e^{pt}\theta(1+U(t))^{\theta-1}U^2(t)\sigma_2(t)y(t)\mathrm{d}B_2(t).
\end{aligned}$$

Here,

$$\begin{aligned}
LW(t) &= e^{pt}(1+U(t))^{\theta-2}\Big(p(1+U(t))^2 + \theta(1+U(t))LU(t)\\
&\quad + \frac{\theta(\theta-1)}{2}U^4(t)(\sigma_1^2(t)x^2(t) + \sigma_2^2(t)y^2(t))\Big)\\
&= e^{pt}(1+U(t))^{\theta-2}\Big[p(1+U(t))^2\\
&\quad - \theta U^2(t)x(t)\left(a(t) - b(t)x(t) - \frac{c(t)y(t)}{(1+\alpha(t)x(t))(1+\beta(t)y(t))}\right)\\
&\quad - \theta U^2(t)y(t)\left(r(t) - h(t)y(t) - \frac{f(t)y(t)}{m(t)+x(t)}\right)\\
&\quad - \theta U^3(t)x(t)\left(a(t) - b(t)x(t) - \frac{c(t)y(t)}{(1+\alpha(t)x(t))(1+\beta(t)y(t))}\right)\\
&\quad - \theta U^3(t)y(t)\left(r(t) - h(t)y(t) - \frac{f(t)y(t)}{m(t)+x(t)}\right)
\end{aligned}$$

$$+ \theta U^3(t)(\sigma_1^2(t)x^2(t) + \sigma_2^2(t)y^2(t)) + \frac{\theta(\theta+1)}{2}U^4(t)(\sigma_1^2(t)x^2(t) + \sigma_2^2(t)y^2(t))\Bigg].$$

In the following, we will make some estimations for the above $LW(t)$. For the last two items, we have

$$\begin{aligned}
&\theta U^3(t)(\sigma_1^2(t)x^2(t) + \sigma_2^2(t)y^2(t)) \\
&\le \theta U^3(t)\max\{(\sigma_1^u)^2, (\sigma_2^u)^2\}(x^2(t) + y^2(t)) \\
&\le \theta U(t)\max\{(\sigma_1^u)^2, (\sigma_2^u)^2\},
\end{aligned}$$

and

$$\frac{\theta(\theta+1)}{2}U^4(t)(\sigma_1^2(t)x^2(t) + \sigma_2^2(t)y^2(t)) \le \frac{\theta(\theta+1)}{2}U^2(t)\max\{(\sigma_1^u)^2, (\sigma_2^u)^2\}.$$

Moreover, the estimations on the expressions involving $U^2(t)$ are

$$\begin{aligned}
&\theta U^2(t)b(t)x^2(t) + \theta U^2(t)h(t)y^2(t) + \theta U^2(t)\frac{f(t)y^2(t)}{m(t)+x(t)} \\
&\le \theta\max\Big\{b^u, h^u + \frac{f^u}{m^\ell}\Big\}U^2(t)(x^2(t) + y^2(t)) \\
&\le \theta\max\Big\{b^u, h^u + \frac{f^u}{m^\ell}\Big\},
\end{aligned}$$

and

$$\begin{aligned}
&-\theta U^2(t)x(t)\left(a(t) - \frac{c(t)y(t)}{(1+\alpha(t)x(t))(1+\beta(t)y(t))}\right) - \theta U^2(t)y(t)r(t) \\
&\le -\theta U(t)\min\Big\{a^\ell - \frac{c^u}{\beta^\ell}, r^\ell\Big\}.
\end{aligned}$$

We proceed to estimate the rest expressions involving $U^3(t)$

$$\theta U^3(t)b(t)x^2(t) + \theta U^3(t)h(t)y^2(t) + \theta U^3(t)\frac{f(t)y^2(t)}{m(t)+x(t)} \le \theta\max\Big\{b^u, h^u + \frac{f^u}{m^\ell}\Big\}U(t),$$

and

$$\begin{aligned}
&-\theta U^3(t)x(t)\left(a(t) - \frac{c(t)y(t)}{(1+\alpha(t)x(t))(1+\beta(t)y(t))}\right) - \theta U^3(t)y(t)r(t) \\
&\le -\theta U^2(t)\min\Big\{a^\ell - \frac{c^u}{\beta^\ell}, r^\ell\Big\}.
\end{aligned}$$

Summarizing the above estimations, we have

$$\begin{aligned}
LW(t) &\le e^{pt}(1+U(t))^{\theta-2}\Bigg[p + \theta\max\Big\{b^u, h^u + \frac{f^u}{m^\ell}\Big\} \\
&\quad + (2p + \theta\max\{(\sigma_1^u)^2, (\sigma_2^u)^2\} + \theta\max\Big\{b^u, h^u + \frac{f^u}{m^\ell}\Big\} \\
&\quad - \theta\min\Big\{a^\ell - \frac{c^u}{\beta^\ell}, r^\ell\Big\})U(t) + (p - \theta\min\Big\{a^\ell - \frac{c^u}{\beta^\ell}, r^\ell\Big\} \\
&\quad + \frac{\theta(\theta+1)}{2}\max\{(\sigma_1^u)^2, (\sigma_2^u)^2\})U^2(t)\Bigg] \\
&\le H_1 e^{pt}.
\end{aligned}$$

Where

$$H_1=\frac{4e_1e_3+e_2^2}{4e_1},\ e_1=\theta\min\Big\{a^\ell-\frac{c^u}{\beta^\ell},r^\ell\Big\}-\frac{1}{2}\theta(\theta+1)\max\{(\sigma_1^u)^2,(\sigma_2^u)^2\}-p,$$

$$e_2=2p+\theta\max\{(\sigma_1^u)^2,(\sigma_2^u)^2\}+\theta\max\Big\{b^u,h^u+\frac{f^u}{m^\ell}\Big\}-\theta\min\Big\{a^\ell-\frac{c^u}{\beta^\ell},r^\ell\Big\},$$

$$e_3=p+\theta\max\Big\{b^u,h^u+\frac{f^u}{m^\ell}\Big\}.$$

Thus

$$\mathbb{E}[e^{pt}(1+U(t))^\theta]\le(1+U(0))^\theta+\frac{H_1(e^{pt}-1)}{p}.$$

Therefore

$$\limsup_{t\to\infty}\mathbb{E}[U^\theta(t)]\le\limsup_{t\to\infty}\mathbb{E}[(1+U(t))^\theta]\le\frac{H_1}{p}.$$

Since

$$\frac{1}{x^2(t)+y^2(t)}=\frac{2}{2(x^2(t)+y^2(t))}\le\frac{2}{(x(t)+y(t))^2},$$

then

$$\frac{1}{\sqrt{x^2(t)+y^2(t)}}\le\frac{2}{x(t)+y(t)}.$$

That is

$$\frac{1}{X(t)}\le 2U(t).$$

In other words,

$$\mathbb{E}\left[\frac{1}{|X(t)|^\theta}\right]\le 2^\theta\mathbb{E}[U^\theta(t)]\le 2^\theta\frac{H_1}{p}:=H.$$

Hence, for any $\varepsilon>0$, letting $\delta=(\frac{\varepsilon}{H})^{\frac{1}{\theta}}$, by the Chebyshev inequality, we obtain

$$\mathbb{P}\{|X(t)|<\delta\}=\mathbb{P}\left\{\frac{1}{|X(t)|}>\frac{1}{\delta}\right\}\le\left(\frac{1}{\delta}\right)^{-\theta}\mathbb{E}\left[\frac{1}{|X(t)|^\theta}\right]=\delta^\theta\mathbb{E}\left[\frac{1}{|X(t)|^\theta}\right]\le\frac{\varepsilon}{H}\cdot H=\varepsilon.$$

Consequently

$$\liminf_{t\to\infty}P\{|X(t)|<\delta\}\le\varepsilon,$$

which indicates

$$\liminf_{t\to\infty}\mathbb{P}\{|X(t)|\ge\delta\}\ge 1-\varepsilon.$$

In the second part, we need to prove that for any $\varepsilon>0$, there exists a constant $\chi>0$ such that

$$\liminf_{t\to\infty}\mathbb{P}\{|X(t)|\le\chi\}\ge 1-\varepsilon.$$

By Theorem 3.2, we obtain that the solution of the system (1.1) is stochastically ultimately bounded, that is

$$\mathbb{E}(|X(t)|^p)\le K(p).$$

Thus, for any given $\varepsilon>0$, choosing $\chi=(\frac{K(p)}{\varepsilon})^{\frac{1}{p}}$, by the Chebyshev inequality, we get

$$\mathbb{P}\{|X(t)|>\chi\}\le\frac{\mathbb{E}\left(|X(t)|^p\right)}{\chi^p}\le\varepsilon.$$

Therefore

$$\liminf_{t\to+\infty} \mathbb{P}\{|X(t)| > \chi\} \le \varepsilon,$$

which shows

$$\liminf_{t\to+\infty} \mathbb{P}\{|X(t)| \le \chi\} \ge 1-\varepsilon.$$

So far, we complete all the proofs. □

6. Numerical simulations

This section presents a numerical simulation to verify the theoretical analysis of our system. By means of the Milstein method mentioned in Higham [7], we obtain the following discretized equations:

$$\begin{aligned} x_{i+1} &= x_i + x_i\left(a(i\Delta t) - b(i\Delta t)x_i - \frac{c(i\Delta t)y_i}{(1+\alpha(i\Delta t)x_i)(1+\beta(i\Delta t)y_i)}\right)\Delta t \\ &\quad + x_i\sigma_1(i\Delta t)\sqrt{\Delta t}\xi_i + \frac{\sigma_1^2(i\Delta t)}{2}x_i(\xi_i^2-1)\Delta t, \\ y_{i+1} &= y_i + y_i\left(r(i\Delta t) - h(i\Delta t)y_i - \frac{f(i\Delta t)y_i}{x_i + m(i\Delta t)}\right)\Delta t \\ &\quad + y_i\sigma_2(i\Delta t)\sqrt{\Delta t}\eta_i + \frac{\sigma_2^2(i\Delta t)}{2}y_i(\eta_i^2-1)\Delta t. \end{aligned}$$

In Figure 1(a), we take

$$\begin{aligned} &a(t) = 0.4 + 0.1\sin t,\ r(t) = 0.5 + 0.05\sin t,\ \sigma_1(t) = 1 + 0.02\sin t, \\ &\sigma_2(t) = 1.2 + 0.01\sin t,\ \alpha(t) = 0.46 + 0.04\sin t,\ \beta(t) = 0.46 + 0.04\sin t. \end{aligned}$$

By calculation, we have

$$\left\langle a(t) - \frac{\sigma_1^2(t)}{2}\right\rangle^* < 0, \quad \text{and} \quad \left\langle r(t) - \frac{\sigma_2^2(t)}{2}\right\rangle^* < 0.$$

By Theorem 4.1 and Theorem 4.2, both of the prey and predator populations (x and y, respectively) end in extinction. In Figure 1(b)-1(d), the other parameters are the same as those in Figture 1(a) expect $\alpha(t)$, $\beta(t)$. By comparing Figures 1(a)-(d), we observe that the effects of handling time $\alpha(t)$ and the magnitude of interference among predators $\beta(t)$ do not influence the extinction of the system.

In Figure 2, we choose

$$\begin{aligned} &a(t) = 1.5 + 0.02\sin t,\ r(t) = 0.4 + 0.02\sin t,\ \sigma_1(t) = 0.5 + 0.02\sin t, \\ &\sigma_2(t) = 1.2 + 0.02\sin t,\ \alpha(t) = 0.46 + 0.04\sin t,\ \beta(t) = 0.46 + 0.04\sin t. \end{aligned}$$

Then, we have

$$\left\langle a(t) - \frac{\sigma_1^2(t)}{2}\right\rangle^* > 0, \quad \text{and} \quad \left\langle r(t) - \frac{\sigma_2^2(t)}{2}\right\rangle^* < 0.$$

By Theorem 4.1 and Theorem 4.2, we get that the prey population x is weakly persistent in the mean and the predator population y goes to extinction.

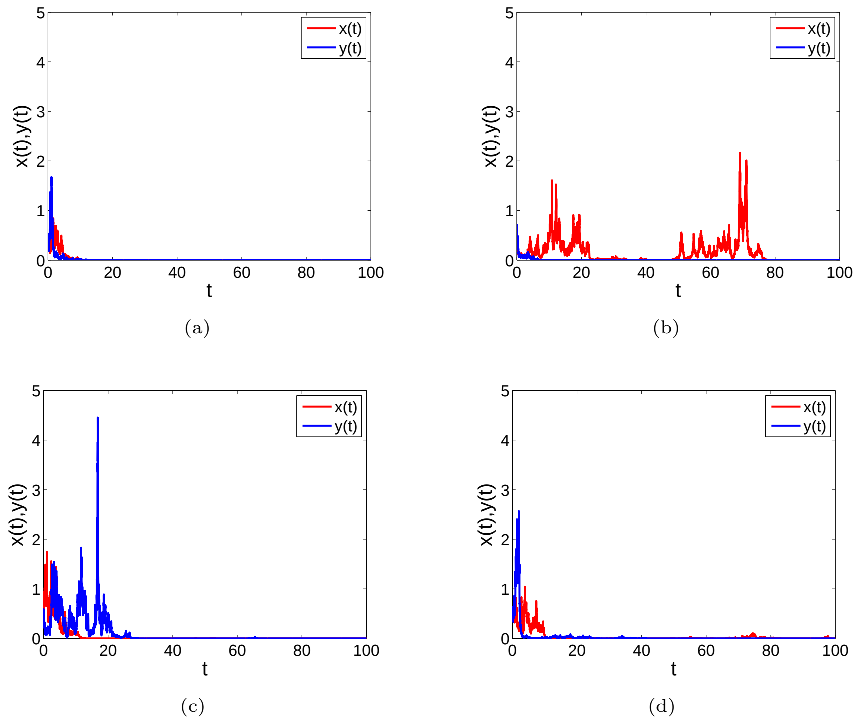

Figure 1. The figures (a)-(d) depict the extinction of the prey and predator species. (**a**) $\alpha(t) = 0.46 + 0.04\sin t,\ \beta(t) = 0.46 + 0.04\sin t$. (**b**) $\alpha(t) = 0,\ \beta(t) = 0.86 + 0.04\sin t$. (**c**) $\alpha(t) = 0.86 + 0.04\sin t,\ \beta(t) = 0$. (**d**) $\alpha(t) = 0,\ \beta(t) = 0$. Both of the prey and predator populations go to extinction.

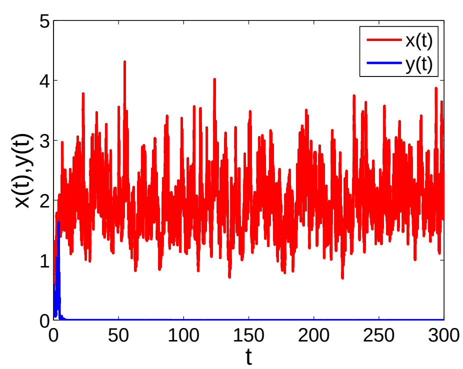

Figure 2. The prey population is weakly persistent in the mean, whereas the predator species y is extinct.

In Figure 3, we make

$$a(t) = 0.8 + 0.1\sin t,\ r(t) = 0.4 + 0.02\sin t,\ \sigma_1(t) = 0.2 + 0.02\sin t,$$
$$\sigma_2(t) = 0.4 + 0.02\sin t, \alpha(t) = 0.46 + 0.04\sin t,\ \beta(t) = 0.5 + 0.04\sin t,$$
$$c(t) = 0.1 + 0.02\sin t.$$

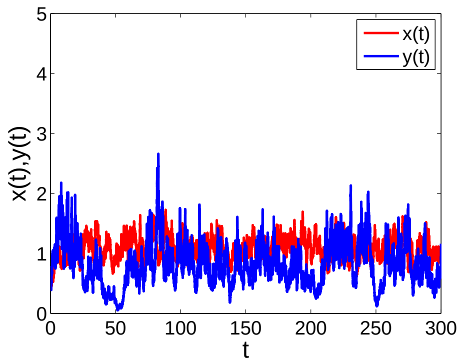

Figure 3. The prey population is strongly persistent in the mean and the predator species y is strongly persistent in the mean.

Direct calculation produces

$$\left\langle a(t) - \frac{\sigma_1^2(t)}{2} \right\rangle_* - \left\langle \frac{c(t)}{\beta(t)} \right\rangle^* > 0, \quad \text{and} \quad \left\langle r(t) - \frac{\sigma_2^2(t)}{2} \right\rangle^* > 0.$$

By Theorem 4.1 and Theorem 4.2, we know that the prey population x is strongly persistent in the mean and the predator population y is strongly persistent in the mean.

In Figure 4, we select

$$\begin{aligned}
&a(t) = 0.7 + 0.02\sin t,\ r(t) = 1.0 + 0.02\sin t,\ \sigma_1(t) = 1.2 + 0.02\sin t,\\
&\sigma_2(t) = 0.4 + 0.02\sin t, \alpha(t) = 0.46 + 0.04\sin t,\ \beta(t) = 0.5 + 0.04\sin t,\\
&c(t) = 0.1 + 0.02\sin t.
\end{aligned}$$

Then, we have

$$\left\langle a(t) - \frac{\sigma_1^2(t)}{2} \right\rangle^* < 0, \quad \text{and} \quad \left\langle r(t) - \frac{\sigma_2^2(t)}{2} \right\rangle_* > 0.$$

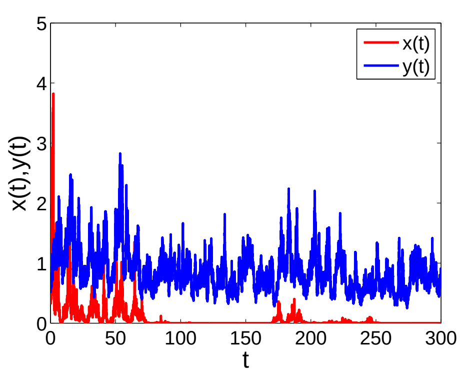

Figure 4. The prey population is extinct but the predator species y is strongly persistent in the mean.

By Theorem 4.1 and Theorem 4.2, we get that the prey population x goes to extinction and the predator population y is weakly persistent in the mean. Although x goes to extinction due to the too large diffusion coefficients σ_1^2, the predator do not extinct because the predator can seek other food.

In Figure 5, we let

$$a(t) = 1.5 + 0.02 \sin t,\ r(t) = 0.4 + 0.02 \sin t,\ \sigma_1(t) = 0.4 + 0.02 \sin t,$$
$$\sigma_2(t) = 0.3 + 0.02 \sin t,\ \alpha(t) = 0.46 + 0.04 \sin t,\ \beta(t) = 0.5 + 0.04 \sin t,$$
$$c(t) = 0.1 + 0.02 sint.$$

Then, we have

$$\min\left\{a^\ell - \frac{c^u}{\beta^\ell}, r^\ell\right\} - \frac{1}{2}\max\{(\sigma_1^u)^2, (\sigma_2^u)^2\} > 0.$$

By Theorem 5.1, we get that the system (1.1) is stochastically permanent.

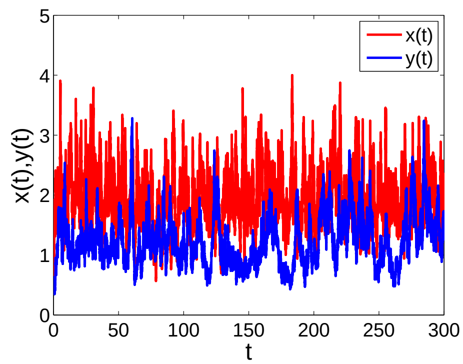

Figure 5. The system (1.1) is stochastically permanent.

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References

[1] R. Arditi and L. R. Ginzburg, *Coupling in predator-prey dynamics: ratio-dependence*, J. Theoret. Biol., 1989, 139, 311–326.

[2] M. Bandyopadhyay and J. Chattopadyay, *Ratio-dependent predator-prey model, effect of environmental fluctuation and stability*, Nonlinearity, 2005, 18, 913–936.

[3] J. R. Beddington, *Mutual interference between parasites or predators and its effect on searching efficiency*, 1975, J. Anim. Ecol., 44, 331–340.

[4] H. I. Freedman, *Deterministic Mathematical Models in Population Ecology*, Marcel Dekker, New York, 1980.

[5] S. J. Gao, Y. J. Liu, J. J. Nieto and H. Andrade, *Seasonality and mixed vaccination strategy in an epidemic model with vertical transmission*, Math. Comput. Simul., 2011, 81, 1855–1868.

[6] Z. K. Guo, H. F. Huo, Q. Y. Ren and H. Xiang, *Bifurcation of a modified Leslie-Gower system with discrete and distributed delays*, J. Nonlinear Model. Anal., 2019, 1(1), 73–91.

[7] D. J. Higham, *An algorithmic introduction to numerical simulation of stochastic differential equations*, SIAM Rev., 2001, 43, 525–546.

[8] Z. Y. Hu, Z. D. Teng, C. J. Jia, L. Zhang and X. Chen, *Complex dynamical behaviors in a discrete eco-epidemiological model with disease in prey*, Adv. Differ. Equ., 2014, 2014, 1–19.

[9] C. S. Holling, *The components of predation as revealed by a study of small mammal predation of the European pine sawfly*, Can. Entomol., 1959, 91, 293–320.

[10] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, North-Holland, New York, 1989.

[11] C. Y. Ji, D. Q. Jiang and N. Z. Shi, *Analysis of a predator-prey model with modified Leslie-Gower and Holling-type II schemes with stochastic perturbation*, J. Math. Anal. Appl., 2009, 359, 482–498.

[12] D. Q. Jiang and N. Z. Shi, *A note on nonautonomous logistic equation with random perturbation*, J. Math. Anal. Appl., 2005, 303, 164–172.

[13] P. H. Leslie, *Some further notes on the use of matrices in population mathematic*, Biometrica, 1948, 35, 213–245.

[14] P. H. Leslie and J. C. Gower, *The properties of a stochastic model for the predator-prey type of interaction between two species*, Biometrica, 1960, 47, 219–234.

[15] X. Y. Li and X. R. Mao, *Population dynamical behavior of nonautonomous Lotka-Volterra competitive system with random perturbation*, Discret. Contin. Dyn. Syst., 2009, 24, 523–545.

[16] Y. G. Lin, D. Q. Jiang and S. Wang, *Stationary distribution of a stochastic SIS epidemic model with vaccination*, Physica A, 2014, 394, 187–197.

[17] M. Liu, C. X. Du and M. L. Deng, *Persistence and extinction of a modified Leslie-Gower Holling-type II stochastic predator-prey model with impulsive toxicant input in polluted environments*, Nonlinear Anal.-Hybrid Syst., 2018, 27, 177–190.

[18] M. Liu and K Wang, *Persistence and extinction in stochastic nonautonomous logistic systems*, J. Math. Anal. Appl., 2011, 375, 443–457.

[19] M. Liu, K. Wang and Q. Wu, *Survival analysis of stochastic competitive models in a polluted enviornment and stochastic competitive exclusion principle*, Bull. Math. Biol., 2011, 73, 1969–2012.

[20] X. Q. Liu, S. M. Zhong, B. D. Tian and F. X. Zheng, *Asymptotic properties of a stochastic predator-prey model with Crowley-Martin functional response*, J. Appl. Math. Comput., 2013, 43, 479–490.

[21] X. R. Mao, *Stochastic Different Equations and Application*, 2nd edition, Horwood, Chichester, 2008.

[22] X. R. Mao and C. G. Yuan, *Stochastic differential equations with Markovian switching*, Imperial College Press, London, 2006.

[23] E. C. Pielou, *An Introduction to Mathematical Ecology*, Wiley, New York, 1969.

[24] X. Y. Song and Y F Li, *Dynamic behaviors of the periodic predatorCprey model with modified Leslie-Gower Holling-type II schemes and impulsive effect*, Nonlinear Anal.-Real World Appl., 2008, 9(1), 64–79.

[25] L. Wang, Z. D. Teng and H. J. Jiang, *Global attractivity of a discrete SIRS epidemic model with standard incidence rate*, Math. Method. Appl. Sci., 2013, 36, 601–619.

[26] P. J. Witbooi, *Stability of an SEIR epidemic model with independent stochastic perturbations*, Physica A, 2013, 392, 4928–4936.

[27] Y. Zhang, S. H. Chen and S. J. Gao, *Analysis of a nonautonomous stochastic predator-prey model with Crowley-Martin functional response*, Adv. Differ. Equ., 2016, 2016, 1–28.