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# TRAVELING WAVES OF THE (3+1)-DIMENSIONAL KADOMTSEV-PETVIASHVILI-BOUSSINESQ EQUATION*

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**Abstract** In this paper, the bifurcation theory of dynamical system is applied to study the traveling waves of the (3+1)-dimensional Kadomtsev-Petviashvili-Boussinesq (KP-Boussinesq) equation. By transforming the traveling wave system of the KP-Boussinesq equation into a dynamical system in $R^3$, we derive various parameter conditions which guarantee the existence of its bounded and unbounded orbits. Furthermore, by calculating complicated elliptic integrals along these orbits, we obtain exact expressions of all possible traveling wave solutions of the (3+1)-dimensional KP-Boussines equation.

**Keywords** KP-Boussinesq equation, traveling waves, bifurcation, dynamical system.

**MSC(2010)** 58F15, 58F17, 53C35.

## 1. Introduction

Throughout the last several decades, many nonlinear partial differential equations (NPDEs) have been put forward to model a wide variety of nonlinear phenomena in various fields such as plasma physics, nonlinear optics, fluid dynamics, solid state physics, electromagnetic waves. As the first glimpse, traveling wave solutions (TWS) play vital roles in the study of NPDEs. From the mathematical point of view, TWS can be well used to describe the long time behaviour of a nonlinear partial differential equation. From the physical point of view, TWS is helpful to understand the complicated nonlinear wave phenomena and wave propagation. Especially, the soliton pulse, as an important class of traveling waves, indicates a perfect balance between nonlinearity and dispersion effects and is investigated widely [5–8, 20, 22].

This paper considers the following (3+1)-dimensional Kadomtsev-Petviashvili-

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Boussinesq (KP-Boussinesq) equation [23, 25]

$$U_{xxxy} + 3(U_xU_y)_x + U_{ty} + U_{tx} + U_{tt} - U_{zz} = 0, \tag{1.1}$$

where the real scalar function $u = u(x, y, z, t)$ is the height of the wave at the point $(x, y, z)$ in $\mathbb{R}^3$ and time $t$. If the term $u_{tt}$ is removed from equation (1.1), the (3+1)-dimensional KP-Boussinesq equation reduces to the well known generalized (3+1)-dimensional Kadomtsev-Petviashvili (KP) equation [17, 23–25]

$$U_{xxxy} + 3(U_xU_y)_x + U_{tx} + U_{ty} - U_{zz} = 0 \tag{1.2}$$

which can be used to describe the growth of quasi-one dimensional shallow water waves when the impact of surface tension and viscosity are minimal and is widely applied in various physics fields such as the internal and surface oceanic waves, ferromagnetics, nonlinear optics, Bose-Einstein condensation. Wazwaz [23] pointed out that this slight change, by adding the term $u_{tt}$ to the generalized (3+1)- dimensional KP equation (1.2), would make a drastic impact on the dispersion relation and the phase shift. Equation (1.1) can also be regarded as a new combination of equation (1.2) and the following generalized Boussinesq equation [25]

$$U_{xxxy} + 3(U_xU_y)_x + U_{tt} - U_{zz} = 0.$$

Due to the Boussinesq structure added to this new model, the (3+1)-dimensional KP-Boussinesq equation can model both right and left-going waves, as in the Boussinesq equation [25]. In addition, equation (1.1) has some advantages when it is applied to solve engineering problems. For example, during the research on the dynamics of the water it can provide a much more precise approximation than the KP equation and do not require any zero mass assumption [25].

In 2017, by applying a simplified Hirotas method to equation (1.1), Wazwaz [23] derived its one and two-soliton solutions where the coefficients of the spatial variables were left as free parameters. Yu [25] used a direct bilinear Bäcklund transformation to present some classes of exponential and rational traveling wave solutions with arbitrary wave numbers. Later, based on the Hirotas bilinear form, Kaur [10] got exact lump solutions of equation (1.1) under some restriction conditions. Subsequently, Verma and Kaur [19] explored that this equation could pass Painlevé test and was completely integrable. With the Bell polynomials approach and novel test function, they also constructed abundant new exact solutions in uniform manner. With the Hirota method, Lü [16] got the lump solution, interaction solution and breather-wave solution under certain constraints. Recently, by means of the Hirotas bilinear method combined with the perturbation expansion, Sun [18] constructed the general N-soliton solutions of the (3 +1)-dimensional KP-Boussinesq equation.

Although there are so many interesting results about the multi-soliton solutions of the (3 +1)-dimensional KP-Boussinesq equation, we note that, for the single wave solutions of this equation, only one type of exponential solution is given. It means that many single wave solutions could be still missing. As shown in [7, 21–23], the single wave solutions play important roles in constructing muti-soliton solutions. So, in this paper, our aim is to seek all possible single wave solutions of equation (1.1). In general, a traveling wave solution of a PDE usually corresponds to a orbit of its traveling wave system [2, 9, 11]. It is just the fact that makes the bifurcation theory of dynamical system [1, 3] become a powerful approach to investigate traveling waves of a PDE. In recent decades, many works have been done in the

field [4, 12–15, 26–29, 31]. Motivated by them, our strategy is to transform the traveling wave equation of equation (1.1) into a dynamical system in $\mathbb{R}^3$. Fortunately, there exists a 2-dimensional invariant manifold which determines most of dynamical behaviours. Then, bifurcation analysis is applied to seek the parameter bifurcation sets which determine various qualitatively different phase portraits. According to them, each orbit is identified clearly and investigated in detailed including bounded and unbounded one. Finally, by calculating complicated elliptic integrals along these orbits, we obtain analytic expressions of all traveling wave solutions of the (3 +1)-dimensional KP-Boussinesq equation without any loss. The obtained solutions well complement the types of traveling wave solutions of the (3+1)-dimensional KP-Boussinesq equation and are helpful to understand the complicated nonlinear wave phenomena and wave propagation, as well as help to construct more exact solutions of this equation including the multi-soliton solutions.

## 2. Traveling wave system and bifurcation analysis

With the traveling wave transformation

$$U = U(t, x, y, z) = u(\xi) = u(x + ay + bz - ct),$$

equation (1.1) can be converted into its traveling wave equation

$$au^{''''} + 3(a(u^{'})^2)^{'} - acu^{''} + c^2u^{''} - cu^{''} - b^2u^{''} = 0, \tag{2.1}$$

where $'$ stands for $d/d\xi$, $a \neq 0$ and $b \neq 0$ represent the wave numbers in the $y$ and $z$ direction respectively and $c \neq 0$ is the wave speed. Integrating (2.1) once, we have

$$au^{'''} + 3a(u^{'})^2 - (ac - c^2 + c + b^2)u^{'} = e, \tag{2.2}$$

where parameter $e$ is the integral constant. Equation (2.2) has the following equivalent form

$$\begin{cases} u^{'} = v, \\ v^{'} = y, \\ y^{'} = -3v^2 + \dfrac{ac - c^2 + c + b^2}{a}v + \dfrac{e}{a}, \end{cases} \tag{2.3}$$

which is a dynamical system in $\mathbb{R}^3$. Obviously, system (2.3) has a 2-dimensional invariant manifold in $\mathbb{R}^3$. Flows on it are governed by the last two equations in system (2.3), i.e.

$$\begin{cases} v^{'} = y, \\ y^{'} = -3v^2 + \dfrac{ac - c^2 + c + b^2}{a}v + \dfrac{e}{a}, \end{cases} \tag{2.4}$$

which is exactly a Hamiltonian system with the energy function

$$H(v, y) = \frac{1}{2}y^2 + v^3 - \frac{1}{2}Av^2 - \frac{e}{a}v, \tag{2.5}$$

where $A = \frac{ac - c^2 + c + b^2}{a}$.

Firstly, we start with equilibria of system (2.4).

**Theorem 2.1.** *If $A^2+\frac{12e}{a}>0$, system (2.4) has a center $B_1(\frac{A+\sqrt{A^2+\frac{12e}{a}}}{6},0)$ and a saddle $B_2(\frac{A-\sqrt{A^2+\frac{12e}{a}}}{6},0)$. If $A^2+\frac{12e}{a}=0$, system (2.4) has a unique cusp $B_3(\frac{A}{6},0)$. If $A^2+\frac{12e}{a}<0$, system (2.4) has no equilibrium.*

**Proof.** By the theory of dynamic system [1, 3, 30] and the method used in [26], it is not difficult to check that system (2.4) has a center $B_1$ and a saddle $B_2$ if $A^2+\frac{12e}{a}>0$, whereas it has no equilibrium if $A^2+\frac{12e}{a}<0$

If $A^2+\frac{12e}{a}=0$, system (2.4) has only one high order equilibrium $B_3(\frac{A}{6},0)$ with the degenerate Jacobian matrix

$$M(B_3)=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}.$$

In order to determine its type further, we make the following homeomorphic transformation

$$\varphi=v-\frac{A}{6},\psi=y,$$

which converts system (2.4) into the equivalent form below

$$\begin{cases}\varphi^{'}=\psi,\\ \psi^{'}=a_k\varphi^k[1+p(\varphi)]+b_n\varphi^k\psi[1+q(\varphi)]+\psi^2g(\varphi,\psi)=-3\varphi^2.\end{cases}$$

According to the qualitative theory of differential equation [30, Theorem 7.3, Chapter 2], we have $k=2$, $a_k=-3$ and $b_n=0$, which indicates that $B_3$ is a cusp. □

Next, we need to discuss global phase portraits of system (2.4) in different parameter bifurcation sets $\{(a,b,c,e)|A^2+\frac{12e}{a}>0\}$, $\{(a,b,c,e)|A^2+\frac{12e}{a}=0\}$ and $\{(a,b,c,e)|A^2+\frac{12e}{a}<0\}$. According to the properties of Hamiltonian system [1] and energy function (2.5), we have the following results.

**Case 1.** For $A^2+\frac{12e}{a}>0$, there is a homoclinic orbit $\gamma$ connecting the saddle $B_2$. Inside the homoclinic loop $\gamma\cup B_2$ there exists a family of periodic orbits

$$\Gamma(h)=\{H(v,y)=h,h\in(h(B_1),h(B_2))\},$$

which surround center $B_1$, where

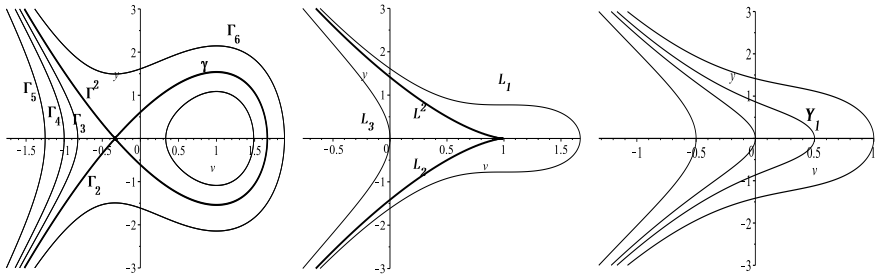
$$h(B_1)=\frac{-2A^3-(2A^2+\frac{24e}{a})\sqrt{A^2+\frac{12e}{a}}-\frac{36e}{a}A}{216},$$

$$h(B_2)=\frac{-2A^3+(2A^2+\frac{24e}{a})\sqrt{A^2+\frac{12e}{a}}-\frac{36e}{a}A}{216}.$$

Moreover, $\Gamma(h)$ tends to $B_1$ as h$\to h(B_1)$ and tends to $\gamma$ as h$\to h(B_2)$. Outside of the homoclinic loop $\gamma\cup B_2$, all orbits are unbounded, as shown in figure 1(a).

**Case 2.** For $A^2+\frac{12e}{a}=0$, all orbits of system (2.4) are unbounded. There are two special orbits $L_2$ and $L^2$ which are different from others. The $\omega$-limit set of $L^2$ and the $\alpha$-limit set of $L_2$ correspond to the same equilibrium $B_3$, as shown in figure 1(b).

**Case 3.** For $A^2+\frac{12e}{a}<0$, system (2.4) has only one type of orbits. All of them are unbounded, as shown in figure 1(c).




(a) $A = 2,\ \frac{e}{a} = 1$ (b) $A = 6,\ \frac{e}{a} = -3$ (c) $A = 2,\ \frac{e}{a} = -1$

**Figure 1.** The phase portraits of system (2.4) in different parameter bifurcation sets.

# 3. Exact solutions of system (2.4)

In this section, we try to seek explicit expressions of all solutions of system (2.4), including bounded and unbounded ones.

## 3.1. Bounded solutions of system (2.4)

For $A^2 + \frac{12e}{a} > 0$, the bounded orbits of system (2.4) exist. They comprise of the homoclinic orbit $\gamma$ and the family of periodic orbits $\Gamma(h)$ inside the homoclinic loop $\gamma \cup B_2$. Firstly, we consider the periodic orbits.

(B1) According to the energy function (2.5), any one of the periodic orbits $\Gamma(h)$ can be expressed by

$$y = \pm\sqrt{2}\sqrt{(v - v_1)(v - v_2)(v_3 - v)},$$

where $v_1$, $v_2$ and $v_3$ are reals and satisfy the constraint condition $v_1 < v_2 < v < v_3$. Assuming that the period is 2T and choosing initial value $v(0) = v_2$, we have

$$\int_{v_2}^{v} \frac{dv}{\sqrt{2}\sqrt{(v - v_1)(v - v_2)(v_3 - v)}} = \int_0^{\xi} d\xi, \quad 0 < \xi < T,$$

$$-\int_{v}^{v_2} \frac{dv}{\sqrt{2}\sqrt{(v - v_1)(v - v_2)(v_3 - v)}} = \int_{\xi}^{0} d\xi, \quad -T < \xi < 0,$$

which can be rewritten as

$$\int_{v_2}^{v} \frac{dv}{\sqrt{2}\sqrt{(v - v_1)(v - v_2)(v_3 - v)}} = \mid \xi \mid, \quad -T < \xi < T.$$

By calculating the elliptic integral

$$\int_{v_2}^{v} \frac{dv}{\sqrt{(v - v_1)(v - v_2)(v_3 - v)}} = g \cdot sn^{-1}\left(\sqrt{\frac{(v_3 - v_1)(v - v_2)}{(v_3 - v_2)(v - v_1)}}, k\right),$$

where $g = \frac{2}{\sqrt{v_3 - v_1}}$, $k^2 = \frac{v_3 - v_2}{v_3 - v_1}$, we get the expression of periodic solution of system (2.4)

$$v_{b1}(\xi) = v_1 + \frac{(v_2 - v_1)(v_3 - v_1)}{(v_3 - v_1) - (v_3 - v_2)sn^2\left(\sqrt{\frac{v_3 - v_1}{2}}\xi\right)}, \quad -T < \xi < T.$$

(B2) Similarly, the homoclinic orbit $\gamma$ can be expressed by

$$y = \pm\sqrt{2}\sqrt{(v-v_4)^2(v_5-v)},$$

where $v_4 = \frac{A-\sqrt{A^2+\frac{12e}{a}}}{6}$ and $v_5 = \frac{A+2\sqrt{A^2+\frac{12e}{a}}}{6}$ satisfy the condition that $v_4 < v < v_5$. Choosing $v(0) = v_5$, we have

$$\int_v^{v_5} \frac{dv}{\sqrt{2}(v-v_4)\sqrt{v_5-v}} = \int_\xi^0 d\xi, \quad \xi < 0,$$

$$-\int_{v_5}^{v} \frac{dv}{\sqrt{2}(v-v_4)\sqrt{v_5-v}} = \int_0^\xi d\xi, \quad \xi > 0,$$

which can be rewritten as

$$\int_{v_5}^{v} \frac{dv}{\sqrt{2}(v-v_4)\sqrt{v_5-v}} = -\mid \xi \mid, \quad -\infty < \xi < +\infty.$$

Noting that

$$\int_{v_5}^{v} \frac{dv}{(v-v_4)\sqrt{v_5-v}} = \frac{1}{\sqrt{v_5-v_4}} \ln \frac{\sqrt{v_5-v_4}-\sqrt{v_5-v}}{\sqrt{v_5-v_4}+\sqrt{v_5-v}},$$

we get the bell-shaped bounded solution of the system (2.4)

$$v_{b2}(\xi) = v_5 - \frac{(v_5-v_4)(1-\exp(\sqrt{2(v_5-v_4)}|\xi|))^2}{(1+\exp(\sqrt{2(v_5-v_4)}|\xi|))^2}, \quad -\infty < \xi < +\infty. \tag{3.1}$$

It is not difficult to check that expression (3.1) can be further simplified to

$$v_{b2}(\xi) = v_5 - \frac{(v_5-v_4)(1-\exp(\sqrt{2(v_5-v_4)}\xi))^2}{(1+\exp(\sqrt{2(v_5-v_4)}\xi))^2}, \quad -\infty < \xi < +\infty. \tag{3.2}$$

## 3.2. Unbounded solutions of system (2.4)

Except the homoclinic orbit and periodic orbits, all orbits of system (2.4) are unbounded. We need to discuss them in three cases.

**(I)** First of all, we start with the case that $A^2 + \frac{12e}{a} > 0$. This case incudes five subcases (U1-U5) according to different level curves of energy function (2.5).

(U1) Consider the first type of unbounded orbits $\Gamma^2$ and $\Gamma_2$ as shown in figure 1(a), whose energy is equal to the energy of the saddle $B_2$, as well as the energy of the homoclinic orbit $\gamma$. They can be expressed respectively by

$$y = \pm\sqrt{2}\sqrt{(v_4-v)^2(v_5-v)},$$

where $-\infty < v < v_4 < v_5$. Firstly, we take $\Gamma^2$ for example to calculate its corresponding solution. Given initial value $v(0) = -\infty$, we have

$$\int_{-\infty}^{v} \frac{dv}{\sqrt{2}(v_4-v)\sqrt{v_5-v}} = \int_0^\xi d\xi, \quad \xi > 0.$$

Noting that

$$\int_{-\infty}^{v} \frac{dv}{(v_4-v)\sqrt{v_5-v}} = -\frac{1}{\sqrt{v_5-v_4}} \ln \frac{\sqrt{v_5-v}-\sqrt{v_5-v_4}}{\sqrt{v_5-v}+\sqrt{v_5-v_4}},$$

we get the expression of the first type of unbounded solution of system (2.4)

$$v_{u1}(\xi) = v_5 - \frac{(v_5-v_4)(1+\exp(\sqrt{2(v_5-v_4)}\xi))^2}{(1-\exp(\sqrt{2(v_5-v_4)}\xi))^2}, \quad \xi > 0. \tag{3.3}$$

For orbit $\Gamma_2$, similar calculation can be used to derive its corresponding unbounded solution. One can check that it has the same form as $v_{u1}(\xi)$.

(U2) Consider the second type of unbounded orbits, for example $\Gamma_3$ shown in figure 1(a), whose energy is lower than energy of saddle $B_2$, but higher than energy of center $B_1$. Any one of them can be expressed by

$$y = \pm\sqrt{2}\sqrt{(v_6-v)(v_7-v)(v_8-v)},$$

where $v_6, v_7, v_8$ are reals and relationship $-\infty < v < v_6 < v_7 < v_8$ holds. Similar to the discussion in (U1), we only need to consider the upper branch of $\Gamma_3$. Choosing $v(0) = -\infty$, we have

$$\int_{-\infty}^{v} \frac{dv}{\sqrt{(v_6-v)(v_7-v)(v_8-v)}} = \int_0^{\xi} d\xi, \quad \xi > 0.$$

By calculating the elliptic integral

$$\int_{-\infty}^{v} \frac{dv}{\sqrt{(v_6-v)(v_7-v)(v_8-v)}} = g \cdot sn^{-1}(\sqrt{\frac{(v_8-v_6)}{(v_8-v)}}, k),$$

where $g = \frac{2}{\sqrt{v_8-v_6}}$, $k^2 = \frac{v_8-v_7}{v_8-v_6}$, we get the second type of unbounded solution of system (2.4)

$$v_{u2}(\xi) = v_8 - \frac{v_8-v_6}{sn^2(\sqrt{\frac{v_8-v_6}{2}}\xi)}, \quad 0 < \xi < \xi_1, \tag{3.4}$$

where $\xi_1 = \frac{2\sqrt{2}}{\sqrt{v_8-v_6}} \cdot \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\frac{v_8-v_7}{v_8-v_6}\cdot\sin^2\theta}}$.

(U3) Consider the unbounded orbit $\Gamma_4$ shown in figure 1(a), whose energy is equal to energy of center $B_1$. It can be expressed by

$$y = \pm\sqrt{2}\sqrt{(v_{10}-v)^2(v_9-v)},$$

where $v_9 = \frac{A-2\sqrt{A^2+\frac{12e}{a}}}{6}$ and $v_{10} = \frac{A+\sqrt{A^2+\frac{12e}{a}}}{6}$ are reals and relationship $-\infty < v_9 < v_{10}$ holds. Similarly, taking the upper branch of $\Gamma_4$ and choosing the initial value $v(0) = -\infty$, we have the following integral expression

$$\int_{-\infty}^{v} \frac{dv}{\sqrt{2}(v_{10}-v)\sqrt{(v_9-v)}} = \int_0^{\xi} d\xi, \quad \xi > 0.$$

Noting that

$$\int_{-\infty}^{v} \frac{dv}{(v_{10}-v)\sqrt{(v_9-v)}} = \frac{1}{\sqrt{v_{10}-v_9}}(\pi - \arctan\sqrt{\frac{v_9-v}{v_{10}-v_9}}),$$

we get the third type of unbounded solution of system (2.4)

$$v_{u3}(\xi) = v_9 - (v_{10} - v_9)\tan^2(\sqrt{2(v_{10} - v_9)}\xi), \quad 0 < \xi < \xi_2, \tag{3.5}$$

where $\xi_2 = \frac{\pi}{\sqrt{2(v_{10}-v_9)}}$.

(U4) Consider the fourth type of unbounded orbits, for example $\Gamma_5$ shown in figure 1(a), whose energy is lower than energy of center $B_1$. Any one of them has the form

$$y = \pm\sqrt{2}\sqrt{(v_{11} - v)[v^2 + (v_{11} - \frac{1}{2}A)v + (v_{11}^2 - \frac{1}{2}Av_{11} - \frac{e}{a})]},$$

where $-\infty < v < v_{11} < v_9$. Taking $v(0) = -\infty$, we have

$$\int_{-\infty}^{v} \frac{dv}{\sqrt{2}\sqrt{(v_{11} - v)[v^2 + (v_{11} - \frac{1}{2}A)v + (v_{11}^2 - \frac{1}{2}Av_{11} - \frac{e}{a})]}} = \int_0^{\xi} d\xi, \quad \xi > 0.$$

By calculating the elliptic integral

$$\int_{-\infty}^{v} \frac{dv}{\sqrt{(v_{11} - v)[v^2 + (v_{11} - \frac{1}{2}A)v + (v_{11}^2 - \frac{1}{2}Av_{11} - \frac{e}{a})]}} = g \cdot sn^{-1}(\frac{v_{11} - B - v}{v_{11} + B - v}, k),$$

where $B^2 = 3v_{11}^2 - Av_{11} - \frac{e}{a}, g = \frac{1}{\sqrt{B}}$ and $k^2 = \frac{4B+6v_{11}-A}{8B}$. We can get the fourth unbounded solution of system (2.4)

$$\begin{aligned} v_{u4}(\xi) =& v_{11} + \sqrt{3v_{11}^2 - Av_{11} - \frac{e}{a}} \\ & - \frac{2\sqrt{3v_{11}^2 - Av_{11} - \frac{e}{a}}}{1 - cn(\sqrt[4]{12v_{11}^2 - 4Av_{11} - \frac{4e}{a}}\xi)}, \quad 0 < \xi < \xi_3, \end{aligned} \tag{3.6}$$

where $\xi_3 = \frac{4}{\sqrt[4]{12v_{11}^2 - 4Av_{11} - \frac{4e}{a}}} \cdot \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - \frac{4\sqrt{3v_{11}^2 - Av_{11} - \frac{e}{a}} + 6v_{11} - A}{8\sqrt{3v_{11}^2 - Av_{11} - \frac{e}{a}}} \cdot \sin^2\theta}}$.

(U5) Consider the fifth type of unbounded orbits, for example $\Gamma_6$ shown in figure 1(a), whose energy is higher than energy of saddle $B_2$. Any one of them can be expressed by

$$y = \pm\sqrt{2}\sqrt{(v_{12} - v)[v^2 + (v_{12} - \frac{1}{2}A)v + (v_{12}^2 - \frac{1}{2}Av_{12} - \frac{e}{a})]},$$

where $v_{12} > \frac{A+2\sqrt{A^2+\frac{12e}{a}}}{6}$ and relationship $-\infty < v < v_{12}$ holds. Choosing $v(0) = -\infty$, we have

$$\int_{-\infty}^{v} \frac{dv}{\sqrt{2}\sqrt{(v_{12} - v)[v^2 + (v_{12} - \frac{1}{2}A)v + (v_{12}^2 - \frac{1}{2}Av_{12} - \frac{e}{a})]}} = \int_0^{\xi} d\xi, \quad \xi > 0.$$

By similar calculation to that in subcase (U4), we get the fifth unbounded solution of system (2.4)

$$\begin{aligned} v_{u5}(\xi) =& v_{12} + \sqrt{3v_{12}^2 - Av_{12} - \frac{e}{a}} \\ & - \frac{2\sqrt{3v_{12}^2 - Av_{12} - \frac{e}{a}}}{1 - cn(\sqrt[4]{12v_{12}^2 - 4Av_{12} - \frac{4e}{a}}\xi)}, \quad 0 < \xi < \xi_4, \end{aligned} \tag{3.7}$$

where $\xi_4 = \frac{4}{\sqrt[4]{12v_{12}^2 - 4Av_{12} - \frac{4e}{a}}} \cdot \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - \frac{4\sqrt{3v_{12}^2 - Av_{12} - \frac{e}{a}} + 6v_{12} - A}{8\sqrt{3v_{12}^2 - Av_{12} - \frac{e}{a}}} \cdot \sin^2\theta}}$.

**(II)** Next, we discuss the case that $A^2 + \frac{12e}{a} = 0$. This case includes two subcases (U6-U7) according to different level curves of energy function.

(U6) Consider the orbits $L_2$ and $L^2$ shown in figure 1(b), whose energy is equal to energy of the cusp $B_3$, which can be expressed by

$$y = \pm\sqrt{2}(\frac{A}{6} - v)\sqrt{\frac{A}{6} - v},$$

where $-\infty < v < \frac{A}{6}$. Similarly, we only need to discuss the orbit $L^2$. choosing $v(0) = -\infty$, we have

$$\int_{-\infty}^{v} \frac{dv}{\sqrt{2}(\frac{A}{6} - v)\sqrt{\frac{A}{6} - v}} = \int_0^{\xi} d\xi, \quad \xi > 0.$$

By a direct calculation, we get the sixth unbounded solution of system (2.4)

$$v_{u6}(\xi) = \frac{A}{6} - \frac{2}{\xi^2}, \quad \xi > 0. \tag{3.8}$$

(U7) Consider other unbounded orbits, for example $L_1$ and $L_3$ shown in figure 1(b), which can be uniformly expressed by

$$y = \pm\sqrt{2}\sqrt{(v_{13} - v)[v^2 + (v_{13} - \frac{1}{2}A)v + (v_{13}^2 - \frac{1}{2}Av_{13} - \frac{e}{a})]},$$

where $-\infty < v < v_{13}$ and $v_{13} \neq \frac{A}{6}$. Choosing $v(0) = -\infty$, we have

$$\int_{-\infty}^{v} \frac{dv}{\sqrt{2}\sqrt{(v_{13} - v)[v^2 + (v_{13} - \frac{1}{2}A)v + (v_{13}^2 - \frac{1}{2}Av_{13} - \frac{e}{a})]}} = \int_0^{\xi} d\xi, \quad \xi > 0.$$

A direct calculation leads to the seventh unbounded solution of system (2.4)

$$\begin{aligned} v_{u7}(\xi) =& v_{13} + \sqrt{3v_{13}^2 - Av_{13} - \frac{e}{a}} \\ & - \frac{2\sqrt{3v_{13}^2 - Av_{13} - \frac{e}{a}}}{1 - cn(\sqrt[4]{12v_{13}^2 - 4Av_{13} - \frac{4e}{a}}\xi)}, \quad 0 < \xi < \xi_5, \end{aligned} \tag{3.9}$$

where $\xi_5 = \frac{4}{\sqrt[4]{12v_{13}^2-4Av_{13}-\frac{4e}{a}}} \cdot \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\frac{4\sqrt{3v_{13}^2-Av_{13}-\frac{e}{a}}+6v_{13}-A}{8\sqrt{3v_{13}^2-Av_{13}-\frac{e}{a}}} \cdot \sin^2\theta}}$.

**(III)** Finally, we discuss the case that $A^2 + \frac{12e}{a} < 0$.

(U8) In this case, system (2.4) has only one type of unbounded orbits. Any one of them can be expressed by

$$y = \pm\sqrt{2}\sqrt{(v_{14}-v)[v^2+(v_{14}-\frac{1}{2}A)v+(v_{14}^2-\frac{1}{2}Av_{14}-\frac{e}{a})]},$$

where $-\infty < v < v_{14}$. Choosing $v(0) = -\infty$, we have

$$\int_{-\infty}^{v} \frac{dv}{\sqrt{2}\sqrt{(v_{14}-v)[v^2+(v_{14}-\frac{1}{2}A)v+(v_{14}^2-\frac{1}{2}Av_{14}-\frac{e}{a})]}} = \int_0^{\xi} d\xi, \quad \xi > 0.$$

Thus, we obtain the last type of unbounded solutions of system (2.4)

$$\begin{aligned} v_{u8}(\xi) =& v_{14} + \sqrt{3v_{14}^2 - Av_{14} - \frac{e}{a}} \\ & - \frac{2\sqrt{3v_{14}^2 - Av_{14} - \frac{e}{a}}}{1 - cn(\sqrt[4]{12v_{14}^2 - 4Av_{14} - \frac{4e}{a}}\xi)}, \quad 0 < \xi < \xi_6, \end{aligned} \tag{3.10}$$

where $\xi_6 = \frac{4}{\sqrt[4]{12v_{14}^2-4Av_{14}-\frac{4e}{a}}} \cdot \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\frac{4\sqrt{3v_{14}^2-Av_{14}-\frac{e}{a}}+6v_{14}-A}{8\sqrt{3v_{14}^2-Av_{14}-\frac{e}{a}}} \cdot \sin^2\theta}}$.

# 4. Traveling wave solutions of equation (1.1)

According to the relationship between (2.3) and (2.4), we can get the traveling wave solutions $u(\xi)$ of equation (1.1) by integrating the obtained solutions of system (2.4) with respect to $\xi$.

(S1) For system (2.4), any one of the periodic orbits can be expressed by

$$v_{b1}(\xi) = v_1 + \frac{(v_2-v_1)(v_3-v_1)}{(v_3-v_1)-(v_3-v_2)sn^2(\sqrt{\frac{v_3-v_1}{2}}\xi)}, \quad -T < \xi < T.$$

Noting that

$$\int \frac{du}{1 \pm k \cdot sn(u)} = \frac{1}{k'^2}[E(u) + k(1 \mp k \cdot sn(u))cd(u)],$$

where $k' = \sqrt{1-k^2}$, we calculate the first type of traveling wave solution of equation (1.1) as follows

$$u_1(\xi) = \int v_{b1}(\xi)d\xi = \int [v_1 + \frac{(v_2-v_1)(v_3-v_1)}{(v_3-v_1)-(v_3-v_2)sn^2(\sqrt{\frac{v_3-v_1}{2}}\xi)}]d\xi$$

$$
\begin{aligned}
&= \int [v_1 + \frac{v_2 - v_1}{1 - \frac{v_3 - v_2}{v_3 - v_1} \cdot sn^2(\sqrt{\frac{v_3 - v_1}{2}}\xi)}]d\xi \\
&= \int [v_1 + \frac{v_2 - v_1}{1 - k^2 \cdot sn^2(\sqrt{\frac{v_3 - v_1}{2}}\xi)}]d\xi \\
&= \int \{v_1 + \frac{v_2 - v_1}{2}[\frac{1}{1 - k \cdot sn(\sqrt{\frac{v_3 - v_1}{2}}\xi)} + \frac{1}{1 + k \cdot sn(\sqrt{\frac{v_3 - v_1}{2}}\xi)}]\}d\xi \\
&= v_1 \cdot \xi + \sqrt{2(v_3 - v_1)}[E(\sqrt{\frac{v_3 - v_1}{2}}\xi) + \sqrt{\frac{v_3 - v_2}{v_3 - v_1}} \cdot cd(\sqrt{\frac{v_3 - v_1}{2}}\xi)] + C_1,
\end{aligned}
$$

where $-T < \xi < T$ and $C_1$ is a constant.

(S2) Integrating (3.2) directly, we get the second type of traveling wave solution of equation (1.1)

$$
\begin{aligned}
u_2(\xi) &= \int v_{b2}(\xi)d\xi \\
&= \int [v_5 - \frac{(v_5 - v_4)(1 - \exp(\sqrt{2(v_5 - v_4)}\xi))^2}{(1 + \exp(\sqrt{2(v_5 - v_4)}\xi))^2}]d\xi \\
&= v_5 \cdot \xi + \sqrt{\frac{v_5 - v_4}{2}}[\frac{\sqrt{2(v_5 - v_4)}\xi - \exp(\sqrt{2(v_5 - v_4)}\xi) + \frac{1}{2}\exp(2\sqrt{2(v_5 - v_4)}\xi)}{1 + \exp(\sqrt{2(v_5 - v_4)}\xi)}] + C_2,
\end{aligned}
$$

where $-\infty < \xi < \infty$ and $C_2$ is a constant.

(S3) Integrating (3.3), we get the third type of traveling wave solution of equation (1.1)

$$
\begin{aligned}
u_3(\xi) &= \int v_{u1}(\xi)d\xi \\
&= \int [v_5 - \frac{(v_5 - v_4)(1 + \exp(\sqrt{2(v_5 - v_4)}\xi))^2}{(1 - \exp(\sqrt{2(v_5 - v_4)}\xi))^2}]d\xi \\
&= v_5 \cdot \xi + \sqrt{\frac{v_5 - v_4}{2}}[\frac{\sqrt{2(v_5 - v_4)}\xi + \exp(\sqrt{2(v_5 - v_4)}\xi) + \frac{1}{2}\exp(2\sqrt{2(v_5 - v_4)}\xi)}{1 - \exp(\sqrt{2(v_5 - v_4)}\xi)}] + C_3,
\end{aligned}
$$

where $\xi > 0$ and $C_3$ is a constant.

(S4) Integrating (3.4) leads to

$$
u_4(\xi) = \int v_{u2}(\xi)d\xi = \int [v_8 - \frac{v_8 - v_6}{sn^2(\sqrt{\frac{v_8 - v_6}{2}}\xi)}]d\xi.
$$

From the fact that

$$
\int \frac{du}{sn^2(u)} = \int ns^2(u)du = u - E(u) - dn(u) \cdot cs(u),
$$

we have the fourth type of traveling wave solution of equation (1.1)

$$
u_4(\xi) = v_6 \cdot \xi + \sqrt{2(v_8 - v_6)}[E(\sqrt{\frac{v_8 - v_6}{2}}\xi) + dn(\sqrt{\frac{v_8 - v_6}{2}}\xi) \cdot cs(\sqrt{\frac{v_8 - v_6}{2}}\xi)] + C_4,
$$

where $0 < \xi < \xi_1$ and $C_4$ is a constant.

(S5) Integrating (3.5) directly, we get the fifth type of traveling wave solution of equation (1.1)

$$\begin{aligned} u_5(\xi) &= \int v_{u3}(\xi)d\xi \\ &= \int [v_9 - (v_{10} - v_9)\tan^2(\sqrt{2(v_{10} - v_9)}\xi)]d\xi \\ &= v_{10} \cdot \xi - \sqrt{\frac{v_{10} - v_9}{2}}\tan(\sqrt{2(v_{10} - v_9)}\xi) + C_5, \end{aligned}$$

where $0 < \xi < \xi_2$ and $C_5$ is a constant.

(S6) Integrating (3.6) leads to

$$\begin{aligned} u_6(\xi) &= \int v_{u4}(\xi)d\xi \\ &= \int [v_{11} + \sqrt{3v_{11}^2 - Av_{11} - \frac{e}{a}} - \frac{2\sqrt{3v_{11}^2 - Av_{11} - \frac{e}{a}}}{1 - cn(\sqrt[4]{12v_{11}^2 - 4Av_{11} - \frac{4e}{a}})}]d\xi. \end{aligned}$$

Noting that

$$\int \frac{du}{1 - cn(u)} = u - E(u) - \frac{dn(u) \cdot sn(u)}{1 - cn(u)},$$

we have the sixth type of traveling wave solution of equation (1.1)

$$\begin{aligned} u_6(\xi) = &(v_{11} - \sqrt{3v_{11}^2 - Av_{11} - \frac{e}{a}})\xi + \sqrt[4]{12v_{11}^2 - 4Av_{11} - \frac{4e}{a}}[E(\sqrt[4]{12v_{11}^2 - 4Av_{11} - \frac{4e}{a}}\xi) \\ &+ \frac{dn(\sqrt[4]{12v_{11}^2 - 4Av_{11} - \frac{4e}{a}}\xi)sn(\sqrt[4]{12v_{11}^2 - 4Av_{11} - \frac{4e}{a}}\xi)}{1 - cn(\sqrt[4]{12v_{11}^2 - 4Av_{11} - \frac{4e}{a}}\xi)}] + C_6, \end{aligned}$$

where $0 < \xi < \xi_3$ and $C_6$ is a constant.

(S7) Integrating (3.7), we get the seventh type of traveling wave solution of equation (1.1)

$$\begin{aligned} u_7(\xi) &= \int v_{u5}(\xi)d\xi \\ &= \int [v_{12} + \sqrt{3v_{12}^2 - Av_{12} - \frac{e}{a}} - \frac{2\sqrt{3v_{12}^2 - Av_{12} - \frac{e}{a}}}{1 - cn(\sqrt[4]{12v_{12}^2 - 4Av_{12} - \frac{4e}{a}})}]d\xi. \\ &= (v_{12} - \sqrt{3v_{12}^2 - Av_{12} - \frac{e}{a}})\xi + \sqrt[4]{12v_{12}^2 - 4Av_{12} - \frac{4e}{a}}[E(\sqrt[4]{12v_{12}^2 - 4Av_{12} - \frac{4e}{a}}\xi) \\ &+ \frac{dn(\sqrt[4]{12v_{12}^2 - 4Av_{12} - \frac{4e}{a}}\xi)sn(\sqrt[4]{12v_{12}^2 - 4Av_{12} - \frac{4e}{a}}\xi)}{1 - cn(\sqrt[4]{12v_{12}^2 - 4Av_{12} - \frac{4e}{a}}\xi)}] + C_7, \end{aligned}$$

where $0 < \xi < \xi_4$ and $C_7$ is a constant.

(S8) Integrating (3.8) directly leads to the eighth type of traveling wave solution of equation (1.1)

$$u_8(\xi) = \int v_{u6}(\xi)d\xi = \int (\frac{A}{6} - \frac{2}{\xi^2})d\xi = \frac{A}{6}\cdot\xi + \frac{2}{\xi} + C_8,$$

where $\xi > 0$ and $C_8$ is a constant.

(S9) Integrating (3.9), we get the ninth type of traveling wave solution of equation (1.1)

$$\begin{aligned} u_9(\xi) &= \int v_{u7}(\xi)d\xi \\ &= \int [v_{13} + \sqrt{3v_{13}^2 - Av_{13} - \frac{e}{a}} - \frac{2\sqrt{3v_{13}^2 - Av_{13} - \frac{e}{a}}}{1 - cn(\sqrt[4]{12v_{13}^2 - 4Av_{13} - \frac{4e}{a}})}]d\xi. \\ &= (v_{13} - \sqrt{3v_{13}^2 - Av_{13} - \frac{e}{a}})\xi + \sqrt[4]{12v_{13}^2 - 4Av_{13} - \frac{4e}{a}}[E(\sqrt[4]{12v_{13}^2 - 4Av_{13} - \frac{4e}{a}}\xi) \\ &+ \frac{dn(\sqrt[4]{12v_{13}^2 - 4Av_{13} - \frac{4e}{a}}\xi)sn(\sqrt[4]{12v_{13}^2 - 4Av_{13} - \frac{4e}{a}}\xi)}{1 - cn(\sqrt[4]{12v_{13}^2 - 4Av_{13} - \frac{4e}{a}}\xi)}] + C_9, \end{aligned}$$

where $0 < \xi < \xi_5$ and $C_9$ is a constant.

(S10) At last, integrating (3.10), we get the tenth type of traveling wave solution of equation (1.1)

$$\begin{aligned} u_{10}(\xi) &= \int v_{u8}(\xi)d\xi \\ &= \int [v_{14} + \sqrt{3v_{14}^2 - Av_{14} - \frac{e}{a}} - \frac{2\sqrt{3v_{14}^2 - Av_{14} - \frac{e}{a}}}{1 - cn(\sqrt[4]{12v_{14}^2 - 4Av_{14} - \frac{4e}{a}})}]d\xi. \\ &= (v_{14} - \sqrt{3v_{14}^2 - Av_{14} - \frac{e}{a}})\xi + \sqrt[4]{12v_{14}^2 - 4Av_{14} - \frac{4e}{a}}[E(\sqrt[4]{12v_{14}^2 - 4Av_{14} - \frac{4e}{a}}\xi) \\ &+ \frac{dn(\sqrt[4]{12v_{14}^2 - 4Av_{14} - \frac{4e}{a}}\xi)sn(\sqrt[4]{12v_{14}^2 - 4Av_{14} - \frac{4e}{a}}\xi)}{1 - cn(\sqrt[4]{12v_{14}^2 - 4Av_{14} - \frac{4e}{a}}\xi)}] + C_{10}, \end{aligned}$$

where $0 < \xi < \xi_6$ and $C_{10}$ is a constant.

## 5. Discussion and conclusion

In this paper, we investigate traveling wave system of the (3+1)-dimensional KP-Boussinesq equation. Although it is a high dimensional dynamical system, we find that there exists a 2-dimensional invariant manifold which makes it possible to completely investigate all bounded and unbounded orbits of it by detailed analysing the phase space geometry. By dynamical system methods and theory of elliptic integral, we obtain all single wave solutions of this equation without any loss. The strategy used in this paper can be applied to other similar high dimensional nonlinear wave

field. The obtained solution are helpful to understand nonlinear wave phenomena and wave propagation in high dimensional space, as well as facilitate the verification of numerical solutions. More importantly, as shown in [7, 19, 21–23], one can combine them with other types of solutions to construct more exact solution including the muti-soliton solutions.

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