

OSCILLATION BEHAVIOR OF SOLUTION OF IMPULSIVE FRACTIONAL DIFFERENTIAL EQUATION*

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Abstract In this paper, we study the oscillation of impulsive Caputo fractional differential equation. Sufficient conditions for the asymptotic and oscillation of the equation are obtained by using the inequality principle and Bihari Lemma. An example is given to illustrate the results. This is the first time to study the oscillation of impulsive fractional differential equation with Caputo derivative.

Keywords Oscillation theory, fractional differential equation, impulsive.

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1. Introduction

Fractional differential equations appear more and more frequently in various research areas, such as in modeling mechanical and electrical properties of real materials, as well as in rheological theory and other physical problems, see [1, 7, 9]. For articles on the oscillation of fractional differential equations, readers can refer to literatures [4, 14, 15, 17–19].

Meanwhile, many evolution processes are subject to short term perturbations whose durations are negligible in comparison with the duration of the processes. Consequently, it is natural to assume that these perturbations act instantaneously, that is, act in the form of impulses. Due to the intensive development about the theory of impulsive differential equations and fractional calculus and their widely applications in diverse fields, impulsive fractional differential equations have become a new hot topic. Very recently, more and more researchers show great interest in the field of impulsive problems for fractional differential equations, see [6, 11, 13, 16].

Because of the difficulties caused by impulsive perturbations, only a small amount of literature have been done on the oscillation of impulsive fractional differential equations.

In 2016, Jessada Tariboon and Sotiris K. Ntouyas [12] investigated oscillation results for the solutions of impulsive fractional differential equations with conformable

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derivative of the form

$$\begin{cases} {}_{t_k}D^\alpha(p(t)[{}_{t_k}D^\alpha x(t) + r(t)x(t)] + q(t)x(t) = 0, & t > t_0, \quad t \neq t_k, \quad \alpha \in (1, 2), \\ x(t_k^+) = a_k x(t_k^-), \quad {}_{t_k}D^\alpha x(t_k^+) = b_{k t_{k-1}} D^\alpha x(t_k^-), & k = 1, 2, \dots \end{cases}$$

They obtained some new oscillatory results by using the equivalence transformation and the associated Riccati techniques.

The definition of conformable derivative is only related to the limit form and is similar to form of integer derivative. Therefore, the methods for oscillation of integer differential equation can be applied to conformable derivative only through a simple transformation. There are still some gaps between conformable derivative and classical fractional derivative.

In 2017, A. Raheem, Md. Maqbul [10] considered the oscillatory behavior of solutions on the differential equation with Riemann-Liouville fractional derivative, for $t \neq t_j$

$$D_{+,t}^\beta u(x,t) + a(t)D_{+,t}^{\beta-1}u(x,t) = b(t)\Delta u(x,t) + \sum_{k=1}^m c_k(t)\Delta u(x,t - \tau_k) - F(x,t),$$

under the impulsive condition

$$D_{+,t}^{\beta-1}u(x,t_j^+) - D_{+,t}^{\beta-1}u(x,t_j^-) = \sigma(x,t_j)D_{+,t}^{\beta-1}u(x,t_j), \quad j = 1, 2, \dots, (x,t) \in \Omega \times \mathbb{R}_+.$$

With two kind of boundary conditions

$$\frac{\partial u(x,t)}{\partial N} + f(x,t)u(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}_+, \quad t \neq t_j$$

and

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}_+, \quad t \neq t_j,$$

where $a, b, c_k \in PC[\mathbb{R}_+, \mathbb{R}_+]$, forcing term $F \in PC[\overline{\Omega} \times \mathbb{R}_+, \mathbb{R}_+]$, $f \in PC[\partial\Omega, \mathbb{R}^+]$, and PC denotes the class of functions which are piecewise continuous functions in t with discontinuities of first kind only at $t = t_j$, $j = 1, 2, \dots$ and left continuous at $t = t_j$, $\beta \in (1, 2)$ is a constant, Δ is the Laplacian operator in \mathbb{R}^n , Ω is a bounded domain in \mathbb{R}^n with a smooth boundary $\partial\Omega$, $\Omega = \Omega \cup \partial\Omega$, N is the unit out normal vector to $\partial\Omega$.

The ingenuity of impulsive fractional partial differential is that the equation has two variables. Impulse phenomenon occurs on the variable t , and the equation is still continuous for the variable x . And the authors skillfully solved the problem of discontinuity at the impulse points by using the boundary condition.

In 2019, Mouffak Benchohra, Samira Hamani and Yong Zhou [3] dealt with the existence of oscillatory and nonoscillatory solutions for the following class of initial value problems for impulsive fractional differential with Caputo–Hadamard derivative inclusion,

$$\begin{cases} {}^{HC}D_{t_k}^\alpha y(t) \in F(t, y(t)), & t \in J = (t_k, t_{k+1}), \\ y(t_k^+) = I_k(y(t_k^-)), & k = 1, 2, \dots, \\ y(1) = y_*. \end{cases}$$

By using the concept of upper and lower solutions and the fixed point theorem, the authors obtained the existence theorems of oscillatory and non-oscillatory solutions of the above equation.

Motivated by the above papers, we consider the oscillatory behavior of solutions of following fractional impulsive differential equation

$$\begin{cases} {}^c D_a^\alpha x(t) = e(t) + f(t, x(t)), & a > 1, t \in J' := J / \{t_1, \dots, t_m\}, J := [a, \infty), \\ \Delta x(t_k) = y_k, \quad \Delta x'(t_k) = \bar{y}_k, & k = 1, 2, \dots, \\ x(a) = x_0, x'(a) = \bar{x}, \end{cases} \quad (1.1)$$

where ${}^c D_a^\alpha$ is the Caputo derivative of the order $\alpha \in (1, 2)$, $x_0, \bar{x}, y_k, \bar{y}_k \in \mathbb{R}$, t_k satisfy $a = t_0 < t_1 < \dots < t_m \rightarrow \infty$ as $m \rightarrow \infty$. $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$ with $u(t_k^+) = \lim_{\varepsilon \rightarrow 0^+} u(t_k + \varepsilon)$ and $u(t_k^-) = \lim_{\varepsilon \rightarrow 0^-} u(t_k + \varepsilon)$ represent the right and left limits of $u(t)$ at $t = t_k$,

The purpose of this paper is to study the oscillation of impulsive fractional differential equation. We analyze the integral expression of (1.1) to obtain oscillatory sufficient condition for (1.1). Since this is the first study of oscillation of impulsive fractional differential equation with Caputo derivative, we choose a relatively simple impulsive condition,

$$\Delta x(t_k) = y_k, \quad \Delta x'(t_k) = \bar{y}_k, \quad k = 1, 2, \dots,$$

which has the advantage of getting the expression of the solution. Although the central idea of our method is similar to Grace [5], the detailed technique is different from the one given in [5].

This paper is structured as follows. In Section 2, we present necessary notation, lemma and definition. In Section 3, we state and prove our main results. At last, one illustrative example are proposed.

2. Preliminaries

In this section, we will present some necessary knowledge and notation.

Lemma 2.1 (Lemma 2.6, [8]). *Let β, γ and p be positive constants such that $[p(\beta - 1) + 1] > 0$, $p(\gamma - 1) + 1 > 0$. Then*

$$\int_0^t (t-s)^{p(\beta-1)} s^{p(\gamma-1)} ds = t^\theta B(p(\gamma-1) + 1, p(\beta-1) + 1), \quad t \geq 0,$$

where $B(\cdot, \cdot)$ is Beta function of form

$$B(\xi, \eta) = \int_0^1 s^{\xi-1} (1-s)^{\eta-1} ds,$$

$$\xi > 0, \quad \eta > 0 \quad \text{and} \quad \theta = p(\beta + \gamma - 2) + 1.$$

Lemma 2.2 (Lemma 3.1, [13]). *Let $\alpha \in (1, 2)$ and $\{e(s) + f(s, x(s))\} : J \rightarrow \mathbb{R}$ be continuous. A function x given by*

$$x(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} [e(s) + f(s, x(s))] ds + x_0 + \bar{x}_0(t-a), & \text{for } t \in [t_0, t_1], \\ \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} [e(s) + f(s, x(s))] ds \\ + \sum_{i=1}^k \bar{y}_i(t-t_i) + \sum_{i=1}^k y_i + x_0 + \bar{x}_0(t-a), & \text{for } t \in (t_k, t_{k+1}], \quad k = 1, 2, \dots \end{cases} \quad (2.1)$$

is the equivalent form of equation (1.1).

We can easily see that

$$x'(t) = \begin{cases} \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} [e(s) + f(s, x(s))] ds + \bar{x}_0, & \text{for } t \in [t_0, t_1], \\ \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} [e(s) + f(s, x(s))] ds \\ + \sum_{i=1}^k \bar{y}_i t + \bar{x}_0, & \text{for } t \in (t_k, t_{k+1}], \quad k = 1, 2, \dots \end{cases}$$

As usual, a solution is called oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is said to be nonoscillatory. In other words, a solution is said to be oscillatory if there exists an increasing divergent sequence $\{\xi_k\}_{k \in \mathbb{N}} \subset [\theta_0, \infty)$ such that $x(\xi_k^+)x(\xi_k^-) \leq 0$ for all $k \in \mathbb{N}$.

3. Main results

We are now in a position to state and prove our main results .

Theorem 3.1. *Suppose that $1 < \alpha < 2$, $p > 1$, $\gamma > 0$, $p(\alpha-2)+1 > 0$, $p(\gamma-1)+1 > 0$, $q = \frac{p}{p-1}$, and the function $e(t) : J \rightarrow \mathbb{R}$ is continuous such that*

$$\frac{1}{t} \int_a^t (t-s)^{\alpha-1} |e(s)| ds \text{ is bounded for all } t \geq a \quad (3.1)$$

and the function $f(t, x)$ satisfies the following conditions.

- (i) $f(t, x)$ is continuous in $D = \{(t, x) : t \in J, x \in \mathbb{R}\}$.
- (ii) There are continuous nonnegative functions $g, h : \mathbb{R}^+ := [a, \infty) \rightarrow \mathbb{R}^+$, g is nondecreasing and let $0 < \gamma \leq 3 - \alpha - 1/p$ such that

$$|f(t, x)| \leq t^{\gamma-1} h(t) g\left(\frac{|x|}{t}\right), \quad t > a, \quad (t, x) \in D, \quad (3.2)$$

and

$$\int_a^\infty s^{\theta q/p} h^q(s) ds < \infty, \quad (3.3)$$

where $\theta := p(\alpha + \gamma - 3) + 1 \leq 0$.

(iii)

$$\int_a^\infty \frac{d\eta}{g^q(\eta)} \rightarrow \infty. \quad (3.4)$$

The impulsive points meet the following condition.

(iv) There is a constant M such that

$$\left| \sum_{i=1}^k \bar{y}_i \right| < M, \quad \left| \sum_{i=1}^k y_i \right| < M, \quad k = 1, 2, \dots \quad (3.5)$$

If $x(t)$ is a solution of (1.1), then

$$\limsup_{t \rightarrow \infty} \frac{|x(t)|}{t} < \infty. \quad (3.6)$$

Proof. We obtain from (2.1) that

$$\begin{aligned} |x(t)| &\leq |x_0| + |\bar{x}_0|(t-a) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} |e(s)| ds + \left| \sum_{i=1}^k \bar{y}_i \right| t + \left| \sum_{i=1}^k y_i \right| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} |f(s, x(s))| ds, \quad t \in (t_k, t_{k+1}]. \end{aligned}$$

Then by applying condition (3.2), we have

$$\begin{aligned} |x(t)| &\leq |x_0| + |\bar{x}_0|(t-a) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} |e(s)| ds + \left| \sum_{i=1}^k \bar{y}_i \right| t + \left| \sum_{i=1}^k y_i \right| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} s^{\gamma-1} h(s) g \left(\frac{|x(s)|}{s} \right) ds, \quad t \in (t_k, t_{k+1}]. \end{aligned}$$

From (3.1), we obtain $\frac{1}{t} \int_a^t (t-s)^{\alpha-1} |e(s)| ds \leq c$ for all $t \geq a$, where d is a constant.

Let $C(k) = |x_0| + |\bar{x}_0| + \left| \sum_{i=1}^k \bar{y}_i \right| + \left| \sum_{i=1}^k y_i \right| + \frac{c}{\Gamma(\alpha)}$. We have

$$\begin{aligned} |x(t)| &\leq C(k)t + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} s^{\gamma-1} h(s) g \left(\frac{|x(s)|}{s} \right) ds \\ &\leq C(k)t + \frac{1}{\Gamma(\alpha)} (t-a) \int_a^t (t-s)^{\alpha-2} s^{\gamma-1} h(s) g \left(\frac{|x(s)|}{s} \right) ds, \quad t \in (t_k, t_{k+1}]. \end{aligned}$$

This yields the inequality

$$\frac{|x(t)|}{t} \leq C(k) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-2} s^{\gamma-1} h(s) g \left(\frac{|x(s)|}{s} \right) ds, \quad t \in (t_k, t_{k+1}]. \quad (3.7)$$

If we denote that $z(t)$ is the right side of the inequality (3.7). We obtain the inequality

$$\frac{|x(t)|}{t} \leq z(t, k), \quad t \in (t_k, t_{k+1}]. \quad (3.8)$$

Since the function g is nondecreasing, the inequality (3.8) yields

$$g \left(\frac{|x(t)|}{t} \right) \leq g(z(t, k)), \quad t \in (t_k, t_{k+1}],$$

and from definition of $z(t, k)$ we get

$$z(t, k) \leq 1 + C(k) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\beta-1} s^{\gamma-1} h(s) g(z(s, k)) ds, \quad t \in (t_k, t_{k+1}], \quad (3.9)$$

where $0 < \beta = \alpha - 1 < 1$.

Applying Hölders inequality and Lemma 2.1, we obtain

$$\begin{aligned} & \int_a^t (t-s)^{\beta-1} s^{\gamma-1} h(s) g(z(s, k)) ds \\ & \leq \left(\int_a^t (t-s)^{p(\beta-1)} s^{p(\gamma-1)} ds \right)^{1/p} \left(\int_a^t h^q(s) g^q(z(s, k)) ds \right)^{1/q} \\ & \leq \left(\int_0^t (t-s)^{p(\beta-1)} s^{p(\gamma-1)} ds \right)^{1/p} \left(\int_a^t h^q(s) g^q(z(s, k)) ds \right)^{1/q} \\ & \leq (Bt^\theta)^{1/p} \left(\int_a^t h^q(s) g^q(z(s, k)) ds \right)^{1/q}, \quad t \in (t_k, t_{k+1}], \end{aligned}$$

where $B := B(p(\gamma - 1) + 1, p(\beta - 1) + 1)$, $\theta = p(\alpha + \gamma - 3) + 1 \leq 0$. Using the fact that $\theta \leq 0$ and $t > s \geq a$, we have

$$\begin{aligned} & \int_a^t (t-s)^{\beta-1} s^{\gamma-1} h(s) g(z(s, k)) ds \\ & \leq B^{1/p} \left(\int_a^t s^{\theta q/p} h^q(s) g^q(z(s, k)) ds \right)^{1/q}, \quad t \in (t_k, t_{k+1}]. \end{aligned} \tag{3.10}$$

Using (3.10) and the elementary inequality

$$(x + y)^q \leq 2^{q-1}(x^q + y^q), \quad x, y \geq 0, \quad q > 1.$$

For $t \in (t_k, t_{k+1}]$, we obtain from (3.9) that

$$z^q(t, k) \leq 2^{q-1} \left((1 + C(k))^q + (B^{1/p} \frac{1}{\Gamma(\alpha)})^q \int_a^t s^{\theta q/p} h^q(s) g^q(z(s, k)) ds \right).$$

If we denote $P_1(k) = 2^{q-1}[(1 + C(k))^q]$, $Q_1 = 2^{q-1}(B^{1/p} \frac{1}{\Gamma(\alpha)})^q$, then

$$z^q(t, k) \leq P_1(k) + Q_1 \int_a^t s^{\theta q/p} h^q(s) g^q(z(s, k)) ds, \quad t \in (t_k, t_{k+1}].$$

Denote

$$\begin{aligned} w(\eta) &= g^q(\eta), \\ G(\xi) &= \int_{z_k}^\xi \frac{d\eta}{w(\eta)}, \quad z_k = z(t_k^+, k). \end{aligned} \tag{3.11}$$

Since $G(z(t, k)) = \int_{z_k}^{z(t, k)} \frac{d\eta}{g^q(\eta)}$, condition (iii) implies that $\lim_{z(t, k) \rightarrow \infty} G(z(t, k)) = \infty$, then by the Bihari Lemma [2] we get

$$z^q(t, k) \leq K(k) := G^{-1} \left(G(P_1(k)) + Q_1 \int_a^t s^{\theta q/p} h^q(s) ds \right), \quad t \in (t_k, t_{k+1}], k = 1, 2, \dots$$

Because of condition (iv) and boundedness of $P_1(k)$. Hence from (ii) and (3.11) we conclude $K(k)$, $k = 1, 2, \dots$ is bounded. Then

$$z^q(t, k) \leq K = \sup_{k \geq 1} K(k), \quad t > t_1, \quad k = 1, 2, \dots$$

We obtain that $z(t, k) \leq K^{1/q}$, and from (3.8), we have

$$\frac{|x(t)|}{t} \leq K^{1/q}, \quad t \geq t_1.$$

We conclude that

$$\limsup_{t \rightarrow \infty} \frac{|x(t)|}{t} < \infty.$$

This completes the proof. □

Remark 3.1. We note that Theorem 3.1 remains valid if $g(z) = z$. In this case condition (3.4) is automatically fulfilled. Also, when $0 < \gamma = 3 - \alpha - 1/p$ we have $\theta := p(\alpha + \gamma - 3) - 1 = 0$.

Theorem 3.2. *Let the constants α, p, q, γ and θ be defined as is in Theorem 3.1, conditions (3.1)-(3.5) hold. If for any constant $\bar{d} \in (M\Gamma(\alpha) + \bar{x}_0\Gamma(\alpha), 1 + M\Gamma(\alpha) + \bar{x}_0\Gamma(\alpha))$,*

$$\liminf_{t \rightarrow \infty} \left[\bar{d}t + \int_a^t (t-s)^{\alpha-1} e(s) ds \right] = -\infty \tag{3.12}$$

or

$$\limsup_{t \rightarrow \infty} \left[\bar{d}t + \int_a^t (t-s)^{\alpha-1} e(s) ds \right] = \infty, \tag{3.13}$$

then (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). We may assume that $x(t) > 0$ for all $t \geq c_0$ for some $c_0 \geq a$. Because of Theorem 3.1 and (3.3), we have

$$\lim_{t \rightarrow \infty} \int_t^\infty s^{\theta q/p} h^q(s) g^q\left(\frac{x(s)}{s}\right) ds = 0.$$

So there is $c_1 \geq c_0$ that satisfies

$$0 < \int_{c_1}^\infty s^{\theta q/p} h^q(s) g^q\left(\frac{x(s)}{s}\right) ds < 1. \tag{3.14}$$

Without loss of generality, we can assume that $a \leq c_0 \leq c_1 < t_1$.

Proceeding similarly to the proof of Theorem 3.1, we obtain

$$\begin{aligned} x(t) &\leq x_0 + \bar{x}_0(t-a) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} e(s) ds + \sum_{i=1}^k \bar{y}_i t + \sum_{i=1}^k y_i \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s, x(s)) ds, \quad t \in (t_k, t_{k+1}], \quad k \geq 1. \end{aligned}$$

Using condition (3.2) in the above equation, we have

$$\begin{aligned} x(t) &\leq x_0 + \bar{x}_0(t-a) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} e(s) ds + \frac{1}{\Gamma(\alpha)} \int_a^{c_1} (t-s)^{\alpha-1} f(s, x(s)) ds \\ &\quad + \sum_{i=1}^k \bar{y}_i t + \sum_{i=1}^k y_i + \frac{1}{\Gamma(\alpha)} \int_{c_1}^t (t-s)^{\alpha-1} s^{\gamma-1} h(s) g\left(\frac{x(s)}{s}\right) ds \\ &\leq x_0 + \bar{x}_0(t-a) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} e(s) ds + \frac{1}{\Gamma(\alpha)} \int_a^{c_1} (t-s)^{\alpha-1} f(s, x(s)) ds \\ &\quad + \sum_{i=1}^k \bar{y}_i t + \sum_{i=1}^k y_i + \frac{1}{\Gamma(\alpha)} t \int_{c_1}^t (t-s)^{\alpha-2} s^{\gamma-1} h(s) g\left(\frac{x(s)}{s}\right) ds, t \in (t_k, t_{k+1}], k \geq 1. \end{aligned}$$

Using inequality (3.10) and condition (iv), we have

$$\begin{aligned} x(t) &\leq \bar{x}_0(t-a) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} e(s) ds + \frac{1}{\Gamma(\alpha)} \int_a^{c_1} (t-s)^{\alpha-1} f(s, x(s)) ds \\ &\quad + \frac{1}{\Gamma(\alpha)} t B^{1/p} \left(\int_{c_1}^t s^{\theta q/p} h^q(s) g^q\left(\frac{x(s)}{s}\right) ds \right)^{1/q} + Mt + M + x_0, \quad t \geq t_1. \end{aligned} \tag{3.15}$$

Clearly, the conclusion of Theorem 3.1 holds, i.e. $\limsup_{t \rightarrow \infty} \frac{|x(t)|}{t} < \infty$. This together with (3.3) imply that the third integral on the right side of (3.15) is bounded and hence one can easily find

$$x(t) \leq d + \frac{1}{\Gamma(\alpha)} \left(\bar{d}t + \int_a^t (t-s)^{\alpha-1} e(s) ds \right) t, \quad t \geq t_1, \tag{3.16}$$

where

$$d = x_0 + M + \frac{1}{\Gamma(\alpha)} \int_a^{c_1} (t-s)^{\alpha-1} f(s, x(s)) ds$$

and

$$\bar{d} := \bar{x}_0 + M\Gamma(\alpha) + \Gamma(\alpha) B^{1/p} \left(\int_{c_1}^t s^{\theta q/p} h^q(s) g^q\left(\frac{x(s)}{s}\right) ds \right)^{1/q}$$

are constants. From (3.14), we have $M\Gamma(\alpha) + \bar{x}_0\Gamma(\alpha) < \bar{d} < 1 + M\Gamma(\alpha) + \bar{x}_0\Gamma(\alpha)$. Finally, taking limit inferior in (3.16) as $t \rightarrow \infty$ and using (3.12) in a contradiction with the fact that $x(t)$ is eventually positive. If $x(t)$ is eventually negative, we set $y = -x$, then we can easily see that y satisfies (1.1) with $e(t)$ being replaced by $-e(t)$ and $f(t, x)$ by $-f(t, -y)$. The proof of this case is the same as above and hence is omitted. This completes the proof of the theorem. \square

4. Example

In this section, we will present an example to illustrate our main results.

Example 4.1. Consider impulsive $\frac{3}{2}$ -order fractional differential equation

$$\begin{cases} {}^c D_2^{3/2} x(t) = e(t) + t^{\gamma-1} h(t) g\left(\frac{x}{t}\right), t \in J' := J / \{t_1, \dots, t_m\}, J := [2, \infty) \quad \alpha = \frac{3}{2}, \\ \Delta x(t_k) = \frac{1}{k(k+1)}, \quad \Delta x'(t_k) = \frac{1}{k(k+1)}, \quad k = 1, 2, \dots, \\ x(2) = x_0, \quad x'(2) = \bar{x}_0, \end{cases} \tag{4.1}$$

where $x_0 = 1$, $\bar{x}_0 < \frac{B(\frac{1}{2}, \frac{3}{2}) - \Gamma(\frac{3}{2}) - 1}{\Gamma(\frac{3}{2})}$, $h(t) = (t+1)^{-\frac{2}{3}}$, $g(\eta) = \frac{1}{\eta^{1/q}}$, $e(t) = -t^{-\frac{1}{2}}$, $s \geq 2$. Let $p = \frac{3}{2}$, $\gamma = \frac{5}{6}$, $a = 2$. Clearly, we see that $p(\alpha - 2) + 1 = \frac{1}{4} > 0$, $p(\gamma - 1) + 1 = \frac{3}{4} > 0$, $q = \frac{p}{p-1} = 3$ and $\theta = p(\alpha + \gamma - 3) + 1 = 0$. Then

$$\begin{aligned} \frac{1}{t} \int_a^t (t-s)^{\alpha-1} |e(s)| ds &= \frac{1}{t} \int_2^t s^{-\frac{1}{2}} (t-s)^{\frac{1}{2}} ds \\ &= \frac{1}{t} \int_2^t s^{-\frac{1}{2}} t^{\frac{1}{2}} \left(1 - \frac{s}{t}\right)^{\frac{1}{2}} ds, \quad t \geq 2. \end{aligned} \tag{4.2}$$

Let $v := \frac{s}{t}$. Then (4.2) can be written as

$$\begin{aligned} \frac{1}{t} \int_a^t (t-s)^{\alpha-1} |e(s)| ds &= \frac{1}{t} \int_{\frac{2}{t}}^1 v^{-\frac{1}{2}} t^{-\frac{1}{2}} t^{\frac{1}{2}} (1-v)^{\frac{1}{2}} t dv \\ &= \int_{\frac{2}{t}}^1 v^{-\frac{1}{2}} (1-v)^{\frac{1}{2}} dv \\ &\leq \int_0^1 v^{-\frac{1}{2}} (1-v)^{\frac{1}{2}} dv = B\left(\frac{1}{2}, \frac{3}{2}\right), \quad t \geq 2. \end{aligned}$$

And

$$\int_a^\infty s^{\theta q/p} h^q(s) ds = \int_2^\infty (s+1)^{-2} ds = \frac{1}{3}.$$

We can obtain

$$\int_a^\infty \frac{d\eta}{g^q(\eta)} = \int_2^\infty \eta d\eta \rightarrow \infty,$$

and

$$\left| \sum_{i=1}^k \bar{y}_i \right| < 1, \quad \left| \sum_{i=1}^k y_i \right| < 1, \quad k = 1, 2, \dots$$

Then all conditions of Theorem 3.1 are satisfied and hence every solution of (4.1) satisfies

$$\limsup_{t \rightarrow \infty} \frac{|x(t)|}{t} < \infty.$$

From (4.2), we can obtain

$$\bar{d}t + \int_a^t (t-s)^{\alpha-1} e(s) ds = \bar{d}t - t \int_{\frac{2}{t}}^1 v^{-\frac{1}{2}} (1-v)^{\frac{1}{2}} dv = t \left(\bar{d} - \int_{\frac{2}{t}}^1 v^{-\frac{1}{2}} (1-v)^{\frac{1}{2}} dv \right).$$

Since

$$\lim_{t \rightarrow \infty} \int_{\frac{2}{t}}^1 v^{-\frac{1}{2}} (1-v)^{\frac{1}{2}} dv = B\left(\frac{1}{2}, \frac{3}{2}\right).$$

And from $M\Gamma(\alpha) + \bar{x}_0\Gamma(\alpha) < \bar{d} < 1 + M\Gamma(\alpha) + \bar{x}_0\Gamma(\alpha)$, we obtain

$$\bar{d} < B\left(\frac{1}{2}, \frac{3}{2}\right).$$

So

$$\bar{d} - \int_{\frac{2}{t}}^1 v^{-\frac{1}{2}} (1-v)^{\frac{1}{2}} dv < 0 \quad \text{for sufficiently large } t.$$

Therefore

$$\lim_{t \rightarrow \infty} \left(\bar{d}t + \int_a^t (t-s)^{\alpha-1} e(s) ds \right) \rightarrow -\infty.$$

Thus (4.1) satisfies (3.12). By Theorem 3.2, all solutions of (4.1) are oscillatory.

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