A THREE-DIMENSIONAL NONLINEAR SYSTEM WITH A SINGLE HETEROCLINIC TRAJECTORY

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Abstract The study for singular trajectories of three-dimensional (3D) nonlinear systems is one of recent main interests. To the best of our knowledge, among the study for most of Lorenz or Lorenz-like systems, a pair of symmetric heteroclinic trajectories is always found due to the symmetry of those systems. Whether or not does there exist a 3D system that possesses a single heteroclinic trajectory? In the present note, based on a known Lorenz-type system, we introduce such a 3D nonlinear system with two cubic terms and one quadratic term to possess a single heteroclinic trajectory. To show its characters, we respectively use the center manifold theory, bifurcation theory, Lyapunov function and so on, to systematically analyse its complex dynamics, mainly for the distribution of its equilibrium points, the local stability, the expression of locally unstable manifold, the Hopf bifurcation, the invariant algebraic surface, and its homoclinic and heteroclinic trajectories, etc. One of the major results of this work is to rigorously prove that the proposed system has a single heteroclinic trajectory under some certain parameters. This kind of interesting phenomenon has not been previously reported in the Lorenz system family (because the huge amount of related research work always presents a pair of heteroclinic trajectories due to the symmetry of studied systems). What's more key, not like most of Lorenz-type or Lorenz-like systems with singularly degenerate heteroclinic cycles and chaotic attractors, the new proposed system has neither singularly degenerate heteroclinic cycles nor chaotic attractors observed. Thus, this work represents an enriching contribution to the understanding of the dynamics of Lorenz attractor.

Keywords Three-dimensional nonlinear system, single heteroclinic trajectory, Hopf bifurcation, Lyapunov function.

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1. Introduction

During past two decades, the investigations on singular trajectories and bifurcations have been turned out to be the lands of great promise for nonlinear dynamical systems. This is because these investigations not only characterize the complex dynamics [13,25,36,37] of those models themselves, but also involve with numerous

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applications [3, 4, 9, 10, 38, 39], such as electrophysics, heart tissue, neurons, cell signalling, planetary field and so on.

Homoclinic and heteroclinic trajectories are an important part of global bifurcation of dynamical system, to which many scientists and engineers [25, 36, 37] have paid a great deal of attention. Their occurrence often leads to the birth of chaos in corresponding systems. How many kinds of chaos are there under their occurrence? In the sense of Ši'lnikov type [23, 25–27], it is traditionally suggested that chaos occurring in 3D quadratic autonomous differential systems may be mainly classified into the following four cases:

 (i) chaos of the Ši'lnikov homoclinic-trajectory type;

 (ii) chaos of the Ši'lnikov heteroclinic-trajectory type;

(iii) chaos of the hybrid type with both Ši'lnikov homoclinic and heteroclinic trajectories;

(iv) chaos of other types.

Meanwhile, many important theories [25, 36, 37] and effective methods [12, 14, 28] have been developed and formulated in the course of the continuous research for detecting homoclinic and heteroclinic trajectories, for example, Poincaré map, Melnikov method, Lyapunov function, Fishing principle, a method of tracing the stable and unstable manifolds, etc.

Here, what we emphasize is the method Li et al formulated in [14] for proving the existence of heteroclinic trajectories: Lyapunov function combining the definitions of the $\alpha-$ and ω-limit set. Moreover, it has been demonstrated that this method can be effectively applied to Lorenz-type systems family: the Chen system [14], the Yang - Chen system or Yang system [18, 40] (in the sense of similar Vanecek and Celikovsky [26] , Yang-Chen system also connects the original Lorenz system and the original Chen system and represents a transition from one to the other), the T and Lü system [29], the general Lorenz family [19], other Lorenz-type systems [2, 15–17, 30–32], the unified Lorenz-type system [33], the complex Lorenz system [34] and the 5D hyperchaotic system [35]. Precisely speaking, it is noticed that those aforementioned models have two symmetric or infinitely many heteroclinic trajectories to origin and the non-zero symmetric equilibria. In addition, as Fishing principle, this method itself has the advantage: one need not consider the mutual disposition of stable and unstable manifolds of a saddle equilibrium in contrast with another technique of proving the existence of homoclinic and heteroclinic trajectories. For related work, see also [1, 6–8, 20–22, 41–45].

However, in neighboring Lorenz-type systems, the scenario for an asymmetric heteroclinic trajectory has not been considered in any publications to the best of our knowledge. Therefore, the following questions naturally arise:

(1) Whether does there exist such a model with a single heteroclinic trajectory or not?

(2) If there is such a model, whether or not is the aforementioned technique (combining the definitions of the $\alpha-$ and ω-limit set, and Lyapunov function) applicable to prove the existence of its heteroclinic trajectory?

(3) Except for the heteroclinic trajectory, whether do there exist other rich dynamics like the Lorenz-type system family? For example, chaotic attractors, Hopf bifurcation, invariant algebraic surface, etc.

In the present work, we devote to solving these problems one after another. Indeed, the new system proposed is found to have some other interesting dynamics, which are the essential differences with most of Lorenz or Lorenz-like systems.

1. Most of Lorenz or Lorenz-like systems possess singularly degenerate heteroclinic cycles and chaotic attractors whereas the new proposed system has neither singularly degenerate heteroclinic cycles nor chaotic attractors observed.

2. The local dynamics of non-isolate equilibria $(0, 0, z)$ for most of Lorenz or Lorenz-like systems are related to the variable z whereas the local dynamics of nonisolate equilibria $(0, 0, z)$ of the new system has nothing with the variable z.

These discoveries make the new system proposed more interesting.

The rest of this paper is organized as follows. Section 2 introduces the new 3D non-symmetric nonlinear system. Section 3 performs the local stability and bifurcation analysis of this system. Section 4 studies its global bifurcation problem of this system-the existence of homoclinic and heteroclinic trajectories. It is found that the system has a single heteroclinic trajectory but no homoclinic trajectories when $b \geq 3a > 0$ and $c \neq 0$. Finally, some conclusions are drawn in Section 5.

2. New 3D nonlinear system

In [40], the authors studied the Lorenz-type system

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = cx - xz, \\ \dot{z} = -bz + xy, \end{cases} \tag{2.1}$$

Obviously, in this system, the change rates of y and z are linear functions for the variable x. How about replacing x into x^2 ? Are there new dynamics to occur? Based on this idea, we introduce the following new 3D nonlinear system

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = cx^2 - x^2 z, \\ \dot{z} = -bz + x^2 y, \end{cases} \tag{2.2}$$

where $a, b, c \in \mathbb{R}$ with $a \neq 0$. Different from most Lorenz-type system family with only quadratic terms, system (2.2) evidently has two cubic terms and is not symmetric with respect to either x-axis, or y-axis or z-axis.

It is easy to introduce system (2.2). Anyway, what one is really concerned with is whether or not system (2.2) possesses some new unique natures. Fortunately, one finds that system (2.2) indeed possesses some unique characters, which is our real purpose to write this paper. Actually, one will see in the sequel that system (2.2) possesses a unique heteroclinic orbit, which different from most of known Lorenz-type or Lorenz-like systems. Moreover, system (2.2) has no singularly degenerate heteroclinic cycles and chaotic attractors have not been observed under numerical simulation so far.

In order to show in detail those unique characters of system (2.2), in the following, one studies the local and global dynamical behaviors of system (2.2) by using

respectively the center manifold theory, bifurcation theory and Lyapunov function, mainly including the distribution of equilibria, the stability, the expression of locally unstable manifold, Hopf bifurcation, the invariant algebraic surface, and the homoclinic and heteroclinic trajectories, etc.

3. Dynamical analysis of singular points of system (2.2)

First of all, the following assertion holds for the equilibria of system (2.2).

Theorem 3.1. *The distribution of the equilibria of system (2.2) is as follows.*

1. *When $c = 0$ and $b \neq 0$, the origin $S_0 = (0,0,0)$ is the single equilibrium point of system (2.2).*

2. *When $b = 0$, system (2.2) has the non-isolated equilibria $S_z = (0,0,z)$ for any $z \in \mathbb{R}$.*

3. *While $bc \neq 0$, $S' = (\sqrt[3]{bc}, \sqrt[3]{bc}, c)$ is the non-trivial equilibrium point of system (2.2) except for S_0.*

Proof.   The proof follows easily from the algebraic structure equation for equilibrium points of system (2.2). The details for the proof are omitted here. $\square$

3.1. Analysis of S_0

The Jacobian matrix of the linearized system (2.2) at the equilibrium S_0 has the eigenvalues and the corresponding eigenvectors given by:

$$\lambda_1 = 0, \quad \xi_1 = (1,1,0)^T,$$

$$\lambda_2 = -a, \quad \xi_2 = (1,0,0)^T,$$

$$\lambda_3 = -b, \quad \xi_3 = (0,0,1)^T.$$

Therefore, the equilibrium S_0 is non-hyperbolic and unstable for $a < 0$ or $b < 0$. In the following, one considers the stability of S_0 for the case $a > 0$ and $b > 0$ by using the center manifold theorem.

Lemma 3.1. *Suppose $a > 0$, $b > 0$ and $c \neq 0$, then the equilibrium S_0 is unstable.*

Proof.   The transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ s \end{pmatrix}$$

changes system (2.2) into

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{s} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -b \end{pmatrix} \begin{pmatrix} u \\ v \\ s \end{pmatrix} + (u+v)^2 \begin{pmatrix} c - s \\ -(c+s) \\ u \end{pmatrix}. \tag{3.1}$$

In light of the center manifold theory, there exists a center manifold for (3.1) which can be locally represented as follows

$$W_{loc}^c(S_0) = \left\{ \begin{pmatrix} u \\ v \\ s \end{pmatrix} \middle| \begin{array}{l} |u| \ll 1, \\ v = h_1(u), h_1(0) = 0, h_1'(0) = 0, \\ s = h_2(u), h_2(0) = 0, h_2'(0) = 0 \end{array} \right\}. \tag{3.2}$$

We now compute $W_{loc}^c(S_0)$. Assume that $h_{1,2}(u)$ have the following forms

$$v = h_1(u) = \Sigma_{k=2}^\infty a_k u^k, \tag{3.3}$$
$$s = h_2(u) = \Sigma_{k=2}^\infty b_k u^k.$$

Substitute (3.3) into the equation (3.1), where $h_{1,2}(u)$ must satisfy the center manifold. We then equate powers of u, and in that way we can compute $h_{1,2}(u)$ to any desired order of accuracy. In practice, computing only a few terms is usually sufficient to answer the question for the stability of S_0, and it is easy to obtain

$$h_1(u) = -\frac{c}{a}u^2 + \frac{4c^2}{a^2}u^3 + O(u^4), \tag{3.4}$$
$$h_2(u) = \frac{1}{b}u^3 + O(u^4),$$

where $O(u^4)$ expresses those terms with orders being equal or greater than four.

Substituting the expression (3.4) into system (3.1), the vector field restricted to the center manifold is given by

$$\dot{u} = cu^2 + O(u^3). \tag{3.5}$$

So, from (3.5) one can see that, for u sufficient small, $u = 0$ is unstable for $c \neq 0$. Hence, by the center manifold theorem, when $a > 0$, $b > 0$ and $c \neq 0$, the equilibrium S_0 is unstable. $\square$

Considering $a > 0$, $b > 0$ and $c \neq 0$, it follows from Lemma 3.1 that the equilibrium S_0 is non-hyperbolic, and has a one-dimensional unstable manifold $W_{loc}^u(S_0)$ and a two-dimensional stable manifold $W_{loc}^s(S_0)$. The tangent unstable subspace $TW_{loc}^u(S_0)$ is given by

$$TW_{loc}^u(S_0) = \{(x, y, z) \in \mathbb{R}^3 | y = x, z = 0\}.$$

The unstable manifold $W_{loc}^u(S_0)$ contains the equilibrium S_0 and is tangent to $TW_{loc}^u(S_0)$ at S_0.

According to the proof of Lemma 3.1, we get

$$W_{loc}^u(S_0) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \begin{array}{l} |y| \ll 1, \\ x = y - \frac{c}{a}y^2 + \frac{4c^2}{a^2}y^3 + O(y^4), \\ z = \frac{1}{b}y^3 + O(y^4). \end{array} \right\}. \tag{3.6}$$

Note that $W_{loc}^u(S_0)$ is indeed tangent to $TW_{loc}^u(S_0)$ since $x'(0) = 1$, $z'(0) = 0$ and the vector $(x'(0), 1, 0)$ is collinear to the direction vector ξ_2 of the line $TW_{loc}^u(S_0)$. Note also that the z-axis is included in the stable manifold $W_{loc}^s(S_0)$.

The following consequence is easily derived.

Table 1. The dynamical behavior of the equilibrium S_0 of system (2.2).

b	a	c	Property of S_0
	< 0	$\neq 0$	a 3D W_{loc}^u
	< 0	$= 0$	a 1D W_{loc}^c and a 2D W_{loc}^u
< 0	> 0	$\neq 0$	a 1D W_{loc}^s and a 2D W_{loc}^u
	> 0	$= 0$	a 1D W_{loc}^s, a 1D W_{loc}^c and a 1D W_{loc}^u
	< 0		a 2D W_{loc}^c and a 1D W_{loc}^u
$= 0$	> 0		a 1D W_{loc}^s and a 2D W_{loc}^c
	< 0	$\neq 0$	a 1D W_{loc}^s and a 2D W_{loc}^u
	< 0	$= 0$	a 1D W_{loc}^s, a 1D W_{loc}^c and a 1D W_{loc}^u
> 0	> 0	$\neq 0$	a 2D W_{loc}^s and a 1D W_{loc}^u
	> 0	$= 0$	a 2D W_{loc}^s and a 1D W_{loc}^c

Theorem 3.2. *The dynamical behavior of system (2.2) at the equilibrium point S_0 is totally tabulated in the following Table 1 when $a \neq 0$ and $(b,c) \in \mathbb{R}^2$.*

Remark 3.1. It follows from the Table 1 that the dynamical behavior of system (2.2) at the equilibrium point S_0 for $c = 0$ is still open and needs further endeavors.

3.2. Analysis of equilibria S_z and S'

First let us consider the dynamics of equilibria $S_z = 0$. The matrix associated with the vector field (2.2) linearized at every one of S_z has the same eigenvalues $\lambda_1 = -a$, $\lambda_{2,3} = 0$. Correspondingly, S_z has a 2D W_{loc}^c and a 1D W_{loc}^u (resp. W_{loc}^s) when $a < 0$ (resp. $a > 0$).

Remark 3.2. It is easy to notice that the dynamics of the non-isolated equilibria S_z has nothing with the values of the variable z, which is different from most of Lorenz-type system family. Furthermore, the singularly degenerate heteroclinic cycle does not exist at all for system (2.2). Consequently, the route to chaos for most chaotic systems, i.e., the collapse of singularly degenerate heteroclinic cycles leads to the birth of chaotic attractors, can not be found from system (2.2).

Next, one investigates the dynamical behaviors of the equilibrium S', which implies that the parameters a, b and c belong to the set $W = \{(a,b,c) \in \mathbb{R}^3 | abc \neq 0\}$. Notice the characteristic equation for the Jacobian matrix of the linearized system (2.2) at S' to take this form

$$\lambda^3 + (a+b)\lambda^2 + b(a + c\sqrt[3]{bc})\lambda + 3abc\sqrt[3]{bc} = 0. \tag{3.7}$$

For convenience of next discussion, one first divides the set W into $W_1 = \{(a,b,c) \in W : a < 0\}$, and $W_2 = \{(a,b,c) \in W : a > 0\}$, which is split into $W_{21} \cup W_{22}$ with

$$W_{21} = \{(a,b,c) \in W_2 : a + b > 0\},$$

$$W_{22} = \{(a,b,c) \in W_2 : a + b \leq 0\}.$$

W_{21} is continued to make the following divisions:

$$W_{211} = \{(a,b,c) \in W_{21} : b \geq 2a\},$$
$$W_{212} = \{(a,b,c) \in W_{21} : 0 < b < 2a\},$$
$$W_{213} = \{(a,b,c) \in W_{21} : b < 0\}.$$

And W_{212} is written as a union of the following three subsets W_{212}^1, W_{212}^2 and W_{212}^3

$$W_{212}^1 = \{(a,b,c) \in W_{212} : -c_* < c < 0 \quad \text{or} \quad 0 < c < c_*\},$$
$$W_{212}^2 = \{(a,b,c) \in W_{212} : c = c_* \quad \text{or} \quad c = -c_*\},$$
$$W_{212}^3 = \{(a,b,c) \in W_{212} : c < -c_* \quad \text{or} \quad c > c_*\},$$

where $c_* = \sqrt[4]{\frac{[a(a+b)]^3}{b(2a-b)^3}}$.

Concerning with the stability of S', the following assertion holds.

Theorem 3.3. *The equilibrium S' is unstable for $(a,b,c) \in W_1 \cup W_{22} \cup W_{212}^3 \cup W_{213}$ whereas asymptotically stable for $(a,b,c) \in W_{211} \cup W_{212}^1$.*

Proof. According to the Routh-Hurwitz stability criterion [24, p.58], the necessary and sufficient condition for the roots of Eq. (3.7) to have negative real parts is

$$a + b > 0, \, abc\sqrt[3]{bc} > 0, \, b(a+b)(a + c\sqrt[3]{bc}) - 3abc\sqrt[3]{bc} > 0.$$

This is equivalent to

$$b > -a, a > 0, ab(a+b) > (2a-b)bc\sqrt[3]{bc}.$$

Namely, $(a,b,c) \in W_{211} \cup W_{212}^1$. So, the proof follows and the other detail is omitted here. $\square$

Theorem 3 implies a birth of bifurcation at S' for $(a,b,c) \in W_{212}^2$. It is a Hopf bifurcation. So, finally, let us study the Hopf bifurcation of equilibrium point S'. Combining the Hopf bifurcation theory and symbolic computation [11], one arrives at the following result.

Theorem 3.4. *For $(a,b,c) \in W_{212}^2$, system (2.2) undergoes a Hopf bifurcation at the equilibrium S'. Further, the first Lyapunov coefficient of system (2.2) at S' is given by*

$$l_1(a,b) = -\frac{a^2\sqrt{ab}N_1}{\sqrt{\frac{ab(a+b)}{2a-b}}D_1 D_2}, \tag{3.8}$$

where

$$N_1 = 2a^5 - 59a^4 b + 219a^3 b^2 - 118a^2 b^3 + ab^4 + 3b^5,$$
$$D_1 = 2a^3 + 15a^2 b - b^3,$$
$$D_2 = 2a^3 + 6a^2 b - b^3.$$

Noticing $D_{1,2} > 0$ for $(a,b,c) \in W_{212}^2$, the sign of $l_1(a,b)$ depends on the one of N_1. The periodic trajectory bifurcating from the equilibrium S' is stable (resp. unstable)

for $N_1 > 0$ (resp. $N_1 < 0$). When $N_1 = 0$, one has to compute the second or the third or even more higher order Lyapunov coefficient to determine the stability of the bifurcated periodic trajectory.

Proof. The proof can be carried out according to two cases: (1) $c = c_*$, (2) $c = -c_*$. For each case, one only verifies the following two conditions for Hopf bifurcation to occur: (i) transversality, (ii) nondegeneracy.

Firstly, one considers the transversality. It follows from the relation between roots and coefficients of an algebraic equation with order 3 that the Eq. (3.7) has one negative real root $\lambda_1 = -(a + b)$ and a pair of conjugate purely imaginary roots $\lambda_{2,3} = \pm\omega i$ with $\omega = \sqrt{\frac{3a^2b}{2a-b}}$. Then calculating the derivative of both sides of Eq. (3.7) with respect to the parameter c and substituting the λ with ωi yield

$$\frac{dRe(\lambda_2)}{dc}\bigg|_{c=c_*} = \frac{2a^2b^2\sqrt[3]{b}\sqrt[3]{c_*}}{w^2[w^2 + (a+b)^2]} > 0.$$

Hence, the transversal condition holds.

Next, it is time for one to verify the nondegeneracy of S' by employing the project method in [11]. Fellow the following steps.

First of all, take the following linear transformation

$$T : (x, y, z) \rightarrow (x + v, y + v, z + c_*)$$

with $v = \sqrt[3]{bc_*}$ so that system (2.2) can be converted into the following equivalent system

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = -v^2z - 2vxz - x^2z, \\ \dot{z} = 2v^2x + v^2y - bz + vx^2 + 2vxy + x^2y. \end{cases} \tag{3.9}$$

And the equilibrium S' of system (2.2) is transformed into the origin S_0 of system (3.9).

For $c = c_*$, the Jacobian matrix of system (3.9) at S_0 is given by

$$A = \begin{pmatrix} -a & a & 0 \\ 0 & 0 & -v^2 \\ 2v^2 & v^2 & -b \end{pmatrix}$$

and the corresponding eigenvalues are

$$\lambda_1 = -(a + b), \quad \lambda_{2,3} = \pm\omega i.$$

Some tedious calculations display that

$$p = \frac{1}{J} \begin{pmatrix} 2v^2 \\ -\frac{(a-\omega)(b-\omega)}{v^2} \\ a - \omega i \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} a \\ a + \omega i \\ -\frac{i\omega(a+\omega i)}{v^2} \end{pmatrix}$$

satisfy $Aq = i\omega q$, $A^T p = -i\omega p$, $\langle p, q\rangle = \sum_{i=1}^{3} \bar{p}_i q_i = 1$, where

$$J = 2av^2 - \frac{(a + \omega i)^2(b - 2\omega i)}{v^2}.$$

It is also easy to derive

$$B(x, y) = \begin{pmatrix} 0 \\ -2v(x_1 y_3 + x_3 y_1) \\ 2v(x_1 y_2 + x_2 y_1 + x_1 y_1) \end{pmatrix}$$

and

$$C(x, y, z) = \begin{pmatrix} 0 \\ -2(x_1 y_1 z_3 + x_1 y_3 z_1 + x_3 y_1 z_1) \\ 2(x_1 y_1 z_2 + x_1 y_2 z_1 + x_2 y_1 z_1) \end{pmatrix}.$$

Some further computations tell us $h_{11} = (-\frac{2a^2}{v} - \frac{4ab\omega^2}{3v^5}, -\frac{2a^2}{v} - \frac{4ab\omega^2}{3v^5}, -\frac{4a\omega^2}{v^3})$ and $h_{20} = \frac{1}{L}(h_{20}^1, h_{20}^2, h_{20}^3)$, where

$$\begin{cases} L = v(4a\omega^2 i + 2v^4\omega - 8\omega^3 + 2ab\omega + 4b\omega^2 i - 3av^4 i), \\ h_{20}^1 = 2a^2(2b\omega^2 i - 4\omega^3 + 2ab\omega + 4a\omega^2 i) + 2a^2 v^4(-2\omega + 3ai), \\ h_{20}^2 = 2a(-2\omega + ai)(3av^4 + 4a\omega^2 + 2b\omega^2 + 2v^4\omega i + 4\omega^3 i - 2ab\omega i), \\ h_{20}^3 = 4av^2\omega(a + \frac{3\omega i}{2})(6a + 4\omega i). \end{cases}$$

At last, substituting the results calculated above into the expression in [11, Definition, Eq. (3.20), p. 99], one obtains the first Lyapunov coefficient which is just given by (3.8).

Case 2: $c = -c_*$.

The computational procedures for this case are similar to the ones of Case 1 and so omitted here. Notice that the first Lyapunov coefficient for both Case 1 and Case 2 are completely same, i.e. the expression (3.8).

The numerical simulation agrees with the theoretical analysis, see Fig. 1.

Since $(a, b, c) \in W_{212}^2$, $D_1 = 2a^3 + 15a^2 b - b^3 = 2a^3 + 11a^2 b + b(2a - b)(2a + b) > 0$, $D_2 = 2a^3 + 6a^2 b - b^3 = 2a^3 + 2a^2 b + b(2a - b)(2a + b) > 0$, the sign of l_1 depends on the one of $N_1 = 2a^5 - 59a^4 b + 219a^3 b^2 - 118a^2 b^3 + ab^4 + 3b^5$.

Now denote $N_1^+ = \{(a, b, c) \in N_1 | l_1(a, b) > 0\}$, $N_1^0 = \{(a, b, c) \in N_1 | l_1(a, b) = 0\}$, and $N_1^- = \{(a, b, c) \in N_1 | l_1(a, b) < 0\}$. Noticing that $l_1(1.61, 1.05) = 264.7280 > 0$, $l_1(0.04644, 0.08) = 2.9168 \times 1e - 007 \approx 0$ and $l_1(2.88, 0.4) = -452.8436 < 0$ and the continuation of $l_1(a, b)$ in W_{212}^2, it is easy to obtain that N_1^+, N_1^0 and N_1^- are all nonempty. The graph for $l_1(a, b)$ in Fig. 2 also verifies the fact above observed. □

4. Homoclinic and heteroclinic trajectories

This section aims to deal with the global bifurcation of system (2.2) on the existence of homoclinic and heteroclinic trajectories. The main conclusions in this section are as follows:

(a) $c = 4.2729$ (b) $c = -4.2729$

Fig. 1. Phase portraits of system (2.2) with $(a, b) = (2, 3)$ and different values of parameter c.
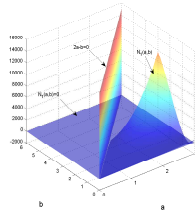

Fig. 2. The graph of $N_1(a, b)$ for $(a, b) = (0, 3) \times (0, 6)$.

Theorem 4.1. *Suppose $b \geq 3a > 0$, system (2.2) has no homoclinic trajectories to any stationary points; while it has a single heteroclinic trajectory to S_0 and S' when $c \neq 0$.*

The procedure for the proof of Theorem 4.1 can be divided into the following two parts. Firstly, one considers the nonexistence of homoclinic trajectories of system (2.2).

4.1. Nonexistence of homoclinic trajectory

For a given solution (x, y, z) of system (2.2), set $Q = z - \frac{1}{3a}x^3$. Then $\dot{Q} = -bQ - \frac{b-3a}{3a}x^3$. Take the first Lyapunov function

$$U_1 = \dot{x}^2 + \frac{a^2}{b(b-3a)}\dot{Q}^2 + \frac{a}{3b}(x^3 - bc)^2$$

with

$$\left.\frac{dU_1}{dt}\right|_{(2.2)} = -2a\dot{x}^2 - \frac{2a^2}{b-3a}\dot{Q}^2 \leq 0 \tag{4.1}$$

for $b > 3a > 0$, and the second one

$$U_2 = \dot{x}^2 + \frac{1}{9}(x^3 - 3ac)^2$$

satisfying

$$\left.\frac{dU_2}{dt}\right|_{(2.2)} = -2a\dot{x}^2 \leq 0 \tag{4.2}$$

for $b = 3a > 0$ (At this time, $Q = 0$, i.e., $x^3 = 3az$. For the proof, see (18) in the sequel).

Lemma 4.1. *Assume that $b \geq 3a > 0$. Then all solutions of system (2.2) tend to either S_0 or S'. Therefore, there do not exist homoclinic trajectories in system (2.2).*

Proof. For any one solution (x, y, z) of system (2.2), it follows from the Lyapunov fucntions $U_{1,2}$ and Eqs. (4.1)-(4.2) that

$$\dot{U}_{1,2} = 0 \Leftrightarrow \dot{x} = \dot{Q} = 0 \Leftrightarrow \dot{x} = \dot{y} = \dot{z} = 0 \Leftrightarrow (x, y, z)$$

is an equilibrium point when $b \geq 3a > 0$. In fact, all solutions tend to S' (resp. S_0) when $c \neq 0$ (resp. $c = 0$) according to the LaSalle theorem [5]. Therefore, system (2.2) has no homoclinic trajectories.   □

Remark 4.1. When $b - 3a = 0$, $x^3 - 3az = 0$ is also an invariant algebraic surface of system (2.2) with the cofactor $k(x, y, z) = -3a$.

In light of the simulation result (see Fig. 3) that has not been theoretically proved yet, we present the following problem.

Conjecture 4.1. *When the parameters a, b, c satisfy $(a, b, c) \in W_{212}^3$, there exists a homoclinic trajectory to S_0 in system (2.2).*

(a) $c = 5.45$ (b) $c = -5.45$

Fig. 3. Phase portraits of system (2.2) with $(a, b) = (2, 3)$, (a) $c = 5.45$, $(x_0, y_0, z_0) = (1.3, 1.6, 1.6) \times 1e - 4$, (b) $c = -5.45$ and $(x_0, y_0, z_0) = -(1.3, 1.6, 1.6) \times 1e - 4$. The figures illustrate that system (2.2) has a homoclinic trajectory to S_0 when $(a, b, c) \in W_{212}^3$.

Next, one devotes to investigating the existence of the heteroclinic trajectory of system (2.2) by employing the concepts of both α, ω-limit set and Lyapunov function.

4.2. Existence of heteroclinic trajectory

In order to detect the existence of heteroclinic trajectory of system (2.2), one recalls the following fact and makes some numerical simulations with MATLAB software.

Fact 4.1. When $c \neq 0$ and $(a, b, c) \in W_{211} \cup W_{212}^1$, it follows from Lemma 3.1 and Theorem 3.3 that S_0 is an unstable non-hyperbolic equilibrium point with 1D W_{loc}^u and 2D W_{loc}^s but S' is locally asymptotically stable. Furthermore, the 1D $W^u(S_0)$ of S_0 tends to the stable manifolds $W^s(S')$ of S' as $t \to \infty$, forming a heteroclinic trajectory to S_0 and S', see Figs. 4–8.

Denote in the following by $\phi_t(q_0) = (x(t; q_0), y(t; q_0), z(t; q_0))$ a solution of the system (2.2) through the initial point $q_0 = (x_0, y_0, z_0)$ and by W_+^u (resp. W_-^u) the positive (resp. negative) branch of the unstable manifold $W^u(S_0)$ corresponding to $x > 0$ (resp. $x < 0$).

(a)   (b)

Fig. 4. Phase portraits of system (2.2) with (a) $(a, c, b) = (2, 19, 6)$, $(x_0, y_0, z_0) = (1.3, 1.6, 1.6) \times 1e-4$, (b) $(a, c, b) = (2, -19, 6)$, $(x_0, y_0, z_0) = -(1.3, 1.6, 1.6) \times 1e-4$. These figures illustrate that system (2.2) has one and only one heteroclinic trajectory to S_0 and S' when $(a, b, c) \in W_{211}$.

(a)   (b)
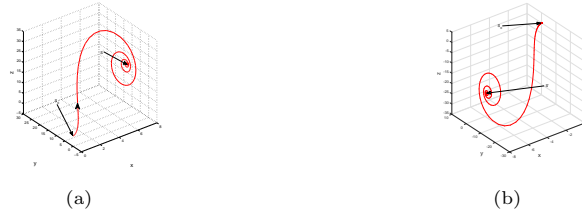
Fig. 5. Phase portraits of system (2.2) with (a) $(a, c, b) = (2, 19, 18)$, $(x_0, y_0, z_0) = (1.3, 1.6, 1.6) \times 1e-4$, (b) $(a, c, b) = (2, -19, 18)$, $(x_0, y_0, z_0) = -(1.3, 1.6, 1.6) \times 1e-4$. These figures illustrate that system (2.2) has one and only one heteroclinic trajectory to S_0 and S' when $(a, b, c) \in W_{211}$.

Set $c \neq 0$. Define the first Lyapunov function

$$V_1(x, y, z) = 3ab(b - 3a)(y - x)^2 + (b - 3a)(x^3 - bc)^2 + 3a(x^3 - bz)^2 \qquad (4.3)$$

for $b > 3a > 0$, and the second one

$$V_2(x, y, z) = (y - x)^2 + \frac{1}{9a^2}(x^3 - 3ac)^2 \qquad (4.4)$$

for $b = 3a > 0$ (At this time $x^3 \equiv 3az$).

Some lengthy computations display that

$$\frac{dV_1}{dt} = -6a^2 b(b - 3a)(y - x)^2 - 6ab(x^3 - bz)^2 \leq 0 \qquad (4.5)$$

and

$$\frac{dV_2}{dt} = -2a(y - x)^2 \leq 0, \qquad (4.6)$$

respectively.

Proposition 4.1. *Suppose $c \neq 0$ and $b \geq 3a > 0$. One has the assertions as follows.*

(i) *If there exist t_1 and t_2 such that $t_1 < t_2$ and $V_{1,2}$ satisfy $V_{1,2}(\phi_{t_1}(q_0)) = V_{1,2}(\phi_{t_2}(q_0))$, then q_0 is one of the equilibria of system (2.2).*

(ii) *If $\lim_{t \to -\infty} \phi_t(q_0) = S_0$ and $x(t; q_0) > 0$ (resp. $x(t; q_0) < 0$) for some $t \in \mathbb{R}$, then $V_{1,2}(S_0) > V_{1,2}(\phi_t(q_0))$ and $x(t; q_0) > 0$ (resp. $x(t; q_0) < 0$) for all $t \in \mathbb{R}$. Consequently, $q_0 \in W_+^u$ (resp. $q_0 \in W_-^u$).*

(a)   (b)

Fig. 6. Phase portraits of system (2.2) with (a) $(a, c, b) = (2, 4, 4)$, $(x_0, y_0, z_0) = (1.3, 1.6, 1.6) \times 1e - 4$, (b) $(a, c, b) = (2, -4, 4)$, $(x_0, y_0, z_0) = -(1.3, 1.6, 1.6) \times 1e - 4$. These figures illustrate that system (2.2) has one and only one heteroclinic trajectory to S_0 and S' when $(a, b, c) \in W_{211}$.

(a)   (b)

Fig. 7. Phase portraits of system (2.2) with (a) $(a, c, b) = (2, 4, 5)$, $(x_0, y_0, z_0) = (1.3, 1.6, 1.6) \times 1e - 4$, (b) $(a, c, b) = (2, -4, 5)$, $(x_0, y_0, z_0) = -(1.3, 1.6, 1.6) \times 1e - 4$. These figures illustrate that system (2.2) has one and only one heteroclinic trajectory to S_0 and S' when $(a, b, c) \in W_{211}$.

Proof.   (i) From (4.5)-(4.6) and the hypothesis of (i), one gets $\frac{dV_{1,2}}{dt}(\phi_t(q_0)) = 0$ for all $t \in (t_1, t_2)$, which implies

$$x'(\phi_t(q_0)) \equiv y'(\phi_t(q_0)) \equiv z'(\phi_t(q_0)) \equiv 0 \qquad (4.7)$$

for all $t \in (t_1, t_2)$. Therefore, q_0 is one of the equilibria of system (2.2), or $b \geq 3a$ and the trajectory $\phi_t(q_0)$ is contained in the intersection of the plane $x = y$ and $bz = x^3$. But the latter leads again to (4.7), which still indicates that q_0 is one of the equilibria of system (2.2), because from $\phi_t(q_0) \in \{x = y\} \cap \{bz = x^3\}$ for all t, one gets $x'(t, q_0) \equiv z'(t, q_0) \equiv 0$. Hence $x(t) = x_0$, but $y(t) = x(t)$, $\forall t \in \mathbb{R}$, i.e., $y'(t, q_0) = 0$. One notices that all the two equilibria lie on the noninvariant curve $\{x = y\} \cap \{bz = x^3\}$.

Lastly, the condition $b = 3a > 0$ in the hypothesis of (i) leads to $x^3 \equiv 3az$. In fact, if $b = 3a$, then $\frac{d}{dt}Q(\phi_t(q_0)) = -3aQ(\phi_t(q_0))$. Consequently, one obtains

$$Q(\phi_t(q_0)) = Q(\phi_\tau(q_0))e^{-3a(t-\tau)} \quad \text{for all} \quad \tau, t \in \mathbb{R}. \qquad (4.8)$$

Since $\phi_\tau(q_0)$ is bounded as $\tau \to -\infty$, Eq. (4.8) yields

$$Q(\phi_t(q_0)) \equiv 0,$$

i.e., $x^3 \equiv 3az$.

(ii) Firstly, one proves $V_{1,2}(S_0) > V_{1,2}(\phi_t(q_0))$, $\forall t \in \mathbb{R}$. Since $b \geq 3a$, $V_{1,2}(S_0) > 0$. To this end, assume by contrary that there exists a $t_0 \in \mathbb{R}$ such that $0 < V_{1,2}(S_0) \leq V_{1,2}(\phi_{t_0}(q_0))$. From $\phi_t(q_0) \to S_0$ as $t \to -\infty$ and the continuity of $V_{1,2}$ on t, it follows that there exists a sequence $t_n \to -\infty$ and two positive integer numbers

(a)   (b)

Fig. 8. Phase portraits of system (2.2) with (a) $(a, c, b) = (2, 4, 5)$, $(x_0, y_0, z_0) = (1.3, 1.6, 1.6) \times 1e - 4$, (b) $(a, c, b) = (2, -4, 5)$, $(x_0, y_0, z_0) = -(1.3, 1.6, 1.6) \times 1e - 4$. These figures illustrate that system (2.2) has one and only one heteroclinic trajectory to S_0 and S' when $(a, b, c) \in W_{212}^1$.

$n_{1,2}$ such that $|V_{1,2}(\phi_{t_n}(q_0)) - V_{1,2}(S_0)| < \varepsilon_{1,2}$ for $\forall \varepsilon_{1,2} > 0$ and $n > \max\{n_1, n_2\}$. Since $t_n \to -\infty$ and $t_0 \in \mathbb{R}$, there are two positive integer number $n_{3,4}$ such that $t_n < t_0$ for $\forall n > \max\{n_3, n_4\}$. Denote further $n_0 = \max\{n_1, n_2, n_3, n_4\}$ and take $\varepsilon_{1,2} = \frac{1}{2}[V_{1,2}(\phi_{t_0}(q_0)) - V_{1,2}(S_0)]$. It is clear that $\varepsilon_{1,2} \geq 0$. Accordingly, one can see

$$V_{1,2}(\phi_{t_n}(q_0)) - V_{1,2}(\phi_{t_0}(q_0)) = V_{1,2}(\phi_{t_n}(q_0)) - V_{1,2}(S_0) + V_{1,2}(S_0) - V_{1,2}(\phi_{t_0}(q_0))$$

$$< \varepsilon_{1,2} + V_{1,2}(S_0) - V_{1,2}(\phi_{t_0}(q_0))$$

$$= -\varepsilon_{1,2} \leq 0.$$

On the other hand $V_{1,2}(t)$ are decreasing with respect to t, which, by definition, leads to $V_{1,2}(\phi_{t_n}(q_0)) \geq V_{1,2}(\phi_{t_0}(q_0))$ for $\forall t_n < t_0$ and $n > n_0$. Therefore, $V_{1,2}(\phi_{t_n}(q_0)) = V_{1,2}(\phi_{t_0}(q_0))$ and by virtue of (i) we get that q_0 is an equilibrium point of system (2.2). Since $\phi_t(q_0) \to S_0$, one has $q_0 \equiv S_0$ and $x(t, q_0) \equiv 0$, $\forall t \in \mathbb{R}$. But this contradicts the hypothesis $x(t, q_0) > 0$ (resp. $x(t, q_0) < 0$) for some t. Hence, $V_{1,2}(S_0) > V_{1,2}(\phi_t(q_0))$, $\forall t \in \mathbb{R}$.

Next, let us prove now that $x(t, q_0) > 0$ (resp. $x(t, q_0) < 0$), $\forall t \in \mathbb{R}$. Otherwise, there exists at least a $t' \in \mathbb{R}$ such that $x(t', q_0) \leq 0$ (resp. $x(t', q_0) \geq 0$). Using $x(t'', q_0) > 0$ (resp. $x(t'', q_0) < 0$) for some $t'' \in \mathbb{R}$ from the hypothesis of (ii), one gets that there exists a $\tau \in \mathbb{R}$ such that $x(\tau, q_0) = 0$. As $V_{1,2}(S_0) > V_{1,2}(\phi_t(q_0))$, $\forall t \in \mathbb{R}$, it follows that $\phi_\tau(q_0) \in \Omega \cap P$, where $\Omega = \{(x, y, z) : V_{1,2}(S_0) > V_{1,2}(x, y, z)\}$ and P is the plane $\{x = 0\}$. However, the element $(x, y, z) \in \Omega \cap P$ satisfies both

$$3ab(b - 3a)y^2 + b^2c^2(b - 3a) + 3ab^2z^2 < b^2c^2(b - 3a)$$

for V_1 and

$$y^2 + c^2 < c^2$$

for V_2. Any one of the above two cases leads to $\Omega \cap P = \emptyset$, which is a contradiction. Therefore $x(t, q_0) > 0$ (resp. $x(t, q_0) < 0$), $\forall t \in \mathbb{R}$. This completes the proof of the proposition. $\qquad \square$

Theorem 4.2. *Consider $b \geq 3a > 0$, $c \neq 0$ and the Lyapunov functions $V_{1,2}$ as stated in (4.3) and (4.4). Then the following statements are true.*

(a) *The ω-limit of any orbit of system (2.2) is an equilibrium point. In particular, system (2.2) has no closed orbits.*

(b) *System (2.2) has no homoclinic trajectories.*

(c) System (2.2) has only one heteroclinic trajectory to S_0 and S'.

Proof.   (a) For $b \geq 3a > 0$ and $c \neq 0$, it follows from (4.5)-(4.6) that the Lyapunov functions $V_{1,2}$ are decreasing along trajectories of system (2.2). That yields that for $\forall t \in \mathbb{R}$,

$$0 \leq V_{1,2}(\phi_t(q_0)) \leq V_{1,2}(q_0), \tag{4.9}$$

where $\phi_t(q_0)$ is a trajectory of system (2.2) through the initial point q_0. Hence, the limits $\lim_{n \to +\infty} V_{1,2}(\phi_t(q_0))$ exist. Denote the two limits by $V_{1,2}^*(q_0)$. From (4.9) one gets that $V_{1,2}(\phi_t(q_0))$ are bounded for $t \geq 0$, which implies further that $x(t, q_0)$, $y(t, q_0)$ and $z(t, q_0)$ are all bounded, i.e., $\phi_t(q_0)$ is bounded for $\forall t \geq 0$. Denote by $\Omega(q_0)$ the ω-limit set of the orbit $\phi_t(q_0)$. It is known that, if $q \in \Omega(q_0)$, then all points of the orbit through q belong to $\Omega(q_0)$, i.e., $\phi_t(q_0) \in \Omega(q_0)$. Therefore, for any point $\phi_t(q)$, $t \geq 0$, there exists a sequence $t_n \to \infty$ for $n \to \infty$ such that $\lim_{n \to +\infty} \phi_{t_n}(q_0) = \phi_t(q)$ which leads to

$$V_{1,2}(\phi_t(q)) = \lim_{n \to +\infty} V_{1,2}(\phi_{t_n}(q_0)) = V_{1,2}^*(q_0) = const$$

for all $t \geq 0$. So, by Proposition 4.1, q is one of the equilibria of system (2.2).

(b) Assume that system (2.2) has a homoclinic trajectory $\gamma(t, q_0)$ to one of the equilibria S_0 or S', through an initial point $q_0 \notin \{S_0, S'\}$. This implies that $\lim_{t \to \pm\infty} \gamma(t, q_0) = u$, $u \in \{S_0, S'\}$. Since $V_{1,2}$ are decreasing along the trajectories of system (2.2), it follows that

$$0 \leq V_{1,2}(u) = V_{1,2}(\gamma(+\infty, q_0)) \leq V_{1,2}(\gamma(t, q_0)) \leq V_{1,2}(\gamma(-\infty, q_0)) = V_{1,2}(u),$$

i.e. $V_{1,2}(\gamma(t, q_0)) = V_{1,2}(u)$ for $\forall t \in \mathbb{R}$. By Proposition 4.1(i), it follows that $q_0 \in \{S_0, S'\}$, which is a contradiction. Hence, system (2.2) has no homoclinic trajectories.

(c) By statement (a), every one-dimensional branch of the unstable manifold W_+^u (resp. W_-^u) has ω-limit, which is an equilibrium point p. Noticing $V_{1,2}(S_0) > V_{1,2}(S')$, the equilibrium point p has to be S', obtaining a single heteroclinic trajectory. This completes the proof of the theorem. $\square$

A numerical case with a single heteroclinic trajectory is illustrated in Figs. 4-5.

Remark 4.2. Figs. 6-8 also illustrate that there exists a heteroclinic trajectory to S_0 and S' in a parameter space other than the one $D := \{(a, b, c)|b \geq 3a > 0, c \neq 0\}$. Theoretically, we can not rigorously prove this case yet. Therefore, this is still an open problem. We hope that interested readers consider it in the future work.

5. Conclusion

Motivated by coining a single heteroclinic trajectory from the Lorenz system family, this paper introduces a new 3D autonomous Lorenz-like system with two cubic terms. Indeed, it is rigorously proved that there exists a single heteroclinic trajectory to S_0 and S' in the proposed system when $b \geq 3a > 0$ and $c \neq 0$ by employing the tools of Lyapunov function and concepts of α-limit set and ω-limit set. Numerical simulation illustrates that there may exist a homoclinic trajectory to S_0 when a, b, c satisfy $(a, b, c) \in W_{212}^1$. Moreover, it is found that the system has no homoclinic (resp. heteroclinic) trajectories when $b \geq 3a > 0$ (resp. $(a, b, c) \in W_{211} \bigcup W_{212}^1$,

or $ab \neq 0$ and $c = 0$). In addition, its other complex dynamical behaviors have been investigated by utilizing the center manifold theory and bifurcation theory, such as the stability and local expression of unstable manifold of S_0, the stability of non-isolated equilibria S_z, and the stability and Hopf bifurcation of S', etc.

Different from most Lorenz-like systems, numerical simulation illustrates neither any singularly degenerate heteroclinic cycles nor chaotic attractors in the proposed system. These differences make the new system more interesting, and also indicate that the system deserves deeper investigations in the future.

Competing interests

The authors declare that they have no competing interests.

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