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HALF-LINEAR VOLTERRA-FREDHOLM TYPE INTEGRAL INEQUALITIES ON TIME SCALES AND THEIR APPLICATIONS*

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Abstract The main aim of this paper is to establish some new half-linear Volterra-Fredholm type integral inequalities on time scales. Our results not only extend and complement some known integral inequalities but also provide an effective tool for the study of qualitative properties of solutions of some dynamic equations.

Keywords Time scales, half-linear, integral inequality, Volterra-Fredholm type.

MSC(2010) 26E70, 26D15, 26D10.

1. Introduction

In 1988, Stefan Hilger [21] introduced the theory of time scales in order to unify and extend the difference and differential calculus in a consistent way. Since this pioneering work, the theory has been growing up and applied to many different fields of mathematics. As one of the most fundamental objects, dynamic equations on time scales has been extensively investigated in recent years, we refer the reader to the books [8,9] and to the papers [2,5,6,10–12,15–17,19,22,27,28,31–36,38,42, 45,47,49–52] and the references therein.

Dynamic inequalities have been used as an important tool in the study of qualitative properties of solutions of dynamic equations on time scales, and established in many directions by several authors. See [1,3,4,7,14,18,23–25,30,39–41,43,48] and the references therein. For example, Saker [39] considered a dynamic inequality of the form

$$x^{\beta}(t) \leq f(t) + g(t)\int_{t_0}^{t}\Big[a(s)x^{\alpha}(s) - b(s)x^{\beta}(\sigma(s))\Big]\Delta s, \tag{1.1}$$

where a, b, f and g are positive rd-continuous functions defined on $[t_0,\infty)_{\mathbb{T}}$, $x(t) \geq 0$, $t \in [t_0,\infty)_{\mathbb{T}}$, α, β are positive constants such that $1 \leq \alpha < \beta$. Sun and Hassan [43] investigated the nonlinear integral inequality on time scales

$$x(t) \leq f(t) + g(t)\int_{t_0}^{t}\Big[a(s)x(s) + b(s)x^{\alpha}(\sigma(s)) - c(s)x^{\beta}(\sigma(s))\Big]\Delta s, \tag{1.2}$$

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where $a, b, c, f, g : \mathbb{T}^\kappa \to \mathbb{R}_+$ are rd-continuous functions, α, β are positive constants such that $0 < \alpha < 1 < \beta$.

Volterra-Fredholm type integral inequalities on time scales are a special type of integral inequalities that contain the definite integral of the unknown function, and useful tools in the study of Volterra-Fredholm type dynamic equations. The inequalities have been paid much attention by many authors, see [13, 20, 26, 29, 37, 44, 46] and the references therein.

In the present paper, we continue our investigation to obtain some new half-linear Volterra-Fredholm type integral inequalities on time scales. Our results not only complement the results established in [43] in the sense that the results can be applied in the cases when $1 < \alpha < \beta$ or $0 < \beta < \alpha < 1$, but also furnish a handy tool for the study of qualitative properties of solutions of some Volterra-Fredholm integral equations and dynamic equations.

2. Preliminaries

Throughout this paper, a knowledge and understanding of time scales and time scale notation is assumed. For an excellent introduction to the calculus on time scales, we refer the reader to [8] and [9] .

List of abbreviations.

In what follows, we always assume that $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+ = [0, \infty)$, $\mathbb{Z}$ denotes the set of integers, C_{rd} denotes the set of all rd-continuous functions, $\mathbb{T}$ is an arbitrary time scale (nonempty closed subset of $\mathbb{R}$), $\mathcal{R}$ denotes the set of all regressive and rd-continuous functions, $\mathcal{R}^+ = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0, \text{ for all } t \in \mathbb{T}\}$ and $I = [t_0, T] \cap \mathbb{T}^\kappa$, $[t_0, \infty)_{\mathbb{T}^\kappa} = [t_0, \infty) \cap \mathbb{T}^\kappa$, where $t_0 \in \mathbb{T}^\kappa$, $T \in \mathbb{T}^\kappa$, $T > t_0$. The "circle plus" addition $\oplus$ defined by $(p \oplus q)(t) := p(t) + q(t) + \mu(t)p(t)q(t)$ for all $t \in \mathbb{T}^\kappa$.

The following lemmas are useful in the proof of the main results of this paper.

Lemma 2.1. *Let $m > 0, n > 0, \alpha > 0$ and $\beta > 0$ be given, then for each $x \geq 0$,*

$$mx^\alpha - nx^\beta \leq \frac{m(\beta - \alpha)}{\beta - 1} \Big(\frac{(\beta - 1)n}{(\alpha - 1)m} \Big)^{(\alpha-1)/(\alpha-\beta)} x \tag{2.1}$$

holds for the cases $1 < \alpha < \beta$ or $0 < \beta < \alpha < 1$.

Proof. If $x = 0$, then it is easy to see that the inequality (2.1) holds. So we only prove the inequality (2.1) holds when $x > 0$. For the case $1 < \alpha < \beta$, set $F(x) = mx^{\alpha-1} - nx^{\beta-1}$, $x > 0$, where $m > 0$ and $n > 0$. Let $F'(x) = 0$, we get $x_0 = \Big(\frac{m(\alpha-1)}{n(\beta-1)}\Big)^{1/(\beta-\alpha)}$. Since $\forall x \in (0, x_0)$, $F'(x) > 0$; $\forall x \in (x_0, +\infty)$, $F'(x) < 0$, F attains its maximum at $x_0 = \Big(\frac{m(\alpha-1)}{n(\beta-1)}\Big)^{1/(\beta-\alpha)}$ and $F_{\max} = F(x_0) = \frac{m(\beta-\alpha)}{\beta-1}\Big(\frac{(\beta-1)n}{(\alpha-1)m}\Big)^{(\alpha-1)/(\alpha-\beta)}$. Thus, (2.1) holds. For the case $0 < \beta < \alpha < 1$, by a similar argument with the case $1 < \alpha < \beta$, we can get (2.1) holds. The proof is complete. □

Lemma 2.2 ([29]). *Let $m \geq n \geq 0$, $m \neq 0$, and $x \geq 0$, then*

$$x^n \leq \frac{n}{m} k^{n-m} x^m + \frac{m-n}{m} k^n, \tag{2.2}$$

for any $k > 0$.

Lemma 2.3 (Theorem 1.117, [8]). *Suppose that for each* $\varepsilon > 0$ *there exists a neighborhood* U *of* t*, independent of* $\tau \in [t_0, \sigma(t)]$*, such that*

$$|w(\sigma(t), \tau) - w(s, \tau) - w_t^{\Delta}(t, \tau)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|, \qquad s \in U, \tag{2.3}$$

where $w : \mathbb{T} \times \mathbb{T}^{\kappa} \to \mathbb{R}_+$ *is continuous at* (t, t)*,* $t \in \mathbb{T}^{\kappa}$ *with* $t > t_0$*, and* $w_t^{\Delta}(t, \cdot)$ *are rd-continuous on* $[t_0, \sigma(t)]$*. Then*

$$y(t) := \int_{t_0}^{t} w(t, \tau)\Delta\tau$$

implies

$$y^{\Delta}(t) = \int_{t_0}^{t} w_t^{\Delta}(t, \tau)\Delta\tau + w(\sigma(t), t), \qquad t \in \mathbb{T}^{\kappa}. \tag{2.4}$$

Lemma 2.4 (Theorem 6.1, [8]). *Suppose* $y, f \in C_{rd}(\mathbb{T}, \mathbb{R})$ *and* $p \in \mathcal{R}^+$*. Then*

$$y^{\Delta}(t) \leq p(t)y(t) + f(t), \qquad \text{for all } t \in \mathbb{T}$$

implies

$$y(t) \leq y(t_0)e_p(t, t_0) + \int_{t_0}^{t} e_p(t, \sigma(\tau))f(\tau)\Delta\tau, \qquad t \in \mathbb{T}.$$

Lemma 2.5 ([4]). *Suppose* $y, a, b \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}_+)$*, where* a *is nondecreasing and not identically zero. If*

$$y(t) \leq a(t) + \int_{t_0}^{t} b(s)y(s)\Delta s, \qquad t \in [t_0, \infty)_{\mathbb{T}},$$

then

$$y(t) \leq a(t)e_b(t, t_0), \qquad t \in [t_0, \infty)_{\mathbb{T}}. \tag{2.5}$$

3. Main results

Theorem 3.1. *Assume that* $x, f, a \in C_{rd}(I, \mathbb{R}_+)$*,* $b, c \in C_{rd}(I, (0, \infty))$*,* f *is nondecreasing and not identically zero,* $k \in C_{rd}(I \times I, \mathbb{R}_+)$ *with* $k(t, s)$ *is nondecreasing for* $t \in I$*,* α *and* β *are positive constants satisfying:* $1 < \alpha < \beta$ *or* $0 < \beta < \alpha < 1$*. Suppose that* x *satisfies*

$$\begin{aligned} x(t) \leq f(t) &+ \int_{t_0}^{t} \Big[a(s)x(s) + b(s)x^{\alpha}(s) - c(s)x^{\beta}(s)\Big]\Delta s \\ &+ \int_{t_0}^{T} k(t, s)x(s)\Delta s, \quad t \in I. \end{aligned} \tag{3.1}$$

If

$$\int_{t_0}^{T} k(T, s)e_F(s, t_0)\Delta s < 1, \tag{3.2}$$

then

$$x(t) \leq \frac{f(T)e_F(t, t_0)}{1 - \int_{t_0}^{T} k(T, s)e_F(s, t_0)\Delta s}, \quad t \in I, \tag{3.3}$$

where

$$F(t) := a(t) + M(t), \tag{3.4}$$

$$M(t) := \frac{b(t)(\beta-\alpha)}{\beta-1}\Big(\frac{(\beta-1)c(t)}{(\alpha-1)b(t)}\Big)^{(\alpha-1)/(\alpha-\beta)}. \tag{3.5}$$

Proof. From Lemma 2.1 and (3.1), we have

$$\begin{aligned} x(t) &\le f(t) + \int_{t_0}^{t}\Big[a(s)x(s) + b(s)x^{\alpha}(s) - c(s)x^{\beta}(s)\Big]\Delta s + \int_{t_0}^{T} k(t,s)x(s)\Delta s \\ &\le f(t) + \int_{t_0}^{t}\Big[a(s)x(s) + M(s)x(s)\Big]\Delta s + \int_{t_0}^{T} k(t,s)x(s) \\ &\le f(t) + \int_{t_0}^{t} F(s)x(s)\Delta s + \int_{t_0}^{T} k(t,s)x(s)\Delta s,\ \ t\in I, \end{aligned} \tag{3.6}$$

where $F(t)$ and $M(t)$ are defined as in (3.4) and (3.5). Since x is nonnegative and $k(t,s)$ is nondecreasing for $t \in I$, we get $\int_{t_0}^{T} k(t,s)x(s)\Delta s$ is nondecreasing for $t \in I$. In view of Lemma 2.5, we obtain

$$x(t) \le \Big(f(t) + \int_{t_0}^{T} k(t,s)x(s)\Delta s\Big)e_F(t,t_0). \tag{3.7}$$

Using the monotonicity of f and $k(\cdot,s)$, we have

$$\begin{aligned} x(t) &\le \Big(f(T) + \int_{t_0}^{T} k(T,s)x(s)\Delta s\Big)e_F(t,t_0) \\ &= C(T)e_F(t,t_0), \quad t\in I, \end{aligned} \tag{3.8}$$

where

$$C(t) = f(t) + \int_{t_0}^{t} k(t,s)x(s)\Delta s.$$

Then from the definition of $C(t)$ and (3.8), we obtain

$$\begin{aligned} C(T) &= f(T) + \int_{t_0}^{T} k(T,s)x(s)\Delta s \\ &\le f(T) + \int_{t_0}^{T} k(T,s)C(T)e_F(s,t_0)\Delta s \\ &= f(T) + C(T)\int_{t_0}^{T} k(T,s)e_F(s,t_0)\Delta s. \end{aligned} \tag{3.9}$$

By (3.2) and (3.9), we have

$$C(T) \le \frac{f(T)}{1 - \int_{t_0}^{T} k(T,s)e_F(s,t_0)\Delta s}. \tag{3.10}$$

Then from (3.8) and (3.10), we get (3.3) holds. This completes the proof. □

Theorem 3.2. *Assume that* $x \in C_{rd}([t_0,\infty)_{\mathbb{T}^\kappa},\mathbb{R}_+)$, $f,g,a,k \in C_{rd}(I,\mathbb{R}_+)$, $b,c \in C_{rd}(I,(0,\infty))$, α,β *and* M *are defined the same as in Theorem 3.1. Let* $w(t,s)$ *be defined as in Lemma 2.3 such that* $w_t^\Delta(t,s) \geq 0$ *for* $t \geq s$ *and (2.3) holds. Suppose that* x *satisfies*

$$x(t) \leq f(t) + g(t)\int_{t_0}^{t} w(t,s)\Big[a(s)x(s) + b(s)x^\alpha(\sigma(s)) - c(s)u^\beta(\sigma(s))\Big]\Delta s + \int_{t_0}^{T} k(s)x(s)\Delta s, \quad t \in I. \tag{3.11}$$

If

$$\mu(t)P(t) < 1 \quad \textit{and} \quad \int_{t_0}^{T} k(s)e_{F\oplus G}(s,t_0)\Delta s < 1, \quad t \in I, \tag{3.12}$$

then

$$x(t) \leq f(t) + \frac{\int_{t_0}^{T} k(s)\big[f(s) + \int_{t_0}^{s} e_{F\oplus G}(s,\sigma(\tau))R(\tau)\Delta\tau\big]\Delta s}{1 - \int_{t_0}^{T} k(s)e_{F\oplus G}(s,t_0)\Delta s} e_{F\oplus G}(t,t_0) + \int_{t_0}^{t} e_{F\oplus G}(t,\sigma(s))R(s)\Delta s, \quad t \in I, \tag{3.13}$$

where

$$P(t) := g^\Delta(t)\int_{t_0}^{\sigma(t)} w(\sigma(t),s)\Big[a(s) + M(s)\Big]\Delta s + g(t)\left[\int_{t_0}^{t} w_t^\Delta(t,s)M(s)\Delta s + w(\sigma(t),t)M(t)\right], \tag{3.14}$$

$$F(t) := \frac{P(t)}{1 - \mu(t)P(t)}, \tag{3.15}$$

$$G(t) := g(t)\int_{t_0}^{t} w_t^\Delta(t,s)a(s)\Delta s + g(t)w(\sigma(t),t)a(t), \tag{3.16}$$

$$Q(t) := g^\Delta(t)\int_{t_0}^{\sigma(t)} w(\sigma(t),s)\Big[a(s)f(s) + M(s)f(\sigma(s))\Big]\Delta s + g(t)\int_{t_0}^{t} w_t^\Delta(t,s)\Big[a(s)f(s) + M(s)f(\sigma(s))\Big]\Delta s + g(t)w(\sigma(t),t)\Big[a(t)f(t) + M(t)f(\sigma(t))\Big], \tag{3.17}$$

$$R(t) := (1 + \mu(t)F(t))Q(t). \tag{3.18}$$

Proof. From Lemma 2.1 and (3.11), we have

$$\begin{aligned} x(t) &\leq f(t) + g(t)\int_{t_0}^{t} w(t,s)\Big[a(s)x(s) + b(s)x^\alpha(\sigma(s)) - c(s)x^\beta(\sigma(s))\Big]\Delta s \\ &\quad + \int_{t_0}^{T} k(s)x(s)\Delta s \\ &\leq f(t) + g(t)\int_{t_0}^{t} w(t,s)\Big[a(s)x(s) + M(s)x(\sigma(s))\Big]\Delta s \end{aligned}$$

$$+\int_{t_0}^{T} k(s)x(s)\Delta s, \qquad t \in I. \tag{3.19}$$

Denote

$$z(t) := g(t)\int_{t_0}^{t} w(t,s)\Big[a(s)x(s) + M(s)x(\sigma(s))\Big]\Delta s + \int_{t_0}^{T} k(s)x(s)\Delta s,\ t \in I. \tag{3.20}$$

From the assumptions on g, w, a, x, M, (3.19) and (3.20), we have z is nondecreasing on I,

$$x(t) \le f(t) + z(t), \qquad t \in I, \tag{3.21}$$

and

$$z(t_0) = \int_{t_0}^{T} k(s)x(s)\Delta s. \tag{3.22}$$

By Lemma 2.3, (3.20) and (3.21), we have

$$\begin{aligned}
z^{\Delta}(t) = {} & g^{\Delta}(t)\int_{t_0}^{\sigma(t)} w(\sigma(t),s)\Big[a(s)x(s) + M(s)x(\sigma(s))\Big]\Delta s \\
& + g(t)\int_{t_0}^{t} w_t^{\Delta}(t,s)\Big[a(s)x(s) + M(s)x(\sigma(s))\Big]\Delta s \\
& + g(t)w(\sigma(t),t)\Big[a(t)x(t) + M(t)x(\sigma(t))\Big] \\
\le {} & g^{\Delta}(t)\int_{t_0}^{\sigma(t)} w(\sigma(t),s)\Big[a(s)\Big(f(s)+z(s)\Big) + M(s)\Big(f(\sigma(s)) + z(\sigma(s))\Big)\Big]\Delta s \\
& + g(t)\int_{t_0}^{t} w_t^{\Delta}(t,s)\Big[a(s)\Big(f(s)+z(s)\Big) + M(s)\Big(f(\sigma(s)) + z(\sigma(s))\Big)\Big]\Delta s \\
& + g(t)w(\sigma(t),t)\Big[a(t)\Big(f(t)+z(t)\Big) + M(t)\Big(f(\sigma(t)) + z(\sigma(t))\Big)\Big] \\
\le {} & g^{\Delta}(t)z(\sigma(t))\int_{t_0}^{\sigma(t)} w(\sigma(t),s)\Big[a(s) + M(s)\Big]\Delta s \\
& + g^{\Delta}(t)\int_{t_0}^{\sigma(t)} w(\sigma(t),s)\Big[a(s)f(s) + M(s)f(\sigma(s))\Big]\Delta s \\
& + g(t)z(t)\int_{t_0}^{t} w_t^{\Delta}(t,s)a(s)\Delta s + g(t)z(\sigma(t))\int_{t_0}^{t} w_t^{\Delta}(t,s)M(s)\Delta s \\
& + g(t)\int_{t_0}^{t} w_t^{\Delta}(t,s)\Big[a(s)f(s) + M(s)f(\sigma(s))\Big]\Delta s \\
& + g(t)w(\sigma(t),t)\Big[a(t)z(t) + M(t)z(\sigma(t))\Big] \\
& + g(t)w(\sigma(t),t)\Big[a(t)f(t) + M(t)f(\sigma(t))\Big] \\
\le {} & z(\sigma(t))\Big\{g^{\Delta}(t)\int_{t_0}^{\sigma(t)} w(\sigma(t),s)\Big[a(s) + M(s)\Big]\Delta s \\
& + g(t)\int_{t_0}^{t} w_t^{\Delta}(t,s)M(s)\Delta s + g(t)w(\sigma(t),t)M(t)\Big\}
\end{aligned}$$

$$
\begin{aligned}
&+z(t)\Big[g(t)\int_{t_0}^{t} w_t^{\Delta}(t,s)a(s)\Delta s + g(t)w(\sigma(t),t)a(t)\Big]\\
&+g^{\Delta}(t)\int_{t_0}^{\sigma(t)} w(\sigma(t),s)\Big[a(s)f(s)+M(s)f(\sigma(s))\Big]\Delta s\\
&+g(t)\int_{t_0}^{t} w_t^{\Delta}(t,s)\Big[a(s)f(s)+M(s)f(\sigma(s))\Big]\Delta s\\
&+g(t)w(\sigma(t),t)\Big[a(t)f(t)+M(t)f(\sigma(t))\Big]\\
=\ & P(t)z(\sigma(t))+G(t)z(t)+Q(t), \qquad t\in I,
\end{aligned} \tag{3.23}
$$

where $P(t)$, $G(t)$ and $Q(t)$ are defined as in (3.14), (3.16) and (3.17). From (3.15), we get

$$
P(t)=\frac{F(t)}{1+\mu(t)F(t)},
$$

and by (3.23) we obtain

$$
\begin{aligned}
z^{\Delta}(t) &\le P(t)z(\sigma(t))+G(t)z(t)+Q(t)\\
&= P(t)\big(z(t)+\mu(t)z^{\Delta}(t)\big)+G(t)z(t)+Q(t)\\
&= \frac{F(t)}{1+\mu(t)F(t)}\big(z(t)+\mu(t)z^{\Delta}(t)\big)+G(t)z(t)+Q(t), \qquad t\in I,
\end{aligned} \tag{3.24}
$$

which yields

$$
\frac{1}{1+\mu(t)F(t)}z^{\Delta}(t)\le \frac{F(t)+G(t)+\mu(t)F(t)G(t)}{1+\mu(t)F(t)}z(t)+Q(t), \qquad t\in I, \tag{3.25}
$$

that is

$$
\begin{aligned}
z^{\Delta}(t) &\le \big(F(t)+G(t)+\mu(t)F(t)G(t)\big)z(t)+(1+\mu(t)F(t))Q(t)\\
&= (F\oplus G)(t)z(t)+R(t), \qquad t\in I,
\end{aligned} \tag{3.26}
$$

where $R(t)$ is defined as in (3.18). Note that z is rd-continuous and $F\oplus G\in\mathcal{R}^+$, from Lemma 2.4 and (3.26), we obtain

$$
z(t)\le z(t_0)e_{F\oplus G}(t,t_0)+\int_{t_0}^{t} e_{F\oplus G}(t,\sigma(s))R(s)\Delta s, \qquad t\in I. \tag{3.27}
$$

Using (3.21) and (3.27) on the right side of (3.22), and in the view of (3.14), we get

$$
z(t_0)\le \frac{\int_{t_0}^{T} k(s)\Big[f(s)+\int_{t_0}^{s} e_{F\oplus G}(s,\sigma(\tau))R(\tau)\Delta\tau\Big]\Delta s}{1-\int_{t_0}^{T} k(s)e_{F\oplus G}(s,t_0)\Delta s}. \tag{3.28}
$$

Substituting (3.28) into (3.27), we obtain

$$
\begin{aligned}
z(t)\le\ & \frac{\int_{t_0}^{T} k(s)\Big[f(s)+\int_{t_0}^{s} e_{F\oplus G}(s,\sigma(\tau))R(\tau)\Delta\tau\Big]\Delta s}{1-\int_{t_0}^{T} k(s)e_{F\oplus G}(s,t_0)\Delta s}e_{F\oplus G}(t,t_0)\\
&+\int_{t_0}^{t} e_{F\oplus G}(t,\sigma(s))R(s)\Delta s, \quad t\in I.
\end{aligned}
$$

By (3.21), we get the desired inequality (3.13). This completes the proof. □

If we let $w(t,s)\equiv 1$ in Theorem 3.2 , then we obtain the following corollary.

Corollary 3.1. *Assume that $x, f, g, k, a, b, c, \alpha, \beta$, and M are defined the same as in Theorem 3.2. Suppose that x satisfies*

$$x(t) \le f(t) + g(t)\int_{t_0}^{t}\Big[a(s)x(s) + b(s)x^{\alpha}(\sigma(s)) - c(s)u^{\beta}(\sigma(s))\Big]\Delta s + \int_{t_0}^{T} k(s)x(s)\Delta s, \quad t \in I,$$

then

$$x(t) \le f(t) + \frac{\int_{t_0}^{T} k(s)\big[a(s) + \int_{t_0}^{s} e_{F\oplus G}(s,\sigma(\tau))R(\tau)\Delta\tau\big]\Delta s}{1 - \int_{t_0}^{T} k(s)e_{F\oplus G}(s,t_0)\Delta s}e_{F\oplus G}(t,t_0) + \int_{t_0}^{t} e_{F\oplus G}(t,\sigma(s))R(s)\Delta s, \quad t \in I,$$

where

$$\begin{aligned}
P(t) &:= g^{\Delta}(t)\int_{t_0}^{\sigma(t)}\Big[a(s) + M(s)\Big]\Delta s + g(t)M(t),\\
F(t) &:= \frac{P(t)}{1 - \mu(t)P(t)}, \qquad G(t) := g(t)a(t),\\
Q(t) &:= g^{\Delta}(t)\int_{t_0}^{\sigma(t)}\Big[a(s)f(s) + M(s)f(\sigma(s))\Big]\Delta s\\
&\quad + g(t)\Big[a(t)f(t) + M(t)f(\sigma(t))\Big],\\
R(t) &:= (1 + \mu(t)F(t))Q(t).
\end{aligned}$$

If we let $k(t) \equiv 0$ in Theorem 3.2, then we obtain the following corollary.

Corollary 3.2. *Assume that $x, f, g, w, a, b, c, \alpha, \beta$, P, F, G, Q and R are defined the same as in Theorem 3.2. Suppose that x satisfies*

$$x(t) \le f(t) + g(t)\int_{t_0}^{t} w(t,s)\Big[a(s)x(s) + b(s)x^{\alpha}(\sigma(s)) - c(s)u^{\beta}(\sigma(s))\Big]\Delta s, \ t \in [t_0,\infty)_{\mathbb{T}^{\kappa}},$$

then

$$x(t) \le f(t) + \int_{t_0}^{t} e_{F\oplus G}(t,\sigma(s))R(s)\Delta s, \ t \in [t_0,\infty)_{\mathbb{T}^{\kappa}}.$$

Remark 3.1. Corollary 3.2 extends [[43], Theorem 2] to the cases $1 < \alpha < \beta$ and $0 < \beta < \alpha < 1$.

Theorem 3.3. *Assume that $x, f, g, w, a, b, c, \alpha, \beta$, P, F, and G are defined the same as in Theorem 3.2, f is nondecreasing, and $\lambda \ge 0$ is a constant. Suppose that x satisfies*

$$\begin{aligned}
x(t) \le\, & f(t) + g(t)\int_{t_0}^{t} w(t,s)\Big[a(s)x(s) + b(s)x^{\alpha}(\sigma(s)) - c(s)x^{\beta}(\sigma(s))\Big]\Delta s\\
& + \lambda g(T)\int_{t_0}^{T} w(t,s)\Big[a(s)x(s) + b(s)x^{\alpha}(\sigma(s)) - c(s)x^{\beta}(\sigma(s))\Big]\Delta s, \ t \in I. \quad (3.29)
\end{aligned}$$

If

$$\xi := e_{F\oplus G}(T, t_0) < 1 + \frac{1}{\lambda}, \tag{3.30}$$

then

$$x(t) \le \frac{f(T)}{\lambda + 1 - \xi\lambda} e_{F\oplus G}(t, t_0), \qquad t \in I, \tag{3.31}$$

where we use the convention that $\frac{1}{0} = +\infty$.

Proof. From Lemma 2.1 and (3.29), we have

$$\begin{aligned} x(t) \le f(t) + g(t) \int_{t_0}^{t} w(t,s)\Big[a(s)x(s) + b(s)x^{\alpha}(\sigma(s)) - c(s)x^{\beta}(\sigma(s))\Big]\Delta s \\ +\lambda g(T) \int_{t_0}^{T} w(t,s)\Big[a(s)x(s) + b(s)x^{\alpha}(\sigma(s)) - c(s)x^{\beta}(\sigma(s))\Big]\Delta s \\ \le f(t) + g(t) \int_{t_0}^{t} w(t,s)\Big[a(s)x(s) + M(s)x(\sigma(s))\Big]\Delta s \\ +\lambda g(T) \int_{t_0}^{T} w(t,s)\Big[a(s)x(s) + M(s)x(\sigma(s))\Big]\Delta s, \quad t \in I. \end{aligned} \tag{3.32}$$

Denote

$$\begin{aligned} z(t) := f(T) + g(t) \int_{t_0}^{t} w(t,s)\Big[a(s)x(s) + M(s)x(\sigma(s))\Big]\Delta s \\ +\lambda g(T) \int_{t_0}^{T} w(T,s)\Big[a(s)x(s) + M(s)x(\sigma(s))\Big]\Delta s, \quad t \in I. \end{aligned} \tag{3.33}$$

From the assumptions on x, f, g, w, a, M, (3.32) and (3.33), we get z is nondecreasing on I,

$$x(t) \le z(t), \qquad t \in I, \tag{3.34}$$

and

$$z(t_0) = f(T) + \lambda g(T) \int_{t_0}^{T} w(T,s)\Big[a(s)x(s) + M(s)x(\sigma(s))\Big]\Delta s. \tag{3.35}$$

From Lemma 2.3, (3.33) and (3.34), we have

$$\begin{aligned} z^{\Delta}(t) = g^{\Delta}(t) \int_{t_0}^{\sigma(t)} w(\sigma(t),s)\Big[a(s)x(s) + M(s)x(\sigma(s))\Big]\Delta s \\ +g(t) \int_{t_0}^{t} w_t^{\Delta}(t,s)\Big[a(s)x(s) + M(s)x(\sigma(s))\Big]\Delta s \\ +g(t)w(\sigma(t),t)\Big[a(t)x(t) + M(t)x(\sigma(t))\Big] \\ \le g^{\Delta}(t) \int_{t_0}^{\sigma(t)} w(\sigma(t),s)\Big[a(s)z(s) + M(s)z(\sigma(s))\Big]\Delta s \\ +g(t) \int_{t_0}^{t} w_t^{\Delta}(t,s)\Big[a(s)z(s) + M(s)z(\sigma(s))\Big]\Delta s \\ +g(t)w(\sigma(t),t)\Big[a(t)z(t) + M(t)z(\sigma(t))\Big] \end{aligned}$$

$$\begin{aligned}
&\le g^{\Delta}(t)z(\sigma(t))\int_{t_0}^{\sigma(t)} w(\sigma(t),s)\Big[a(s)+M(s)\Big]\Delta s\\
&\quad +g(t)z(t)\int_{t_0}^{t} w_t^{\Delta}(t,s)a(s)\Delta s+g(t)z(\sigma(t))\int_{t_0}^{t} w_t^{\Delta}(t,s)M(s)\Delta s\\
&\quad +g(t)w(\sigma(t),t)\Big[a(t)z(t)+M(t)z(\sigma(t))\Big]\\
&\le z(\sigma(t))\Bigg[g^{\Delta}(t)\int_{t_0}^{\sigma(t)} w(\sigma(t),s)\Big[a(s)+M(s)\Big]\Delta s\\
&\quad +g(t)\int_{t_0}^{t} w_t^{\Delta}(t,s)M(s)\Delta s+g(t)w(\sigma(t),t)M(t)\Bigg]\\
&\quad +z(t)\Bigg[g(t)\int_{t_0}^{t} w_t^{\Delta}(t,s)a(s)\Delta s+g(t)w(\sigma(t),t)a(t)\Bigg]\\
&= P(t)z(\sigma(t))+G(t)z(t), \qquad t\in I, \qquad (3.36)
\end{aligned}$$

where $P(t)$ and $G(t)$ are defined as in (3.14) and (3.16). By a similar argument with Theorem 3.2, we get

$$z(t)\le z(t_0)e_{F\oplus G}(t,t_0), \qquad t\in I. \qquad (3.37)$$

From (3.33) and (3.35), we obtain

$$(z(t_0)-f(T))\frac{\lambda+1}{\lambda}+f(T)=z(T), \qquad (3.38)$$

that is

$$\frac{\lambda+1}{\lambda}z(t_0)-\frac{1}{\lambda}f(T)=z(T). \qquad (3.39)$$

From (3.30), (3.37) and (3.39), we have

$$\frac{\lambda+1}{\lambda}z(t_0)-\frac{1}{\lambda}f(T)=z(T)\le z(t_0)e_{B\oplus C}(T,t_0)=z(t_0)\xi. \qquad (3.40)$$

In view of (3.30) and (3.40), we get

$$z(t_0)\le \frac{f(T)}{\lambda+1-\xi\lambda}. \qquad (3.41)$$

Substituting (3.41) into (3.37), we obtain

$$z(t)\le \frac{f(T)}{\lambda+1-\xi\lambda}e_{B\oplus C}(t,t_0), \quad t\in I. \qquad (3.42)$$

Noting $x(t)\le z(t)$, we get the desired inequality (3.31). This completes the proof. □

If we let $w(t,s)\equiv 1$ in Theorem 3.3 , then we obtain the following corollary.

Corollary 3.3. *Assume that $x, f, g, a, b, c, M, \alpha, \beta, \lambda, P, F$ and G are defined the same as in corollary 3.1. Suppose that x satisfies*

$$x(t)\le f(t)+g(t)\int_{t_0}^{t}\Big[a(s)x(s)+b(s)x^{\alpha}(\sigma(s))-c(s)x^{\beta}(\sigma(s))\Big]\Delta s$$

$$+\lambda b(T)\int_{t_0}^{T}\Big[a(s)x(s)+b(s)x^{\alpha}(\sigma(s))-c(s)x^{\beta}(\sigma(s))\Big]\Delta s,\qquad t\in I. \quad (3.43)$$

If

$$\xi := e_{F\oplus G}(T,t_0) < 1+\frac{1}{\lambda}, \quad (3.44)$$

then

$$x(t)\le \frac{f(T)}{\lambda+1-\xi\lambda}e_{F\oplus G}(t,t_0),\qquad t\in I. \quad (3.45)$$

4. Applications

In this section, we will present some simple applications for our results.

Example 4.1. Consider the following Volterra-Fredholm type dynamic integral equation on time scales:

$$x(t)=f(t)+g(t)\int_{t_0}^{t}H\Big(s,x(s),x(\sigma(s))\Big)\Delta s+\int_{t_0}^{T}N\Big(s,x(s)\Big)\Delta s,\ t\in I, \quad (4.1)$$

where $x\in C_{rd}([t_0,\infty)_{\mathbb{T}^\kappa},\mathbb{R})$, $f\in C_{rd}(I,\mathbb{R})$, $g\in C_{rd}(I,\mathbb{R}_+)$, $g^{\Delta}(t)\ge 0$, $H\in C(I\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and $N\in C(I\times\mathbb{R},\mathbb{R})$.

The following theorem gives an estimate for the solutions of Eq.(4.1).

Theorem 4.1. *Suppose that the functions H and N in Eq.(4.1) satisfy the conditions*

$$|H(t,u,v)|\le a(t)|u|+b(t)|v|^{\alpha}-c(t)|v|^{\beta},\qquad t\in I,\quad u,v\in\mathbb{R}, \quad (4.2)$$

$$|N(t,u)|\le k(t)|u|,\qquad t\in I,\quad u\in\mathbb{R}, \quad (4.3)$$

where $a,k\in C_{rd}(I,\mathbb{R}_+)$, $b,c\in C_{rd}(I,(0,\infty))$. If x is a solution of Eq.(4.1) and

$$\mu(t)P(t)<1,\qquad \int_{t_0}^{T}k(s)e_{F\oplus G}(s,t_0)\Delta s<1,\quad t\in I, \quad (4.4)$$

then

$$|x(t)|\le |f(t)|+\frac{\int_{t_0}^{T}k(s)\Big[|f(s)|+\int_{t_0}^{s}e_{F\oplus G}(s,\sigma(\tau))R(\tau)\Delta\tau\Big]\Delta s}{1-\int_{t_0}^{T}k(s)e_{F\oplus G}(s,t_0)\Delta s}e_{F\oplus G}(t,t_0)$$

$$+\int_{t_0}^{t}e_{F\oplus G}(t,\sigma(s))R(s)\Delta s,\qquad t\in I, \quad (4.5)$$

where P,F,G and M are defined the same as in Corollary 3.1, and

$$Q(t):=b^{\Delta}(t)\int_{t_0}^{\sigma(t)}\Big[a(s)|f(s)|+M(s)|f(\sigma(s))|\Big]\Delta s$$

$$+g(t)\Big[a(t)|f(t)|+M(t)|f(\sigma(t))|\Big],$$

$$R(t):=(1+\mu(t)F(t))Q(t).$$

Proof. From (4.1)-(4.3), we have

$$\begin{aligned}|x(t)| &\leq |f(t)| + g(t)\int_{t_0}^{t}\Big|H\Big(s, x(s), x(\sigma(s))\Big)\Big|\Delta s \\ &\quad + \int_{t_0}^{T}\Big|N\Big(s, x(s)\Big)\Big|\Delta s \\ &\leq |f(t)| + g(t)\int_{t_0}^{t}\Big[a(s)|x(s)| + b(s)|x(\sigma(s))|^{\alpha} - c(s)|x(\sigma(s))|^{\beta}\Big]\Delta s \\ &\quad + \int_{t_0}^{T} k(s)|x(s)|\Delta s, \qquad t \in I. \end{aligned} \tag{4.6}$$

Then applying Corollary 3.1 to (4.6), we get (4.5). □

Example 4.2. Consider the following Volterra type dynamic integral equation on time scales:

$$x(t) = f(t) + g(t)\int_{t_0}^{t} w(t,s)\Big[a(s)x(s) + b(s)x^3(\sigma(s)) - c(s)x^4(\sigma(s))\Big]\Delta s, \quad t \in [t_0, \infty)_{\mathbb{T}^\kappa}, \tag{4.7}$$

where $x, f \in C_{rd}([t_0, \infty)_{\mathbb{T}^\kappa}, \mathbb{R})$, $a, g \in C_{rd}([t_0, \infty)_{\mathbb{T}^\kappa}, \mathbb{R}_+)$, $b, c \in C_{rd}([t_0, \infty)_{\mathbb{T}^\kappa}, (0, \infty))$, $g^{\Delta}(t) \geq 0$ and $w(t,s)$ is defined the same as in Lemma 2.3 such that $w_t^{\Delta}(t,s) \geq 0$ for $t \geq s$ and (2.3) holds.

Theorem 4.2. *If x is a solution of Eq.(4.7), then*

$$|x(t)| \leq |f(t)| + \int_{t_0}^{t} e_{F\oplus G}(t, \sigma(s))R(s)\Delta s, \quad t \in [t_0, \infty)_{\mathbb{T}^\kappa}, \tag{4.8}$$

where $P(t)$, $F(t)$, $G(t)$ and $M(t)$ are defined as in Theorem 3.2, and

$$\begin{aligned} Q(t) &:= g^{\Delta}(t)\int_{t_0}^{\sigma(t)} w(\sigma(t), s)\Big[a(s)|f(s)| + M(s)|f(\sigma(s))|\Big]\Delta s \\ &\quad + g(t)\int_{t_0}^{t} w_t^{\Delta}(t,s)\Big[a(s)|f(s)| + M(s)|f(\sigma(s))|\Big]\Delta s \\ &\quad + g(t)w(\sigma(t), t)\Big[a(t)|f(t)| + M(t)|f(\sigma(t))|\Big], \\ R(t) &:= (1 + \mu(t)F(t))Q(t). \end{aligned}$$

Proof. From (4.7), we have

$$|x(t)| \leq |f(t)| + g(t)\int_{t_0}^{t} w(t,s)\Big[a(s)|x(s)| + b(s)|x(\sigma(s))|^3 - c(s)|x(\sigma(s))|^4\Big]\Delta s. \tag{4.9}$$

Using Corollary 3.2, we obtain the desired inequality (4.8). □

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