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DYNAMICAL BEHAVIOR AND SOLUTION OF NONLINEAR DIFFERENCE EQUATION VIA FIBONACCI SEQUENCE*

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Abstract In this paper, we study the behavior of the difference equation $x_{n+1} = ax_n + \dfrac{bx_nx_{n-1}}{cx_{n-1}+dx_{n-2}}$, $n = 0, 1, \ldots$, where the initial conditions $x_{-2},\ x_{-1},\ x_0$ are arbitrary positive real numbers and a, b, c, d are positive constants. Also, we give the solution of some special cases of this equation.

Keywords Stability, boundedness, solution of difference equations.

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1. Introduction

In this paper, we deal with the behavior of the solutions of the following difference equation

$$x_{n+1} = ax_n + \frac{bx_nx_{n-1}}{cx_{n-1}+dx_{n-2}}, \quad n = 0, 1, ..., \tag{1.1}$$

where the initial conditions $x_{-2},\ x_{-1},\ x_0$ are arbitrary positive real numbers and a, b, c, d are positive constants. Also, we obtain the solution of some special cases of the same equation.

Let us introduce some basic definitions and some theorems that we need in the sequel.

Let I be some interval of real numbers and let

$$f : I^{k+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{1.2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$ [34].

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Definition 1.1 (Equilibrium Point)**.** A point $\overline{x} \in I$ is called an equilibrium point of Eq. (1.2) if

$$\overline{x} = f(\overline{x}, \overline{x}, ..., \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of Eq. (1.2), or equivalently, $\overline{x}$ is a fixed point of f.

Definition 1.2 (Stability)**.** (i) The equilibrium point $\overline{x}$ of Eq. (1.2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{for all} \quad n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of Eq. (1.2) is locally asymptotically stable if $\overline{x}$ is locally stable solution of Eq. (1.2) and there exists $\gamma > 0$, such that for all x_{-k}, $x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of Eq. (1.2) is global attractor if for all x_{-k}, $x_{-k+1}, ...,$ x_{-1}, $x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of Eq. (1.2) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Eq. (1.2).
(v) The equilibrium point $\overline{x}$ of Eq. (1.2) is unstable if $\overline{x}$ is not locally stable.

The linearized equation of Eq. (1.2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}} y_{n-i}. \tag{1.3}$$

Theorem A ([34]). *Assume that* $p, q \in R$ *and* $k \in \{0, 1, 2, ...\}$. *Then*

$$|p| + |q| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + px_n + qx_{n-k} = 0, \quad n = 0, 1, ... \,.$$

Remark 1.1. Theorem A can be easily extended to a general linear equations of the form

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, \ \ n = 0, 1, ..., \tag{1.4}$$

where $p_1, p_2, ..., p_k \in R$ and $k \in \{1, 2, ...\}$. Then Eq. (1.4) is asymptotically stable provided that

$$\sum_{i=1}^{k} |p_i| < 1.$$

Consider the following equation

$$x_{n+1} = g(x_n, x_{n-1}, x_{n-2}). \tag{1.5}$$

The following theorem will be useful for the proof of our results in this paper.

Theorem B ([35]). *Let $[a, b]$ be an interval of real numbers and assume that*

$$g : [a, b]^3 \to [a, b],$$

is a continuous function satisfying the following properties :

(a) $g(x, y, z)$ is non-decreasing in x and y in $[a, b]$ for each $z \in [a, b]$, and is non-increasing in $z \in [a, b]$ for each x and y in $[a, b]$;

(b) If $(m, M) \in [a, b] \times [a, b]$ is a solution of the system

$$M = g(M, M, m) \quad \text{and} \quad m = g(m, m, M),$$

then

$$m = M.$$

Then Eq. (1.5) has a unique equilibrium $\overline{x} \in [a, b]$ and every solution of Eq. (1.5) converges to $\overline{x}$.

Definition 1.3 (Periodicity)**.** A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

Definition 1.4 (Fibonacci Sequence)**.** The sequence $\{f_m\}_{m=0}^{\infty} = \{1, 2, 3, 5, 8, 13, ...\}$ i.e. $f_m = f_{m-1} + f_{m-2},\ m \geq 0,\ f_{-2} = 0,\ f_{-1} = 1$ is called Fibonacci Sequence.

The study of rational difference equations of order greater than one is quite challenging and rewarding because some prototypes for the development of the basic theory of the global behavior of nonlinear difference equations of order greater than one come from the results for rational difference equations. However, there have not been any effective general methods to deal with the global behavior of rational difference equations of order greater than one so far. Therefore, the study of rational difference equations of order greater than one is worth further consideration.

Recently, Agarwal et al. [4] investigated the global stability, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = a + \frac{dx_{n-l}x_{n-k}}{b - cx_{n-s}}.$$

Aloqeili [6] has obtained the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{a - x_n x_{n-1}}.$$

Cinar [12, 13] deal with the solutions of the following difference equations

$$x_{n+1} = \frac{x_{n-1}}{1 + ax_n x_{n-1}}, \quad x_{n+1} = \frac{x_{n-1}}{-1 + ax_n x_{n-1}}.$$

Elabbasy et al. [17, 18] investigated the global stability, periodicity character and gave the solution of special case of the following recursive sequences

$$x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}, \quad x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}.$$

Ibrahim [27] has got the solutions of the rational difference equation

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}(a + b x_n x_{n-2})}.$$

Karatas et al. [31] studied form of the solution of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2} x_{n-5}}.$$

Simsek et al. [40] obtained the solutions of the following difference equations

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}.$$

See also [1–20]. Other related results on rational difference equations can be found in refs. [21–49].

The study of these equations is quite challenging and rewarding and is still in its infancy. We believe that the nonlinear rational difference equations are of paramount importance in their own right, and furthermore we believe that these results about such equations over prototypes for the development of the basic theory of the global behavior of nonlinear rational difference equations.

2. Local Stability of Eq. (1.1)

In this section we investigate the local stability character of the solutions of Eq. (1.1). Eq. (1.1) has a unique equilibrium point and is given by

$$\overline{x} = a\overline{x} + \frac{b\overline{x}^2}{c\overline{x} + d\overline{x}},$$

or,

$$\overline{x}^2(1-a)(c+d) = b\overline{x}^2,$$

if $(c+d)(1-a) \neq b$, then the unique equilibrium point is $\overline{x} = 0$.

Let $f : (0,\infty)^3 \longrightarrow (0,\infty)$ be a function defined by

$$f(u,v,w) = au + \frac{buv}{cv + dw}. \tag{6}$$

Therefore it follows that

$$f_u(u,v,w) = a + \frac{bv}{cv + dw},$$
$$f_v(u,v,w) = \frac{bduw}{(cv + dw)^2},$$
$$f_w(u,v,w) = \frac{-bduv}{(cv + dw)^2},$$

we see that

$$f_u(\overline{x},\overline{x},\overline{x}) = a + \frac{b}{c+d},$$

$$f_v(\overline{x},\overline{x},\overline{x}) = \frac{bd}{(c+d)^2},$$
$$f_w(\overline{x},\overline{x},\overline{x}) = \frac{-bd}{(c+d)^2}.$$

The linearized equation of Eq. (1.1) about $\overline{x}$ is

$$y_{n+1} - \left(a + \frac{b}{c+d}\right) y_n - \frac{bd}{(c+d)^2} y_{n-1} + \frac{bd}{(c+d)^2} y_{n-2} = 0. \tag{2.1}$$

Theorem 2.1. *Assume that*

$$b(c+3d) < (1-a)(c+d)^2.$$

Then the equilibrium point of Eq. (1.1) *is locally asymptotically stable.*

Proof. It is follows by Theorem A that, Eq. (2.1) is asymptotically stable if

$$\left| a + \frac{b}{c+d} \right| + \left| \frac{bd}{(c+d)^2} \right| + \left| \frac{bd}{(c+d)^2} \right| < 1,$$

or,

$$a + \frac{b}{c+d} + \frac{2bd}{(c+d)^2} < 1,$$

and so,

$$\frac{bc+3bd}{(c+d)^2} < (1-a).$$

The proof is completed. □

3. Global Attractor of the Equilibrium Point of Eq. (1.1)

In this section we investigate the global attractivity character of solutions of E-q. (1.1).

Theorem 3.1. *The equilibrium point $\overline{x}$ of Eq.* (1.1) *is global attractor if* $c(1-a) \neq b$.

Proof. Let p, q are a real numbers and assume that $g : [p,q]^3 \longrightarrow [p,q]$ be a function defined by $g(u,v,w) = au + \frac{buw}{cv+dw}$, then we can easily see that the function $g(u,v,w)$ increasing in u, v and decreasing in w.

Suppose that (m, M) is a solution of the system

$$M = g(M,M,m) \quad \text{and} \quad m = g(m,m,M).$$

Then from Eq. (1.1), we see that

$$M = aM + \frac{bM^2}{cM+dm}, \quad m = am + \frac{bm^2}{cm+dM},$$

or,

$$M(1-a) = \frac{bM^2}{cM+dm}, \quad m(1-a) = \frac{bm^2}{cm+dM},$$

then

$$c(1-a)M^2 + d(1-a)Mm = bM^2, \quad c(1-a)m^2 + d(1-a)Mm = bm^2.$$

Subtracting we obtain

$$c(1-a)(M^2-m^2) = b(M^2-m^2), \quad c(1-a) \neq b.$$

Thus

$$M = m.$$

It follows by Theorem B that $\overline{x}$ is a global attractor of Eq. (1.1) and then the proof is completed. □

4. Boundedness of solutions of Eq. (1.1)

In this section, we study the boundedness of solutions of Eq. (1.1).

Theorem 4.1. *Every solution of Eq.* (1.1) *is bounded if* $(a + \frac{b}{c}) < 1$.

Proof. Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of Eq. (1.1). It follows from Eq. (1.1) that

$$x_{n+1} = ax_n + \frac{bx_n x_{n-1}}{cx_{n-1} + dx_{n-2}} \leq ax_n + \frac{bx_n x_{n-1}}{cx_{n-1}} = \left(a + \frac{b}{c}\right) x_n.$$

Then

$$x_{n+1} \leq x_n \quad \text{for all} \quad n \geq 0.$$

Then the sequence $\{x_n\}_{n=0}^{\infty}$ is decreasing and so are bounded from above by $M = \max\{x_{-2}, x_{-1}, x_0\}$. □

5. Special Cases of Eq. (1.1)

5.1. First Equation

In this subsection, we deal with the following special case of Eq. (1.1)

$$x_{n+1} = x_n + \frac{x_n x_{n-1}}{x_{n-1} + x_{n-2}}, \tag{5.1}$$

where the initial conditions $x_{-2},\ x_{-1},\ x_0$ are arbitrary positive real numbers.

Theorem 5.1. *Let* $\{x_n\}_{n=-2}^{\infty}$ *be a solution of Eq.* (5.1). *Then for* $n = 0, 1, 2, ...$

$$x_{2n} = h \prod_{i=0}^{n-1} \left(\frac{f_{2i+3}h + f_{2i+2}k}{f_{2i+2}h + f_{2i+1}k} \right) \left(\frac{f_{2i+3}k + f_{2i+2}r}{f_{2i+2}k + f_{2i+1}r} \right),$$

$$x_{2n+1} = \prod_{i=0}^{n} \left(\frac{f_{2i+1}h + f_{2i}k}{f_{2i}h + f_{2i-1}k} \right) \left(\frac{f_{2i+3}k + f_{2i+2}r}{f_{2i+2}k + f_{2i+1}r} \right),$$

where $x_{-2} = r,\ x_{-1} = k,\ x_0 = h,\ \{f_m\}_{m=-1}^{\infty} = \{0, 0, 1, 1, 2, 3, 5, 8, 13, ...\}$.

Proof. For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n-1,\ n-2$. That is;

$$x_{2n-2} = h\prod_{i=0}^{n-2}\left(\frac{f_{2i+3}h + f_{2i+2}k}{f_{2i+2}h + f_{2i+1}k}\right)\left(\frac{f_{2i+3}k + f_{2i+2}r}{f_{2i+2}k + f_{2i+1}r}\right),$$

$$x_{2n-1} = \prod_{i=0}^{n-1}\left(\frac{f_{2i+1}h + f_{2i}k}{f_{2i}h + f_{2i-1}k}\right)\left(\frac{f_{2i+3}k + f_{2i+2}r}{f_{2i+2}k + f_{2i+1}r}\right),$$

$$x_{2n-3} = \prod_{i=0}^{n-2}\left(\frac{f_{2i+1}h + f_{2i}k}{f_{2i}h + f_{2i-1}k}\right)\left(\frac{f_{2i+3}k + f_{2i+2}r}{f_{2i+2}k + f_{2i+1}r}\right).$$

Now, it follows from Eq. (5.1) that

$$\begin{aligned}
x_{2n} =& x_{2n-1} + \frac{x_{2n-1}x_{2n-2}}{x_{2n-2}+x_{2n-3}} = \prod_{i=0}^{n-1}\left(\frac{f_{2i+1}h + f_{2i}k}{f_{2i}h + f_{2i-1}k}\right)\left(\frac{f_{2i+3}k + f_{2i+2}r}{f_{2i+2}k + f_{2i+1}r}\right) \\
&+ \frac{\prod_{i=0}^{n-1}\left(\frac{f_{2i+1}h+f_{2i}k}{f_{2i}h+f_{2i-1}k}\right)\left(\frac{f_{2i+3}k+f_{2i+2}r}{f_{2i+2}k+f_{2i+1}r}\right)h\prod_{i=0}^{n-2}\left(\frac{f_{2i+3}h+f_{2i+2}k}{f_{2i+2}h+f_{2i+1}k}\right)\left(\frac{f_{2i+3}k+f_{2i+2}r}{f_{2i+2}k+f_{2i+1}r}\right)}{h\prod_{i=0}^{n-2}\left(\frac{f_{2i+3}h+f_{2i+2}k}{f_{2i+2}h+f_{2i+1}k}\right)\left(\frac{f_{2i+3}k+f_{2i+2}r}{f_{2i+2}k+f_{2i+1}r}\right)+\prod_{i=0}^{n-2}\left(\frac{f_{2i+1}h+f_{2i}k}{f_{2i}h+f_{2i-1}k}\right)\left(\frac{f_{2i+3}k+f_{2i+2}r}{f_{2i+2}k+f_{2i+1}r}\right)} \\
=& \prod_{i=0}^{n-1}\left(\frac{f_{2i+1}h + f_{2i}k}{f_{2i}h + f_{2i-1}k}\right)\left(\frac{f_{2i+3}k + f_{2i+2}r}{f_{2i+2}k + f_{2i+1}r}\right) \\
&+ \frac{\prod_{i=0}^{n-1}\left(\frac{f_{2i+1}h + f_{2i}k}{f_{2i}h + f_{2i-1}k}\right)\left(\frac{f_{2i+3}k + f_{2i+2}r}{f_{2i+2}k + f_{2i+1}r}\right)\prod_{i=0}^{n-2}\left(\frac{f_{2i+3}h + f_{2i+2}k}{f_{2i+2}h + f_{2i+1}k}\right)}{\prod_{i=0}^{n-2}\left(\frac{f_{2i+3}h + f_{2i+2}k}{f_{2i+2}h + f_{2i+1}k}\right) + \prod_{i=1}^{n-2}\left(\frac{f_{2i+1}h + f_{2i}k}{f_{2i}h + f_{2i-1}k}\right)} \\
=& \prod_{i=0}^{n-1}\left(\frac{f_{2i+1}h + f_{2i}k}{f_{2i}h + f_{2i-1}k}\right)\left(\frac{f_{2i+3}k + f_{2i+2}r}{f_{2i+2}k + f_{2i+1}r}\right) \\
&+ \frac{\prod_{i=0}^{n-1}\left(\frac{f_{2i+1}h + f_{2i}k}{f_{2i}h + f_{2i-1}k}\right)\left(\frac{f_{2i+3}k + f_{2i+2}r}{f_{2i+2}k + f_{2i+1}r}\right)\left(\frac{f_{2n-1}h + f_{2n-2}k}{f_{2n-2}h + f_{2n-3}k}\right)}{\left(\frac{f_{2n-1}h + f_{2n-2}k}{f_{2n-2}h + f_{2n-3}k}\right) + 1} \\
=& \prod_{i=0}^{n-1}\left(\frac{f_{2i+1}h + f_{2i}k}{f_{2i}h + f_{2i-1}k}\right)\left(\frac{f_{2i+3}k + f_{2i+2}r}{f_{2i+2}k + f_{2i+1}r}\right) \\
&+ \frac{\prod_{i=0}^{n-1}\left(\frac{f_{2i+1}h + f_{2i}k}{f_{2i}h + f_{2i-1}k}\right)\left(\frac{f_{2i+3}k + f_{2i+2}r}{f_{2i+2}k + f_{2i+1}r}\right)(f_{2n-1}h + f_{2n-2}k)}{f_{2n}h + f_{2n-1}k} \\
=& \prod_{i=0}^{n-1}\left(\frac{f_{2i+1}h + f_{2i}k}{f_{2i}h + f_{2i-1}k}\right)\left(\frac{f_{2i+3}k + f_{2i+2}r}{f_{2i+2}k + f_{2i+1}r}\right)\left(1 + \frac{f_{2n-1}h + f_{2n-2}k}{f_{2n}h + f_{2n-1}k}\right) \\
=& \prod_{i=0}^{n-1}\left(\frac{f_{2i+1}h + f_{2i}k}{f_{2i}h + f_{2i-1}k}\right)\left(\frac{f_{2i+3}k + f_{2i+2}r}{f_{2i+2}k + f_{2i+1}r}\right)\left(\frac{f_{2n+1}h + f_{2n}k}{f_{2n}h + f_{2n-1}k}\right).
\end{aligned}$$

Therefore

$$x_{2n} = h\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}h + f_{2i+2}k}{f_{2i+2}h + f_{2i+1}k}\right)\left(\frac{f_{2i+3}k + f_{2i+2}r}{f_{2i+2}k + f_{2i+1}r}\right).$$

Also, from Eq. (5.1), we see that

$$\begin{aligned}
x_{2n+1} =& x_{2n} + \frac{x_{2n}x_{2n-1}}{x_{2n-1} + x_{2n-2}} = h\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}h + f_{2i+2}k}{f_{2i+2}h + f_{2i+1}k}\right)\left(\frac{f_{2i+3}k + f_{2i+2}r}{f_{2i+2}k + f_{2i+1}r}\right)\\
&+ \frac{h\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}h+f_{2i+2}k}{f_{2i+2}h+f_{2i+1}k}\right)\left(\frac{f_{2i+3}k+f_{2i+2}r}{f_{2i+2}k+f_{2i+1}r}\right)\prod_{i=0}^{n-1}\left(\frac{f_{2i+1}h+f_{2i}k}{f_{2i}h+f_{2i-1}k}\right)\left(\frac{f_{2i+3}k+f_{2i+2}r}{f_{2i+2}k+f_{2i+1}r}\right)}{\prod_{i=0}^{n-1}\left(\frac{f_{2i+1}h+f_{2i}k}{f_{2i}h+f_{2i-1}k}\right)\left(\frac{f_{2i+3}k+f_{2i+2}r}{f_{2i+2}k+f_{2i+1}r}\right)+h\prod_{i=0}^{n-2}\left(\frac{f_{2i+3}h+f_{2i+2}k}{f_{2i+2}h+f_{2i+1}k}\right)\left(\frac{f_{2i+3}k+f_{2i+2}r}{f_{2i+2}k+f_{2i+1}r}\right)}\\
=& h\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}h + f_{2i+2}k}{f_{2i+2}h + f_{2i+1}k}\right)\left(\frac{f_{2i+3}k + f_{2i+2}r}{f_{2i+2}k + f_{2i+1}r}\right)\\
&+ \frac{\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}h + f_{2i+2}k}{f_{2i+2}h + f_{2i+1}k}\right)\left(\frac{f_{2i+3}k + f_{2i+2}r}{f_{2i+2}k + f_{2i+1}r}\right)\left(\frac{f_{2n+1}k + f_{2n}r}{f_{2n}k + f_{2n-1}r}\right)}{\left(\frac{f_{2n+1}k + f_{2n}r}{f_{2n}k + f_{2n-1}r}\right) + 1}\\
=& h\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}h+f_{2i+2}k}{f_{2i+2}h+f_{2i+1}k}\right)\left(\frac{f_{2i+3}k+f_{2i+2}r}{f_{2i+2}k+f_{2i+1}r}\right)\left(1+\frac{f_{2n+1}k+f_{2n}r}{f_{2n+1}k+f_{2n}r+f_{2n}k+f_{2n-1}r}\right)\\
=& h\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}h + f_{2i+2}k}{f_{2i+2}h + f_{2i+1}k}\right)\left(\frac{f_{2i+3}k + f_{2i+2}r}{f_{2i+2}k + f_{2i+1}r}\right)\left(1 + \frac{f_{2n+1}k + f_{2n}r}{f_{2n+2}k + f_{2n+1}r}\right)\\
=& h\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}h + f_{2i+2}k}{f_{2i+2}h + f_{2i+1}k}\right)\left(\frac{f_{2i+3}k + f_{2i+2}r}{f_{2i+2}k + f_{2i+1}r}\right)\left(\frac{f_{2n+3}k + f_{2n+2}r}{f_{2n+2}k + f_{2n+1}r}\right).
\end{aligned}$$

Thus

$$x_{2n+1} = \prod_{i=0}^{n}\left(\frac{f_{2i+1}h + f_{2i}k}{f_{2i}h + f_{2i-1}k}\right)\left(\frac{f_{2i+3}k + f_{2i+2}r}{f_{2i+2}k + f_{2i+1}r}\right).$$

Hence, the proof is completed. □

For confirming the results of this section, we consider numerical example for Eq. (5.1) put $x_{-2} = 3,\ x_{-1} = 6,\ x_0 = 7$. [See Fig. 1].

5.2. Second Equation

In this subsection, we give a specific form of the solutions of the difference equation

$$x_{n+1} = x_n + \frac{x_n x_{n-1}}{x_{n-1} - x_{n-2}}, \tag{5.2}$$

where the initial conditions $x_{-2},\ x_{-1},\ x_0$ are arbitrary positive real numbers with $x_{-1} \neq x_0,\ x_{-1} \neq x_{-2}$.

Theorem 5.2. *Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of Eq. (5.2). Then for $n = 0, 1, 2, ...$*

$$x_{2n} = h\prod_{i=0}^{n-1}\left(\frac{f_{i+3}h - f_{i+1}k}{f_{i+1}h - f_{i-1}k}\right)\left(\frac{f_{i+3}k - f_{i+1}r}{f_{i+1}k - f_{i-1}r}\right),$$

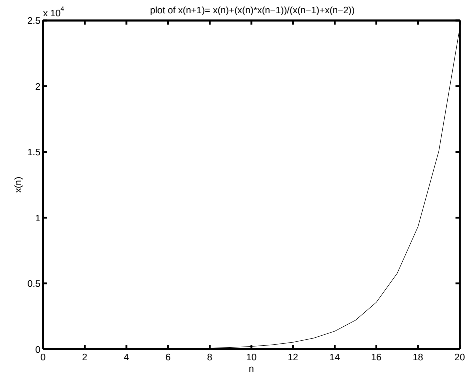

Figure 1.

$$x_{2n+1} = h\left(\frac{2k-r}{k-r}\right)\prod_{i=0}^{n-1}\left(\frac{f_{i+3}h - f_{i+1}k}{f_{i+1}h - f_{i-1}k}\right)\left(\frac{f_{i+4}k - f_{i+2}r}{f_{i+2}k - f_{i}r}\right),$$

where $x_{-2} = r,\ x_{-1} = k,\ x_0 = h,\ \{f_m\}_{m=-1}^{\infty} = \{1, 0, 1, 1, 2, 3, 5, 8, ...\}.$

Proof. As the proof of Theorem 5.1 and will be omitted. □

Assume for Eq. (5.2) that $x_{-2} = 3.6,\ x_{-1} = 2,\ x_0 = 1.4$. [See Fig. 2], and for $x_{-2} = 4,\ x_{-1} = 11,\ x_0 = 3$. [See Fig. 3].

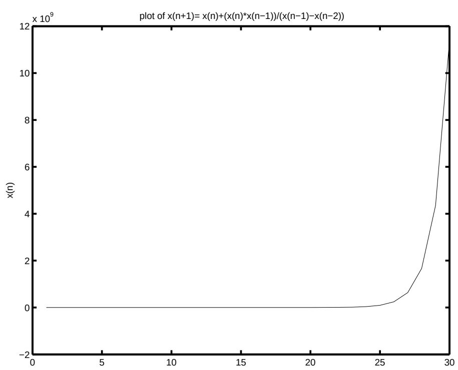

Figure 2.

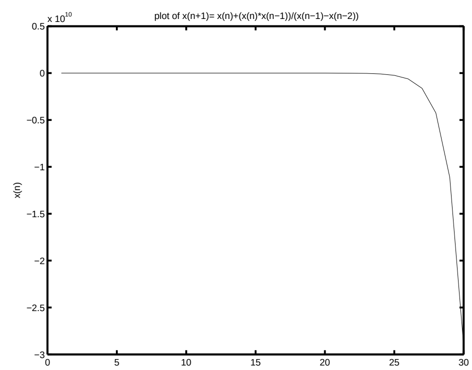

Figure 3.

5.3. Third Equation

In this subsection, we obtain the solution of the following special case of Eq. (1.1)

$$x_{n+1} = x_n - \frac{x_n x_{n-1}}{x_{n-1} + x_{n-2}}, \tag{5.3}$$

where the initial conditions $x_{-2},\ x_{-1},\ x_0$ are arbitrary positive real numbers.

Theorem 5.3. *Let* $\{x_n\}_{n=-2}^{\infty}$ *be a solution of Eq.* (5.3). *Then for* $n = 0, 1, 2, ...$

$$x_{2n} = \frac{hkr}{(f_n k + f_{n+1} r)(f_n h + f_{n+1} k)},$$

$$x_{2n+1} = \frac{hkr}{(f_{n+1}k + f_{n+2}r)(f_n h + f_{n+1}k)}.$$

Proof. For $n = 0, 1$ the result holds. Now suppose that $n > 1$ and that our assumption holds for $n-1,\ n-2$. That is;

$$x_{2n-2} = \frac{hkr}{(f_{n-1}k + f_n r)(f_{n-1}h + f_n k)},$$
$$x_{2n-1} = \frac{hkr}{(f_n k + f_{n+1}r)(f_{n-1}h + f_n k)},$$
$$x_{2n-3} = \frac{hkr}{(f_{n-1}k + f_n r)(f_{n-2}h + f_{n-1}k)}.$$

Now, it follows from Eq. (5.3) that

$$\begin{aligned}
x_{2n} =& x_{2n-1} - \frac{x_{2n-1}x_{2n-2}}{x_{2n-2} + x_{2n-3}} = \frac{hkr}{(f_n k + f_{n+1}r)(f_{n-1}h + f_n k)} \\
& - \frac{\frac{hkr}{(f_n k + f_{n+1}r)(f_{n-1}h + f_n k)} \frac{hkr}{(f_{n-1}k + f_n r)(f_{n-1}h + f_n k)}}{\left(\frac{hkr}{(f_{n-1}k + f_n r)(f_{n-1}h + f_n k)} + \frac{hkr}{(f_{n-1}k + f_n r)(f_{n-2}h + f_{n-1}k)}\right)} \\
=& \frac{hkr}{(f_n k + f_{n+1}r)(f_{n-1}h + f_n k)} - \frac{\frac{hkr}{(f_n k + f_{n+1}r)(f_{n-1}h + f_n k)} \frac{1}{(f_{n-1}h + f_n k)}}{\left(\frac{1}{(f_{n-1}h + f_n k)} + \frac{1}{(f_{n-2}h + f_{n-1}k)}\right)} \\
=& \frac{hkr}{(f_n k + f_{n+1}r)(f_{n-1}h + f_n k)} \left(1 - \frac{1}{1 + \frac{f_{n-1}h + f_n k}{f_{n-2}h + f_{n-1}k}}\right) \\
=& \frac{hkr}{(f_n k + f_{n+1}r)(f_{n-1}h + f_n k)} \left(1 - \frac{f_{n-2}h + f_{n-1}k}{f_{n-2}h + f_{n-1}k + f_{n-1}h + f_n k}\right) \\
=& \frac{hkr}{(f_n k + f_{n+1}r)(f_{n-1}h + f_n k)} \left(1 - \frac{f_{n-2}h + f_{n-1}k}{f_n h + f_{n+1}k}\right) \\
=& \frac{hkr}{(f_n k + f_{n+1}r)(f_{n-1}h + f_n k)} \left(\frac{f_n h + f_{n+1}k - f_{n-2}h - f_{n-1}k}{f_n h + f_{n+1}k}\right) \\
=& \frac{hkr}{(f_n k + f_{n+1}r)(f_{n-1}h + f_n k)} \left(\frac{f_{n-1}h + f_n k}{f_n h + f_{n+1}k}\right).
\end{aligned}$$

Then

$$x_{2n} = \frac{hkr}{(f_n k + f_{n+1}r)(f_n h + f_{n+1}k)}.$$

Also, we see from Eq. (5.3) that

$$\begin{aligned}
x_{2n+1} =& x_{2n} - \frac{x_{2n}x_{2n-1}}{x_{2n-1} + x_{2n-2}} = \frac{hkr}{(f_n k + f_{n+1}r)(f_n h + f_{n+1}k)} \\
& - \frac{\frac{hkr}{(f_n k + f_{n+1}r)(f_n h + f_{n+1}k)} \frac{hkr}{(f_n k + f_{n+1}r)(f_{n-1}h + f_n k)}}{\frac{hkr}{(f_n k + f_{n+1}r)(f_{n-1}h + f_n k)} + \frac{hkr}{(f_{n-1}k + f_n r)(f_{n-1}h + f_n k)}}
\end{aligned}$$

$$=\frac{hkr}{(f_nk+f_{n+1}r)\,(f_nh+f_{n+1}k)}-\frac{\frac{hkr}{(f_nk+f_{n+1}r)\,(f_nh+f_{n+1}k)}\,(f_{n-1}k+f_nr)}{f_{n-1}k+f_nr+f_nk+f_{n+1}r}$$
$$=\frac{hkr}{(f_nk+f_{n+1}r)\,(f_nh+f_{n+1}k)}\left(1-\frac{f_{n-1}k+f_nr}{f_{n+1}k+f_{n+2}r}\right)$$
$$=\frac{hkr}{(f_nk+f_{n+1}r)\,(f_nh+f_{n+1}k)}\left(\frac{f_{n+1}k+f_{n+2}r-f_{n-1}k-f_nr}{f_{n+1}k+f_{n+2}r}\right)$$
$$=\frac{hkr}{(f_nk+f_{n+1}r)\,(f_nh+f_{n+1}k)}\left(\frac{f_nk+f_{n+1}r}{f_{n+1}k+f_{n+2}r}\right).$$

Therefore

$$x_{2n+1}=\frac{hkr}{(f_{n+1}k+f_{n+2}r)\,(f_nh+f_{n+1}k)}.$$

Hence, the proof is completed. □

Fig. 4 shows the solution of Eq. (5.3) when $x_{-2}=9,\ x_{-1}=6,\ x_0=11.$

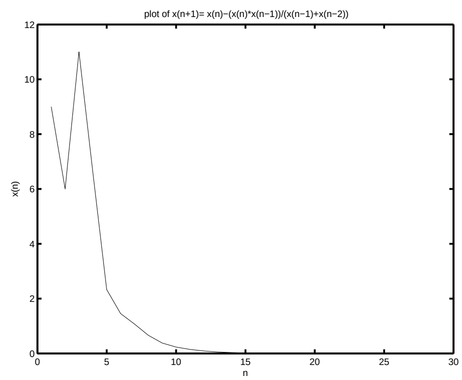

Figure 4.

5.4. Fourth Equation

In this subsection, we study the following special case of Eq. (1.1)

$$x_{n+1}=x_n-\frac{x_nx_{n-1}}{x_{n-1}-x_{n-2}},\tag{5.4}$$

where the initial conditions $x_{-2},\ x_{-1},\ x_0$ are arbitrary non zero real numbers.with $x_{-1}\neq x_0$, $x_{-1}\neq x_{-2}$.

Theorem 5.4. *Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of Eq. (5.4). Then every solution of Eq. (5.4) is periodic with period 6. Moreover $\{x_n\}_{n=-2}^{\infty}$ takes the form*

$$\left\{r,k,h,\frac{hr}{r-k},\frac{hkr}{(h-k)(k-r)},\frac{hr}{h-k},r,k,h,\frac{hr}{r-k},\frac{hkr}{(h-k)(k-r)},\frac{hr}{h-k},...\right\}.$$

Or

$$x_{6n-2}=r,\quad x_{6n-1}=k,\quad x_{6n}=h,\quad x_{6n+1}=\frac{hr}{r-k},$$

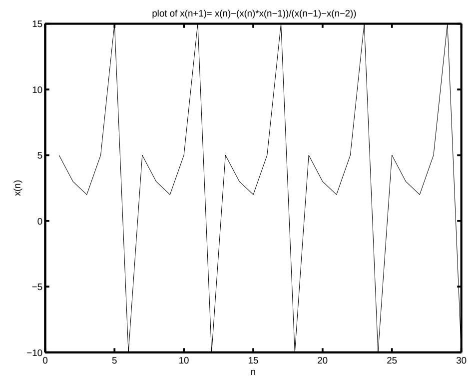

Figure 5.

$$x_{6n+2} = \frac{hkr}{(h-k)(k-r)}, \qquad x_{6n+3} = \frac{hr}{h-k}.$$

Proof. The proof is left to the reader. □

Fig. 5 shows the solution of Eq. (5.4) when $x_{-2} = 5,\ x_{-1} = 3,\ x_0 = 2$.

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