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ORBITAL STABILITY OF PERIODIC TRAVELING WAVE SOLUTIONS TO THE GENERALIZED LONG-SHORT WAVE EQUATIONS*

Xiaoxiao Zheng[1,†], Jie Xin[1] and Xiaoming Peng[2]

Abstract This paper investigates the orbital stability of periodic traveling wave solutions to the generalized Long-Short wave equations $\begin{cases} i\varepsilon_t+\varepsilon_{xx}=n\varepsilon+\alpha|\varepsilon|^2\varepsilon, \\ n_t=(|\varepsilon|^2)_x, x\in R. \end{cases}$ Firstly, we show that there exist a smooth curve of positive traveling wave solutions of dnoidal type with a fixed fundamental period L for the generalized Long-Short wave equations. Then, combining the classical method proposed by Benjamin, Bona et al., and detailed spectral analysis given by using Lamé equation and Floquet theory, we show that the dnoidal type periodic wave solution is orbitally stable by perturbations with period L. As the modulus of the Jacobian elliptic function $k \to 1$, we obtain the orbital stability results of solitary wave solution with zero asymptotic value for the generalized Long-Short equations. In particular, as $\alpha = 0$, we can also obtain the orbital stability results of periodic wave solutions and solitary wave solutions for the long-short wave resonance equations. The results in the present paper improve and extend the previous stability results of long-shore wave equations and its extension equations.

Keywords Generalized Long-Short wave equations, periodic traveling waves, orbital stability.

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1. Introduction

The long-short wave resonance equations

$$\begin{cases} iu_t + u_{xx} = \mu uv, \\ v_t + \beta|u|^2_x = 0, \quad x \in R \end{cases} \tag{1.1}$$

was first derived by Djordjevic and Redekopp in [10] to describe the resonance interaction between the long wave and the short wave. In Eqs.(1.1), $u(x,t)$ is a

†Corresponding author. Email address:xiaoxiaozheng87@163.com(X. Zheng)
1School of Mathematical Sciences, Qufu Normal University, 273155 Qufu, Shandong, China
2School of Statistics and Mathematics, Guangdong University of Finance and Economics, 510320 Guangzhou, China
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complex valued function and denotes the envelope of the short wave, $v(x,t)$ is a real valued function and denotes amplitude of long wave. As pointed out in [10], the physical significance of Eqs.(1.1) is that the dispersion of the short wave is balanced by nonlinear interaction of the long wave with the short wave, while the evolution of the long wave is driven by the self-interaction of the short wave. These equations also appear in an analysis of internal wave [12], as well as Rossby waves. In plasma physics similar equations can be used to describe the resonance between high frequency electron plasma oscillations and associated low frequency ion density perturbations [22]. Ma [20] found that Eqs.(1.1) can be rewritten in Lax's formulation and the Cauchy problem of Eqs.(1.1) can be solved by the inverse scattering method. Laurencot [19] discussed the well-posedness of initial value problem of (1.1) and studied the orbital stability of solitary waves by a modification to method developed by Benjamin, Bona et al.

The generalized Long-Short wave equations

$$\begin{cases} i\varepsilon_t + \lambda\varepsilon_{xx} = \mu n\varepsilon + \alpha|\varepsilon|^2\varepsilon, \\ n_t = \nu(|\varepsilon|^2)_x, \quad x \in R \end{cases} \tag{1.2}$$

was presented by Benney [5] which permitted both long and short wave solution. Here $\alpha, \lambda, \mu, \nu \in R$ with $\lambda\mu\nu \neq 0$ and a complex function ε is the envelope of the short wave, while a real function n is the amplitude of long wave. By virtue of an appropriate change of both independent and dependent variables, one can take $\lambda = \mu = \nu = 1$, to get

$$\begin{cases} i\varepsilon_t + \varepsilon_{xx} = n\varepsilon + \alpha|\varepsilon|^2\varepsilon, \\ n_t = (|\varepsilon|^2)_x, \quad x \in R, \end{cases} \tag{1.3}$$

where $\alpha \in R$. The system (1.3) arises in the study of surface waves with both gravity and capillary modes present and also in plasma physics (see [26]). For a complete orbital stability theory in the space $H^1_{per}([0,L]) \times L^2_{per}([0,L])$, we need to present information about the well-posedness problem for the generalized Long-Short wave equations (1.3). Many authors have investigated the well-posedness of local solution or/and global solution for the initial value problem and periodic initial value problem of the systems (1.3)(see Ref. [11, 15, 24]). Guo and Chen in [16, 17] studied the orbital stability of solitary waves of (1.3) by applying the abstract results of Grillakis et al. [13, 14], while the method of proof is different from the method used in Ref. [19]. Moreover, Angulo et al. [23] considered the existence and orbital stability of solitary wave solutions for an interaction equation of short and long dispersive waves.

Obviously, if $\alpha = 0$, Eqs.(1.3) deduces to Eqs.(1.1). Thus, we can say that Eqs.(1.1) is the special forms of Eqs.(1.3). Based on the abstract stability theory and other method, the stability results of solitary wave solutions of Eqs.(1.3) was obtained. However, to the author's knowledge, no research has been done on the orbital stability for the periodic wave solutions of the generalized long-short wave equations (1.3). As the modulus of the Jacobian elliptic function $k \to 1$, we can obtain the stability results of solitary waves from the study of periodic waves.

Recently, many authors have studied the orbital stability of periodic traveling waves (see Ref. [1–3, 9, 25]). The study of orbital stability for periodic traveling waves is valuable. In this paper, we are interested in the existence and orbital stability of periodic traveling wave solutions for the generalized Long-Short wave equations. We focus on solutions for (1.3) of the form

$$\varepsilon(x,t) = e^{-i\omega t} e^{i\frac{c}{2}(x-ct)} \phi_{\omega,c}(x-ct), \qquad \text{and} \quad n(x,t) = \psi_{\omega,c}(x-ct), \tag{1.4}$$

where $\omega, c \in R$, $\xi = x - ct$, $\phi_{\omega,c}$, $\psi_{\omega,c} : R \to R$ being periodic smooth functions with the same arbitrary fundamental period $L > 0$.

Because the stability in view here refers to perturbations of the periodic-wave profile itself, a study of the initial-value problem for Eqs.(1.3) is necessary. Similar to Theorem [11, 15], we have the following general lemma regarding the existence of solutions to the initial value problem of Eqs.(1.3).

Lemma 1.1. *For any fixed initial values* $(\varepsilon_0, n_0) \in H^{\frac{1}{2}}(0,L) \times L^2(0,L)$*, there exists* $T \geq 0$ *and a unique solution* $(\varepsilon, n) \in C([-T,T]; H^{\frac{1}{2}}) \times C([-T,T]; L^2)$.

Next, based on the classical method proposed by Benjamin [6], Bona [7], and the detailed spectral analysis given by using Lamé equation and Floquet theory, we study orbital stability of the periodic wave solutions for Eqs.(1.3). Firstly, we show that there exist smooth periodic traveling wave solutions (1.4), where $\phi_{\omega,c}, \psi_{\omega,c}$ are smooth function with given period $L > 0$, and ω belongs to a determined interval in R.

Theorem 1.1. *Let* $L > 0$, $c > 0$ *and* $1 - \alpha c > 0$ *be arbitrarily fixed. Consider* $\nu_0 > \frac{2\pi^2}{L^2}$ *and the unique* $\eta_{2,0} = \eta_2(\nu_0) \in (0, \sqrt{\nu_0 \beta})$ *such that* $T_{\phi_{\nu_0}} = T_{\psi_{\nu_0}} = L$. *Then,*
(1) there exist intervals $I(\nu_0)$ *and* $B(\eta_{2,0})$ *around* ν_0 *and* $\eta_{2,0}$ *respectively, and a unique smooth function* $\Pi : I(\nu_0) \to B(\eta_{2,0})$ *such that* $\Pi(\nu_0) = \eta_{2,0}$ *and*

$$\frac{2\sqrt{2\beta}}{\sqrt{2\nu\beta - \eta_2^2}} K(k) = L,$$

for all $\nu \in I(\nu_0)$, $\eta_2 \in \Pi(\nu)$ *and*

$$k^2 = k^2(\nu) = \frac{2\nu\beta - 2\eta_2^2}{2\nu\beta - \eta_2^2} \in (0,1). \tag{1.5}$$

(2) The dnoidal waves $\phi(\cdot;\eta_1,\eta_2)$ *and* $\psi(\cdot;\eta_1,\eta_2)$ *in (2.10) and (2.11) determined by* $\eta_1 \equiv \eta_1(\nu)$, $\eta_2 \equiv \eta_2(\nu) = \Pi(\nu)$, *with* $\eta_1^2 + \eta_2^2 = 2\nu\beta$, *have fundamental period* L *and satisfy (2.1) and (2.4). Moreover, the mapping*

$$\nu \in I(\nu_0) \mapsto (\phi(\cdot;\eta_1(\nu),\eta_2(\nu)), \psi(\cdot;\eta_1(\nu),\eta_2(\nu))) \in H^{n+1}_{per}([0,L]) \times H^n_{per}([0,L]) \tag{1.6}$$

is smooth for all integer $n \geq 1$.
(3) $I(\nu_0)$ *can be chosen as* $(\frac{2\pi^2}{L^2}, +\infty)$.

Then, by applying the Floquet theory related to the linear operator

$$L_1 = -\frac{d^2}{dx^2} - (\omega + \frac{c^2}{4}) + 3(1-\alpha c)\psi \quad \text{and} \quad L_2 = -\frac{d^2}{dx^2} - (\omega + \frac{c^2}{4}) + (1-\alpha c)\psi, \tag{1.7}$$

and the stability framework established in [6, 7, 27], we show that for $1 - \alpha c > 0$, $c > 0$, $\omega + \frac{c^2}{4} < -\frac{2\pi^2}{L^2}$, and $\widetilde{\phi}(\xi) = e^{i\frac{c}{2}\xi}\phi(\xi)$, the orbit

$$\vartheta = \{(e^{i\theta}\widetilde{\phi}(\cdot + y), \psi(\cdot + y) : (\theta, y) \in [0, 2\pi) \times [0, L])\}$$

will be stable in the space $H^1_{per}([0,L]) \times L^2_{per}([0,L])$ by the periodic flow of the system (1.3).

Theorem 1.2. *Let $L>0$, $c>0$ and $1-\alpha c>0$ be arbitrarily fixed. Consider the smooth curve of dnoidal waves $\nu\in(\frac{2\pi^2}{L^2},+\infty)\mapsto(\phi(\cdot;\eta_1(\nu),\eta_2(\nu)),\psi(\cdot;\eta_1(\nu),\eta_2(\nu)))$ determined by Theorem 1.1. Then, for ω and c such that $\nu=-(\omega+\frac{c^2}{4})>\frac{2\pi^2}{L^2}$, the orbit generated by $\Psi(x,t)=(\widetilde{\phi},\psi)$ is orbitally stable in X with regard to the periodic flow generated by system (1.3).*

The rest of this paper is organized as follows. In section 2, we devote to prove the existence of a smooth curve of dnoidal wave solutions for Eqs.(1.3). Section 3 studies the spectral analysis of two certain self-adjoint operators with a crucial role to obtain our stability result. In section 4, we show our stability result of the dnoidal waves solutions for system (1.3).

2. Existence of dnoidal wave solutions for the generalized Long-Short wave equations

In this section, we devote to show the existence of a smooth curve of dnoidal wave solutions of the form (1.4) for the generalized Long-Short wave equations (1.3).

Substituting the form of solutions in (1.4) into the system (1.3), we obtain that $\phi=\phi_{\omega,c},\psi=\psi_{\omega,c}$ have to satisfy the following ordinary differential system

$$\begin{cases}\phi''+(\omega+\dfrac{c^2}{4}-\psi)\phi-\alpha\phi^3=0,\\ -c\psi'=2\phi\phi'.\end{cases}\tag{2.1}$$

Integrating the second equation of (2.1) and assuming the integration constant being zero, we have

$$\psi=-\frac{1}{c}\phi^2.\tag{2.2}$$

Then, substituting (2.2) into the first equation of (2.1), we have

$$\phi''+(\omega+\frac{c^2}{4})\phi+(\frac{1}{c}-\alpha)\phi^3=0.\tag{2.3}$$

Next, we prove that there exists an explicit periodic solution which will depend on Jacobian elliptic functions for Eq.(2.3). Multiplying (2.3) by ϕ' and integrating once, we get that ϕ must satisfy

$$(\phi')^2=\frac{1-\alpha c}{2c}[-\phi^4+\frac{2c}{1-\alpha c}(-\omega-\frac{c^2}{4})\phi^2+\frac{4c}{1-\alpha c}A_\phi],\tag{2.4}$$

where A_ϕ is a needed nonzero integration constant. For convenience, we make $F(t)=-t^4+\frac{2c}{1-\alpha c}(-\omega-\frac{c^2}{4})t^2+\frac{4c}{1-\alpha c}A_\phi$. We know that solutions of Eq.(2.4) depend on the roots of the polynomial $F(\phi)$. Assume that $1-\alpha c>0$, $c>0$, and $\omega+\frac{c^2}{4}<0$, for $-\frac{c}{4(1-\alpha c)}(\omega+\frac{c^2}{4})^2<A_\phi<0$, we have

$$\begin{aligned}F(t)&=-\big(t^2+\frac{c}{1-\alpha c}(\omega+\frac{c^2}{4})\big)^2+\frac{4c}{1-\alpha c}A_\phi+\frac{c^2}{(1-\alpha c)^2}(\omega+\frac{c^2}{4})^2\\&=(t^2+\frac{c}{1-\alpha c}(\omega+\frac{c^2}{4})+\sqrt{\frac{4c}{1-\alpha c}A_\phi+\frac{c^2}{(1-\alpha c)^2}(\omega+\frac{c^2}{4})^2})\end{aligned}$$

$$\cdot(\sqrt{\frac{4c}{1-\alpha c}A_\phi + \frac{c^2}{(1-\alpha c)^2}(\omega + \frac{c^2}{4})^2} - \frac{c}{1-\alpha c}(\omega + \frac{c^2}{4}) - t^2),$$

and

$$\frac{c}{1-\alpha c}(\omega + \frac{c^2}{4}) + \sqrt{\frac{4c}{1-\alpha c}A_\phi + \frac{c^2}{(1-\alpha c)^2}(\omega + \frac{c^2}{4})^2} < 0,$$
$$\sqrt{\frac{4c}{1-\alpha c}A_\phi + \frac{c^2}{(1-\alpha c)^2}(\omega + \frac{c^2}{4})^2} - \frac{c}{1-\alpha c}(\omega + \frac{c^2}{4}) > 0.$$

Hence, $F(t)$ has the real and symmetric roots $\pm\eta_1$ and $\pm\eta_2$. Without loss of generality, we assume that $0 < \eta_2 < \eta_1$. Hence, we can write

$$(\phi')^2 = \frac{1-\alpha c}{2c}(\phi^2 - \eta_2^2)(\eta_1^2 - \phi^2). \tag{2.5}$$

Then, by the above assumption, the left side of (2.5) is not negative. Therefore, we obtain that $\eta_2 \leq \phi \leq \eta_1$ and the $\eta_i^{;}s$ satisfy

$$\begin{cases} \eta_1^2 + \eta_2^2 = -\dfrac{2c}{1-\alpha c}(\omega + \dfrac{c^2}{4}), \\ -\eta_1^2\eta_2^2 = \dfrac{4c}{1-\alpha c}A_\phi. \end{cases} \tag{2.6}$$

Define $\rho = \frac{\phi}{\eta_1}$ and $k^2 = \frac{\eta_1^2 - \eta_2^2}{\eta_1^2}$, then (2.5) becomes

$$(\rho')^2 = \frac{(1-\alpha c)\eta_1^2}{2c}(\rho^2 - \frac{\eta_2^2}{\eta_1^2})(1-\rho^2). \tag{2.7}$$

Define a new variable χ through the relation $\rho^2 = 1 - k^2\sin^2\chi$, from (2.6) and (2.7), we get

$$(\chi')^2 = \frac{(1-\alpha c)\eta_1^2}{2c}(1 - k^2\sin^2\chi). \tag{2.8}$$

Then, according to the definition of the Jacobi elliptic function snoidal, we get that

$$\int_0^{\chi(\xi)} \frac{dt}{\sqrt{1-k^2\sin^2 t}} = \sqrt{\frac{1-\alpha c}{2c}}\eta_1\xi \tag{2.9}$$

has the solution

$$\sin(\chi(\xi)) = sn(\sqrt{\frac{1-\alpha c}{2c}}\eta_1\xi; k).$$

Hence, using the fact that $k^2sn^2 + dn^2 = 1$, we obtain

$$\rho(\xi) = \sqrt{1-k^2\sin^2\chi} = \sqrt{1 - k^2sn^2(\sqrt{\frac{1-\alpha c}{2c}}\eta_1\xi; k)} = dn(\sqrt{\frac{1-\alpha c}{2c}}\eta_1\xi; k),$$

and $\rho(0) = 1$. Substituting the form of $\rho(\xi)$ to the definition $\rho = \frac{\phi}{\eta_1}$, we get the dnoidal wave solution

$$\phi(\xi) = \eta_1 dn(\sqrt{\frac{1-\alpha c}{2c}}\eta_1\xi; k). \tag{2.10}$$

Substituting (2.10) into (2.2), we have

$$\psi(\xi) = -\frac{\eta_1^2}{c} dn^2(\sqrt{\frac{1-\alpha c}{2c}}\eta_1\xi; k). \tag{2.11}$$

Since dn has fundamental period $2K$, namely, $dn(u; k) = dn(u + 2K; k)$, where $K = K(k)$ represents the complete elliptic integral of first kind(see [8] or section 2 in [29]), we obtain that ϕ and ψ have fundamental period

$$T_\phi = T_\psi = \frac{2\sqrt{2c}}{\eta_1\sqrt{1-\alpha c}} K(k). \tag{2.12}$$

For convenience, we define $\nu = -(\omega + \frac{c^2}{4})$, and $\beta = \frac{c}{1-\alpha c}$. Then, from (2.6), we get $0 < \eta_2 < \sqrt{\nu\beta} < \eta_1 < \sqrt{2\nu\beta}$ and fundamental period $T_\phi = T_\psi$ can be seen as a function of variable η_2 only, that is

$$T_\phi(\eta_2) = T_\psi(\eta_2) = \frac{2\sqrt{2\beta}}{\sqrt{2\nu\beta - \eta_2^2}} K(k(\eta_2)), \quad \text{with} \quad k^2(\eta_2) = \frac{2\nu\beta - 2\eta_2^2}{2\nu\beta - \eta_2^2}. \tag{2.13}$$

Next, we will show that $T_\phi = T_\psi > \frac{\sqrt{2}\pi}{\sqrt{\nu}}$. Note that if $\eta_2 \to 0$, we have that $k(\eta_2) \to 1^-$, and then $K(k(\eta_2)) \to +\infty$. Therefore, $T_\phi, T_\psi \to +\infty$ as $\eta_2 \to 0$. On the other hand, if $\eta_2 \to \sqrt{\nu\beta}$, we have that $k(\eta_2) \to 0^+$, which imply that $K(k(\eta_2)) \to \frac{\pi}{2}$. Therefore, $T_\phi, T_\psi \to \frac{\sqrt{2}\pi}{\sqrt{\nu}}$ as $\eta_2 \to \sqrt{\nu\beta}$. Moreover, since the function $\eta_2 \in (0, \sqrt{\nu\beta}) \mapsto T_\phi(\eta_2) = T_\psi(\eta_2)$ is a strictly decreasing function (see proof of Theorem 1.1), it follows that $T_\phi = T_\psi > \frac{\sqrt{2}\pi}{\sqrt{\nu}}$.

For $L > 0$, $c > 0$ and $1 - \alpha c > 0$ fixed, we choose $\nu > 0$ such that $\sqrt{\nu} > \frac{\sqrt{2}\pi}{L}$. From the analysis given above, there exists a unique $\eta_2 \equiv \eta_2(\nu)$ such that the fundamental period of the dnoidal wave $\phi = \phi(\cdot; \eta_1(\nu); \eta_2(\nu))$ and $\psi = \psi(\cdot; \eta_1(\nu); \eta_2(\nu))$, will be $T_\phi(\eta_2) = T_\psi(\eta_2) = L$.

Remark 2.1. If $\eta_2 \to 0^+$, we obtain that $\eta_1 \to \sqrt{2\nu\beta}$, $k(\eta_2) \to 1^-$. Then, on the basis of the limitation $dn(x, 1) = sech(x)$, the (2.10) and (2.11) lose its periodicity in this limit and we obtain a wave form with a single hump and with "infinity period" of the form

$$\phi(\xi; \sqrt{2\nu\beta}, 0) \to \sqrt{2\nu\beta} sech(\sqrt{\nu}\xi), \quad \psi(\xi; \sqrt{2\nu\beta}, 0) \to -\frac{2\nu\beta}{c} sech^2(\sqrt{\nu}\xi),$$

which are exactly the classical ground state solutions for the generalized Long-Short wave equations.

In the following, by applying the implicit function theorem, we show that the Theorem 1.1 holds.

Proof of Theorem 1.1. We consider the open set

$$\Omega = \{(\eta, \nu) \in R^2 : \nu > \frac{2\pi^2}{L^2} \text{ and } \eta \in (0, \sqrt{\nu\beta})\},$$

and define $\Lambda : \Omega \to R$ by

$$\Lambda(\eta, \nu) = \frac{2\sqrt{2\beta}}{\sqrt{2\nu\beta - \eta^2}} K(k) - L, \tag{2.14}$$

where

$$k^2(\eta, \nu) = \frac{2\nu\beta - 2\eta^2}{2\nu\beta - \eta^2}. \tag{2.15}$$

From the hypotheses, we get $\Lambda(\eta_{2,0}, \nu_0) = 0$. Next, we show that $\partial_\eta \Lambda < 0$ in Ω. Firstly, differentiating (2.15) with respect to η, we have

$$\frac{\partial k}{\partial \eta} = -\frac{2\eta\nu\beta}{k(2\nu\beta - \eta^2)^2} < 0. \tag{2.16}$$

Hence, the function $k(\eta, \nu)$ decreases strictly with respect to η. Then, according to the relation

$$\frac{dK(k)}{dk} = \frac{E(k) - k'^2 K(k)}{kk'^2}, \tag{2.17}$$

differentiating (2.14) with respect to η, and combining (2.16), (2.17), we obtain

$$\begin{aligned} \partial_\eta \Lambda(\eta, \nu) &= \frac{2\eta\sqrt{2\beta}}{(2\nu\beta - \eta^2)^{\frac{3}{2}}} K(k) + \frac{2\sqrt{2\beta}}{\sqrt{2\nu\beta - \eta^2}} \frac{dK}{dk} \frac{\partial k}{\partial \eta} \\ &= \frac{2\eta\sqrt{2\beta}}{(2\nu\beta - \eta^2)^{\frac{5}{2}} k^2 k'^2} [k^2 k'^2 (2\nu\beta - \eta^2) K(k) - 2\nu\beta(E(k) - k'^2 K(k))] \\ &= \frac{2\eta\sqrt{2\beta}}{(2\nu\beta - \eta^2)^{\frac{5}{2}} k^2 k'^2} [\frac{4\nu\beta k'^2}{1 + k'^2} K(k) - 2\nu\beta E(k)] \\ &= \frac{4\mu\beta\eta\sqrt{2\beta}}{(2\nu\beta - \eta^2)^{\frac{5}{2}} k^2 k'^2 (1 + k'^2)} [2k'^2 K(\sqrt{1 - k'^2}) - (1 + k'^2) E(\sqrt{1 - k'^2})]. \end{aligned} \tag{2.18}$$

Therefore, from (2.18), we have

$$\frac{\partial \Lambda}{\partial \eta} < 0 \Leftrightarrow f(k') \equiv (1 + k'^2) E(\sqrt{1 - k'^2}) - 2k'^2 K(\sqrt{1 - k'^2}) > 0. \tag{2.19}$$

Since the function $k(\eta, \nu)$ decreases strictly with respect to η and $k' = \sqrt{1 - k^2}$, we obtain k' is a increasing function of $\eta \in (0, \sqrt{\nu\beta})$ with $k' \in (0, 1)$. Differentiating $f(k')$ defined by (2.19) with respect to k' and using the relation $x\frac{dE(x)}{dx} = E(x) - K(x)$ and $E(x) < K(x)$, we have

$$\frac{\partial f(k')}{\partial k'} = \frac{3k'(E - K)(1 - k'^2)}{1 - k'^2} < 0.$$

Thus, $f(k')$ is a decreasing function. Since $f(1) = 0$, we have $f(k') > f(1) = 0$ for all $k' \in (0, 1)$, which show (2.19) and verify $\frac{\partial \Lambda}{\partial \eta} < 0$. Therefore, applying the implicit function theorem, there exist an interval $I(\nu_0)$ around ν_0, an interval $B(\eta_{2,0})$ around

$\eta_{2,0}$ and a unique smooth function $\Lambda : I(\nu_0) \to B(\eta_{2,0})$ such that $\Pi(\nu_0) = \eta_{2,0}$ and $\Lambda(\Pi(\nu), \nu) = 0$, $\forall \nu \in I(\nu_0)$. So, we can obtain (1) of Theorem 1.1.

Since ν_0 was chosen arbitrarily in the interval $I = (\frac{2\pi^2}{L^2}, +\infty)$, from the uniqueness of the function Λ, it follows that we can extend Λ to $(\frac{2\pi^2}{L^2}, +\infty)$. Using the smoothness of the function involved, we can immediately obtain part (2) of Theorem 1.1. □

Next, we give an important result which will be used in the spectral analysis.

Corollary 2.1. *The map $\Pi : I(\nu_0) \to B(\eta_{2,0})$ is a strictly decreasing function. Therefore, from (2.15), $\nu \mapsto k(\nu)$ strictly increases with respect to ν.*

Proof. According to the proof of Theorem 1.1, we know that the function Λ is a strictly decreasing function with respect to η. Note that $\Lambda(\Pi(\nu), \nu) = 0$ for all $\nu \in I(\nu_0)$ in Theorem 1.1, for $\frac{\partial \Pi}{\partial \nu} < 0$, we have to prove $\frac{\partial \Lambda}{\partial \nu} < 0$ in $I(\nu_0)$. Since $\eta^2 = (2\nu\beta - \eta^2)k'^2$, differentiating (2.14) with respect to ν and combining (2.15), (2.17), we have

$$\begin{aligned} \frac{\partial \Lambda}{\partial \nu} &= \frac{2\beta\sqrt{2\beta}}{(2\nu\beta - \eta^2)^{\frac{3}{2}}}[-K + \frac{\eta^2}{k(2\nu\beta - \eta^2)}\frac{dK}{dk}] \\ &= \frac{2\beta\sqrt{2\beta}}{k(2\nu\beta - \eta^2)^{\frac{3}{2}}}[-kK + \frac{E - k'^2K}{k}] \\ &= \frac{2\beta\sqrt{2\beta}}{k^2(2\nu\beta - \eta^2)^{\frac{3}{2}}}(E - K). \end{aligned} \tag{2.20}$$

Since $E(k) < K(k)$ for any $k \in (0, 1)$(see section 2 in [29]), we have $\frac{\partial \Lambda}{\partial \nu} < 0$ from (2.20). Hence, from the relation $\frac{\partial \Lambda}{\partial \eta}\frac{\partial \Pi}{\partial \nu} + \frac{\partial \Lambda}{\partial \nu} = 0$, we obtain $\frac{\partial \Pi}{\partial \nu} = -\frac{\partial \Lambda}{\partial \nu}/\frac{\partial \Lambda}{\partial \eta} < 0$, namely, Π is a strictly decreasing function with respect to ν. Next, differentiating k given by (2.15) with respect to ν, we have

$$\frac{\partial k}{\partial \nu} = \frac{\beta\eta(\eta - 2\eta'\nu)}{k(2\nu\beta - \eta^2)^2} > 0,$$

where $\eta' = \frac{\partial \Pi}{\partial \nu} < 0$. Then, we prove that the function $k(\nu)$ is strictly increasing with respect to ν, which show that Corollary 2.1 holds. □

Corollary 2.2. *Let $L > 0$, $c > 0$ and $1 - \alpha c > 0$. From the dnoidal waves $\nu \in (\frac{2\pi^2}{L^2}, +\infty) \mapsto (\phi(\cdot; \eta_1(\nu), \eta_2(\nu)), \psi(\cdot; \eta_1(\nu), \eta_2(\nu))) = (\phi, \psi)$ given by Theorem 1.1, we have*

$$\frac{d}{d\nu}\int_0^L \phi^2(\xi)d\xi > 0.$$

Proof. Combining $\frac{2\sqrt{2c}}{\sqrt{1-\alpha c}}K = \eta_1 L$ and $\int_0^K dn^2(x)dx = E(k)$, we have

$$\begin{aligned} \int_0^L \phi^2(\xi)d\xi &= \eta_1^2\int_0^L dn^2(\eta_1\sqrt{\frac{1-\alpha c}{2c}}\xi, k)d\xi = 2\eta_1\sqrt{\frac{2c}{1-\alpha c}}\int_0^K dn^2(x)dx \\ &= \frac{8c}{L(1-\alpha c)}KE. \end{aligned}$$

Since function $K(k)E(k)$ and $k(\nu)$ are strictly increasing with respect to k and ν respctively, we arrive at $\frac{d}{d\nu}\int_0^L \phi^2(\xi)d\xi > 0$, which prove that Corollary 2.2 holds. □

3. Spectral analysis

Before starting the spectral analysis of two linear operator, we firstly derive the operators L_1 and L_2. The operators L_1 and L_2 play key roles in proving the orbital stability of dnoidal wave solutions.

Firstly, from the equation (2.3), we have

$$(-\partial_x^2 - (\omega + \frac{c^2}{4}) + (1-\alpha c)\psi)\phi = 0.$$

Then, we define one linear operator $L_2 = -\partial_x^2 - (\omega + \frac{c^2}{4}) + (1-\alpha c)\psi$, that is $L_2\phi = 0$. Next, differentiating (2.3) with respect to x, we have

$$(-\partial_x^2 - (\omega + \frac{c^2}{4}) + 3(1-\alpha c)\psi)\phi' = 0.$$

Then, we get another linear operator $L_1 = \partial_x^2 - (\omega + \frac{c^2}{4}) + 3(1-\alpha c)\psi$, that is, $L_1\phi' = 0$.

Our main purpose in this section is devoted to consider the spectral properties related to the linear operators

$$\begin{aligned} L_1 &= -\frac{d^2}{dx^2} - (\omega + \frac{c^2}{4}) + 3(1-\alpha c)\psi, \\ L_2 &= -\frac{d^2}{dx^2} - (\omega + \frac{c^2}{4}) + (1-\alpha c)\psi, \end{aligned} \tag{3.1}$$

where ψ is the dnoidal wave solution (2.11) with the fundamental period L and $\nu = -(\omega + \frac{c^2}{4}) \in (\frac{2\pi^2}{L^2}, +\infty)$. We write $L_1 = L + M_1$ and $L_2 = L + M_2$, where $L = -\frac{d^2}{dx^2} - (\omega + \frac{c^2}{4})$. Since M_1 and M_2 are relatively compact with respect to L, we have $\sigma_{ess}(L_i) = \sigma_{ess}(L)$ with $i = 1, 2$, following from Wely's essential spectrum theorem. The spectrum properties for operators L_1 and L_2 in the following theorem are obtained with the assistance of the periodic eigenvalue problem considered on $[0, L]$

$$\begin{cases} L_i\chi = \lambda\chi, \\ \chi(0) = \chi(L), \quad \chi'(0) = \chi'(L), \end{cases} \tag{3.2}$$

and a semi-periodic eigenvalue problem considered on $[0, L]$

$$\begin{cases} L_i\chi = \mu\chi, \\ \chi(0) = -\chi(L), \quad \chi'(0) = -\chi'(L), \end{cases} \tag{3.3}$$

(see (4.2)-(4.8) in [29]).

Theorem 3.1. *Consider the dnoidal wave solutions $\phi = \phi(\cdot; \eta_1(\nu); \eta_2(\nu))$ and $\psi = \psi(\cdot; \eta_1(\nu); \eta_2(\nu))$ given by Theorem 1.1. Then, the operator $L_1 : H^2_{per}([0, L]) \to L^2_{per}([0, L])$ defined in (3.1) has first three simple eigenvalues λ_0, λ_1 and λ_2, where $\lambda_1 = 0$ is the second one with associated eigenfunction ϕ'. Moreover, the remainder of the spectrum for operator L_1 consists of a discrete set of eigenvalues which are double.*

Proof. From (4.7) in [29], we will firstly proof that the first two spectrum of L_1 satisfy $0 = \lambda_1 < \lambda_2$. Differentiating (2.3) and combining (2.2), we have $L_1\phi' = 0$, that is, zero is an eigenvalue of L_1 with associated eigenfunction ϕ'. From (2.10), we know that ϕ' has two zeros in $[0, L)$, then the eigenvalue zero is either λ_1 or λ_2 (see (4.8) of [29]). Next, we show that the eigenvalue zero is the second one. By using the transformation $\Psi(x) = \chi(\eta x)$ where $\eta^2 = \frac{2\beta}{\eta_1^2}$ and the relation $k^2 sn^2(x)+dn^2(x) = 1$, we get that problem (3.2) turn to the eigenvalue problem

$$\begin{cases} \Psi'' + (\varrho - 6k^2 sn^2(x; k))\Psi = 0, \\ \Psi(0) = \Psi(2K), \quad \Psi'(0) = \Psi'(2K), \end{cases} \tag{3.4}$$

where the relation between ϱ and λ is given by

$$\varrho = \frac{2\beta}{\eta_1^2}(\lambda + \frac{3\eta_1^2}{\beta} - \nu). \tag{3.5}$$

The second order differential equation in (3.4) is called the Jacobian form of Lam$\acute{e}'$s equation. From the Floquet theory [21, Theorem 7.8], we have that Eq.(3.4) has exactly 3 intervals of instability: $(-\infty, \varrho_0)$, $(\vartheta_0, \vartheta_1)$, (ϱ_1, ϱ_2) (where for $i \geq 0$, ϱ_i are the eigenvalues associated to the periodic problem and ϑ_i are the eigenvalues associated to the semi-periodic problem determined by the Lam$\acute{e}'$s equation). Hence, the first three eigenvalues ϱ_0, ϱ_1, ϱ_2 will be simple and the rest of the eigenvalues $\varrho_3 \leq \varrho_4 < \varrho_5 \leq \varrho_6 < \cdots$ satisfy that $\varrho_3 = \varrho_4, \varrho_5 = \varrho_6, \cdots$.

Now, we give explicit formulas for the eigenvalues ϱ_0, ϱ_1, ϱ_2 and its corresponding eigenfunctions Ψ_0, Ψ_1, Ψ_2. Firstly, we observe that $\varrho_1 = 4 + k^2$ satisfies $L_1\Psi_1 = \varrho_1\Psi_1$ with $\Psi_1(x) = sn(x)cn(x)$. Moreover, from Ince [18], we have that the functions

$$\Psi_0 = 1 - (1 + k^2 - \sqrt{1 - k^2 + k^4})sn^2, \quad \Psi_2 = 1 - (1 + k^2 + \sqrt{1 - k^2 + k^4})sn^2,$$

with period $2K$, satisfy that $L_1\Psi_0 = \varrho_0\Psi_0$ and $L_1\Psi_2 = \varrho_2\Psi_2$, with

$$\varrho_0 = 2(1 + k^2 - \sqrt{1 - k^2 + k^4}), \quad \varrho_2 = 2(1 + k^2 + \sqrt{1 - k^2 + k^4}).$$

Observe that Ψ_0 has no zeros in $[0, 2K]$, Ψ_2 has two zeros in $[0, 2K]$ and $\varrho_0 < \varrho_1 < \varrho_2$ for every $k \in (0, 1)$, then ϱ_0 is the first eigenvalue, ϱ_1 is the second eigenvalue and ϱ_2 is the third eigenvalue. Since the relation between ϱ and λ is given by

$$\lambda = \frac{\eta_1^2}{2\beta}(\varrho - 6) + \nu,$$

we obtain that λ is a increasing function with respect to ϱ, then $\lambda_0 < \lambda_1 < \lambda_2$. Since $k^2 = 2 - \frac{2\beta\nu}{\eta_1^2}$ from (2.15) and $\lambda_0 < \lambda_1 < \lambda_2$, we have that $\lambda(\varrho_1) = 0 = \lambda_1$ and $\lambda_0 < 0$.

To complete the proof of Theorem 3.1, we need to study the semi-periodic problem (3.3) and get the first two eigenvalues ϑ_0, ϑ_1 related to problem (3.3). Similar to Eq.(3.4), the semi-periodic problem (3.3) can also turn to the Lamé equation in (3.4) with the conditions $\Psi(0) = -\Psi(2K), \Psi'(0) = -\Psi'(2K)$, and the eigenvalues ϑ_i associated to semi-periodic problem (3.3) are related to the μ_i by the relation between ϑ_i and μ_i

$$\vartheta_i = \frac{2\beta}{\eta_1^2}(\mu_i + \frac{3\eta_1^2}{\beta} - \nu). \tag{3.6}$$

Firstly, note that $\vartheta_0 = 1 + k^2$ and $\vartheta_1 = 1 + 4k^2$ are the first two eigenvalues to the Lamé's equation in (3.4) in the semi-periodic case, where the eigenfunctions associated to ϑ_0 and ϑ_1 are $\Psi_{0,sm}(x) = cn(x)dn(x)$ and $\Psi_{1,sm}(x) = sn(x)dn(x)$ respectively. Therefore, via the relation $\mu_i = \frac{\eta_1^2}{2\beta}(\vartheta_i - 6) + \nu$ and $\Psi(x) = \chi(\eta x)$, we have the eigenvalues $\mu_0 = -\frac{3\nu}{2-k^2}$ and $\mu_1 = -\frac{3\nu(1-k^2)}{2-k^2}$. Hence, ϱ_0, ϱ_1 and ϱ_2 are simple and the rest of the eigenvalues are double, which finishes the proof of Theorem 3.1. □

Theorem 3.2. *Consider the dnoidal wave solutions $\phi = \phi(\cdot;\eta_1(\nu);\eta_2(\nu))$ and $\psi = \psi(\cdot;\eta_1(\nu);\eta_2(\nu))$ given by Theorem 1.1. Then, the operator $L_2 : H^2_{per}([0,L]) \to L^2_{per}([0,L])$ defined in (3.1) has simple zero eigenvalue, and zero is first one with associated eigenfunction ϕ. Moreover, the remained of the spectrum of operator L_2 consists of a discrete set of eigenvalues which are double.*

Proof. From (2.2) and (2.3), we have $L_2\phi = 0$, that is, zero is an eigenvalue of L_2 with associated eigenfunction ϕ. From (2.10), we know that ϕ has no zeros in $[0, L)$. Then the eigenvalue zero is its first eigenvalue and simple according to (4.8) in [29]. Therefore, the operator L_2 is nonnegative. Similar to the proof of Theorem 3.1, by applying Floquet theory, we get that there are exactly two intervals of instability associated to the periodic eigenvalue problem generated by L_2, and the rest of the eigenvalues are double. This completes the proof of Theorem 3.2. □

Next, we further study the properties of operators L_1 and L_2 which will paly a key role later in the proof of our stability theory.

Theorem 3.3. *Let $L > 0$, $c > 0$, $1 - \alpha c > 0$ and $\nu > \frac{2\pi^2}{L^2}$. Consider the smooth curve of dnoidal waves ϕ, ψ determined by Theorem 1.1. Then*

$$\begin{aligned} &\text{(i)} \inf\{(L_1\varphi,\varphi) : \varphi \in H^1_{per}([0,L]), \|\varphi\| = 1 \text{ and } (\varphi,\phi) = 0\} \equiv \gamma_0 = 0,\\ &\text{(ii)} \inf\{(L_1\varphi,\varphi) : \varphi \in H^1_{per}([0,L]), \|\varphi\| = 1 \ \ (\varphi,\phi) = 0 \text{ and } (\varphi,(\phi\psi)') = 0\} \equiv \gamma > 0. \end{aligned} \tag{3.7}$$

Proof. (1) Firstly, we show (3.7)(i) holds. From (2.10), we know that ϕ is a bounded dnoidal wave solution. Then, γ_0 is finite. Due to $(\phi', \phi) = 0$ and $L_1\phi' = 0$, we have that $\gamma_0 \le 0$. Next, we firstly show that the infimum in (3.7)(i) is attained. Let $\{\varphi_j\}$ be a sequence of $H^1_{per}([0,L])$-functions with $\|\varphi_j\| = 1$, $(\varphi_j, \phi) = 0$, and $(L_1\varphi_j, \varphi_j) \to \gamma_0$ as $j \to \infty$. Therefore, $\{\varphi_j\}$ is bounded in $H^1_{per}([0,L])$. Then, there exists a subsequence of $\{\varphi_j\}$, that we denote again $\{\varphi_j\}$, such that $\varphi_j \to \Phi$ weakly in $H^1_{per}([0,L])$. Since the embedding $H^1_{per}([0,L]) \hookrightarrow L^2_{per}([0,L])$ is compact, we have Φ satisfies $\|\Phi\| = 1$, $(\Phi, \phi_\nu) = 0$. Moreover, since weak convergence is lower semi-continuous, it follows that

$$\gamma_0 \le (L_1\Phi, \Phi) \le \lim_{j\to\infty} \inf(L_1\varphi_j, \varphi_j) = \gamma_0.$$

Therefore, the infimum γ_0 is attained at an admissible function $\Phi \ne 0$.

In the following, using Lemma E.1 in [28], we show that $\gamma_0 \ge 0$. From Theorem 3.1, we have that L_1 has the necessary spectral properties required by Lemma E.1. Therefore, we require to find τ such that $L_1\tau = \phi$ and $(\tau, \phi) \le 0$. From Theorem 1.1, we have that the mapping $\nu \in (\frac{2\pi^2}{L^2}, +\infty) \mapsto \phi \in H^1_{per}$ is of class C^1. Hence, differentiating (2.2) with respect to ν, we obtain that $\tau = -\frac{d\phi}{d\nu}$ satisfies $L_1\tau = \phi$, namely, $\tau = L_1^{-1}\phi$. Therefore, we get that

$$(\tau, \phi) = -\frac{1}{2}\frac{d}{d\nu}\int_0^L \phi^2(\xi)d\xi < 0,$$

using the Corollary 2.2, that is, $(\tau, \phi) < 0$. Then, according to Lemma E.1, we get that $\gamma_0 \geq 0$, which finishes the proof of (3.7)(i).

(2) Secondly, we show that (3.7)(ii) holds. From (3.7)(i), we have that $\gamma \geq 0$. Suppose that $\gamma = 0$. Similar to argument in part(i), we can find a function Φ such that $\|\Phi\| = 1$, $(\Phi, \phi) = 0$, $(\Phi, (\phi\psi)') = 0$ and $(L_1\Phi, \Phi) = 0$. Hence, according to the theory of Lagrange multipliers, there are α, λ, θ such that

$$L_1\Phi = \alpha\Phi + \lambda\phi + \theta(\phi\psi)'. \tag{3.8}$$

Since $(L_1\Phi, \Phi) = 0$, $(\Phi, \phi) = 0$ and $(\Phi, (\phi\psi)') = 0$, we have $\alpha = 0$. From the fact that $L_1\phi' = 0$, we get

$$\theta\int_0^L \phi'(\phi\psi)'d\xi = -\frac{3\theta}{c}\int_0^L (\phi')^2\phi^2 d\xi = 0. \tag{3.9}$$

The equality (3.9) implies $\theta = 0$. Then, according to the above analysis, we get $L_1\Phi = \lambda\phi$. Since $L_1\tau = \phi$ with $\tau = -\frac{d\phi}{d\nu}$, we have $L_1(\Phi - \lambda\tau) = 0$ and there is a $\varsigma \in R\backslash\{0\}$ such that $\Phi - \lambda\tau = \varsigma\phi'$. Since $(\tau, \phi) \neq 0$, $(\phi, \phi') = 0$ and $(\Phi, \phi) = 0$, we have $\lambda = 0$, which is absurd. Therefore, the minimum $\gamma > 0$, which completes the proof of Theorem 3.3. □

Theorem 3.4. *Let $L > 0$, $c > 0$, $1 - \alpha c > 0$ and $\nu > \frac{2\pi^2}{L^2}$. Consider the smooth curve of dnoidal waves ϕ, ψ given by Theorem 1.1. Then*

$$\inf\{(L_2\varphi, \varphi) : \varphi \in H^1_{per}([0, L]), \|\varphi\| = 1 \ \ (\varphi, \phi\psi) = 0\} \equiv \Gamma > 0. \tag{3.10}$$

Proof. From Theorem 3.2, we know that L_2 is a nonnegative operator. Then, we have $\Gamma \geq 0$. Similar to the proof of Theorem 3.3 (ii), we assume that $\Gamma = 0$, and can find a function Φ such that $\|\Phi\| = 1$, $(\Phi, \phi\psi) = 0$ and $(L_2\Phi, \Phi) = 0$. Moreover, from the theory of Lagrange multipliers, there exists α, $\lambda \in R$ such that

$$L_2\Phi = \alpha\Phi + \lambda\phi\psi. \tag{3.11}$$

Since $(L_2\Phi, \Phi) = 0$ and $(\Phi, \phi\psi) = 0$, we have $\alpha = 0$. Since $L_2\phi = 0$, we get

$$\lambda\int_0^L \phi^2\psi d\xi = (L_2\phi, \Phi) = 0. \tag{3.12}$$

The equality (3.12) implies $\lambda = 0$, then $L_2\Phi = 0$. Therefore, there is a $\varsigma \in R\backslash\{0\}$ such that $\Phi = \varsigma\phi$, which is absurd. Therefore, the minimum $\Gamma > 0$, which completes the proof of Theorem 3.4. □

4. Orbital stability of the dnoidal wave solutions for the generalized Long-Short wave equations

In this section, we will prove that Eqs.(1.3) is a Hamiltonian system, and has three conserved functions E, Q_1 and Q_2. Moreover, we shall apply the Lyapunov method to study the orbital stability properties of the periodic traveling wave solution $\Psi(\xi) = (\widetilde{\phi(\xi)}, \psi(\xi))$ with $\widetilde{\phi(\xi)} = e^{i\frac{c}{2}\xi}\phi(\xi)$ given by (2.10) and (2.11).

Let $U = (\varepsilon, n)^T$. The function space in which we shall work is defined by $X = H^1_{per}([0,L]) \times L^2_{per}([0,L])$, with inner product

$$(f,g) = \int_0^L \big(Re(f_1\overline{g_1}) + Re(f_{1x}\overline{g_{1x}}) + f_2 g_2\big)dx, \quad \text{for } f,g \in X. \tag{4.1}$$

The dual space of X is $X^* = H^{-1}_{per}([0,L]) \times L^2_{per}([0,L])$. Define the pairing $< \cdot, \cdot >$ between X and X^* by

$$< f, g >= \int_0^L \big(Re(f_1\overline{g_1}) + f_2 g_2\big)dx. \tag{4.2}$$

Then, there is a natural isomorphism $I : X \to X^*$ defined by $< If, g >= (f,g)$.

By (4.1) and (4.2), we get that

$$I = \begin{pmatrix} 1 - \frac{\partial^2}{\partial x^2} & 0 \\ 0 & 1 \end{pmatrix}.$$

Let T_1, T_2 be one-parameter groups of unitary operator on X defined by

$$T_1(s_1)U(\cdot) = U(\cdot + s_1), \quad \text{for } U(\cdot) \in X,\ s_1 \in R, \tag{4.3}$$

$$T_2(s_2)U(\cdot) = \begin{pmatrix} e^{is_2}\varepsilon(\cdot) \\ n(\cdot) \end{pmatrix}, \quad \text{for } U(\cdot) \in X,\ s_2 \in R. \tag{4.4}$$

Differentiating (4.3) and (4.4) with respect to s_1 and s_2 at $s_1 = 0$, $s_2 = 0$, respectively, we can obtain

$$T_1'(0) = \begin{pmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial x} \end{pmatrix}, \quad \text{and} \quad T_2'(0) = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}.$$

Define a functional on X

$$E(U) = \int_0^L \big(\frac{1}{2}|\varepsilon_x|^2 + \frac{1}{2}n|\varepsilon|^2 + \frac{\alpha}{4}|\varepsilon|^4\big)dx. \tag{4.5}$$

By (4.3), (4.4) and (4.5), we can verify that $E(U)$ is invariant under operators T_1 and T_2, namely,

$$E(T_1(s_1)T_2(s_2)U) = E(U), \quad \text{for any } s_1, s_2 \in R, \tag{4.6}$$

and $E(U)$ is conserved, namely, for any $t \in R$,

$$E(U(t)) = E(U(0)). \tag{4.7}$$

Note that the system (1.3) can be written as the following Hamiltonian system:

$$\frac{dU}{dt} = JE'(U), \tag{4.8}$$

where $U = (\varepsilon, n)^T$, J is a skew-symmetrically linear operator defined by

$$J = \begin{pmatrix} -i & 0 \\ 0 & 2\frac{\partial}{\partial x} \end{pmatrix},$$

and

$$E'(U) = \begin{pmatrix} -\varepsilon_{xx} + n\varepsilon + \alpha|\varepsilon|^2\varepsilon \\ \frac{1}{2}|\varepsilon|^2 \end{pmatrix} \tag{4.9}$$

is the Frechet derivative of $E(U)$.

Let

$$B_1 = \begin{pmatrix} i\frac{\partial}{\partial x} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad \text{and} \quad B_2 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

such that $T_1'(0) = JB_1$, $T_2'(0) = JB_2$. Then, we define the conserved functionals $Q_1(U)$ and $Q_2(U)$ as following

$$Q_1(U) = \frac{1}{2} < B_1U, U >= \frac{1}{4}\int_0^L n^2 dx - \frac{1}{2}\int_0^L Im(\varepsilon_x\overline{\varepsilon})dx, \tag{4.10}$$

and

$$Q_2(U) = \frac{1}{2} < B_2U, U >= -\frac{1}{2}\int_0^L |\varepsilon|^2 dx. \tag{4.11}$$

By (4.3), (4.4), (4.10) and (4.11), we can verify that $Q_1(U)$ and $Q_2(U)$ are invariants under operators T_1 and T_2, namely,

$$Q_1(T_1(s_1)T_2(s_2)U) = Q_1(U), \quad \text{for any } s_1, s_2 \in R, \tag{4.12}$$

$$Q_2(T_1(s_1)T_2(s_2)U) = Q_2(U), \quad \text{for any } s_1, s_2 \in R, \tag{4.13}$$

and for any $t \in R$,

$$Q_1(U(t)) = Q_1(U(0)), \quad Q_2(U(t)) = Q_2(U(0)).$$

Next, we shall consider the orbital stability of periodic waves $T_1(ct)T_2(\omega t)\Psi(x)$ of Eqs.(1.3). Note that the system (1.3) has phase and translations symmetries, that is, if $(\varepsilon(x,t), n(x,t))$ is solution of (1.3), then $(e^{i\theta}\varepsilon(x+y,t), n(x+y,t))$ solves (1.3) for any real constants $(y,\theta) \in R \times [0, 2\pi)$. Then, the stability that we will study in this paper will be modulo these symmetries, we define the orbital stability as follows:

Definition 4.1. We say that the orbit generated by $\Psi(\xi)$,

$$\Theta_\Psi := \{(e^{i\theta}\widetilde{\phi}(\cdot + y), \psi(\cdot + y)) : (y,\theta) \in R \times [0, 2\pi)\} \tag{4.14}$$

is stable in $X = H^1_{per}([0,L]) \times L^2_{per}([0,L])$ by the periodic flow generated by the generated Long-Short equations (1.3), namely, for every $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that for any $(\varepsilon_0(x,t), n_0(x,t)) \in X$ satisfying

$$\|\varepsilon_0 - \widetilde{\phi}\|_{H^1_{per}} < \delta \ \text{ and } \ \|n_0 - \psi\|_{L^2_{per}} < \delta,$$

we have that the solution $(\varepsilon(x,t), n(x,t))$ of the system (1.3) with initial data $(\varepsilon(0), n(0)) = (\varepsilon_0, n_0)$ satisfies

$$(\varepsilon(x,t), n(x,t)) \in C(R; H^1_{per}[0,L]) \times C(R; L^2_{per}[0,L]),$$
$$\inf_{y\in R, \theta\in[0,2\pi)} \|e^{i\theta}\varepsilon(\cdot+y,t) - \widetilde{\phi}\|_{H^1_{per}} < \epsilon \text{ and } \inf_{y\in R} \|n(\cdot+y,t) - \psi\|_{L^2_{per}} < \epsilon, \quad (4.15)$$

for any $t \in R$, $\theta = \theta(t)$ and $y = y(t)$. Otherwise, we say that Ψ is X-unstable.

Next, based on the classical method developed by Benjamin [6], Bona [7], an extension of the Lyapunov techniques in Weinstein [27], we show our result of orbital stability for periodic traveling dnoidal waves (Theorem 1.2).

Proof of Theorem 1.2. Consider the solutions $\widetilde{\phi}$, ψ of (1.3) given by Theorem 1.1. Firstly, for $(\varepsilon_0, n_0) \in H^1_{per}([0,L]) \times L^2_{per}([0,L])$ and the global solution (ε, n) of system (1.3) corresponding to these initial data, we define for $t \geq 0$, $y \in R$ and $\theta \in [0, 2\pi)$,

$$\Omega_t(y,\theta) = \|e^{i\theta}(T_c\varepsilon)'(\cdot+y,t) - \phi'\|^2_{L^2_{per}} + \nu\|e^{i\theta}(T_c\varepsilon)(\cdot+y,t) - \phi\|^2_{L^2_{per}},$$

where the bounded linear operator T_c is defined as

$$(T_c\varepsilon)(x,t) = e^{-\frac{ic}{2}(x-ct)}\varepsilon(x,t).$$

Then, the deviation of the solution $\varepsilon(t)$ from orbit Θ_Ψ is measure by

$$[\rho_\nu(\varepsilon(t), \Theta_\Psi)]^2 \equiv \inf_{(y,\theta)\in[0,L]\times[0,2\pi)} \Omega_t(y,\theta).$$

Therefore, by applying standard argument in [6,7] or Lemma 6.3 in [4], there exists an internal $I = [0,T]$ such that $\inf \Omega_t(y,\theta)$ is attained in $(y,\theta) \equiv (y(t), \theta(t))$ for every $t \in I$. Then, we arrive at

$$[\rho_\nu(\varepsilon(t), \Theta_\Psi)]^2 \equiv \Omega_t(y(t), \theta(t)). \quad (4.16)$$

Next, we consider the following perturbation of the periodic wave $(\widetilde{\phi}, \psi)$

$$\begin{cases} \xi = e^{i\theta}e^{-\frac{ic}{2}(x-ct)}\varepsilon(x+y,t) - \phi, \quad \xi = p + iq, \\ \gamma = n(x+y,t) - \psi, \end{cases} \quad (4.17)$$

for $t \in [0,T]$, $y = y(t)$ and $\theta = \theta(t)$.

By the property of minimum for $(y,\theta) = (y(t),\theta(t))$, we obtain that $\frac{\partial\Omega_t}{\partial y}|_{y=y(t)} = 0$ and $\frac{\partial\Omega_t}{\partial\theta}|_{\theta=\theta(t)} = 0$. Hence, using the fact that ϕ satisfies equation (2.3), we have that

$$\begin{aligned} \frac{\partial\Omega_t}{\partial\theta}|_{\theta=\theta(t)} &= 2i\int_0^L (-i\phi'q' - i\nu\phi q) = 2\int_0^L (-\phi'' + \nu\phi)q dx \\ &= -2(1-\alpha c)\int_0^L \phi\psi q dx = 0, \\ \frac{\partial\Omega_t}{\partial y}|_{y=y(t)} &= -2\int_0^L (\phi' p''' + \nu\phi p')dx = -2\int_0^L (\phi''' - \nu\phi')p dx \end{aligned}$$

$$= -2(1-\alpha c)\int_0^L (\phi\psi)' p dx = 0,$$

which provide us with compatibility relation on $p(x,t)$ and $q(x,t)$, namely,

$$\int_0^L (\phi(x)\psi(x))' p(x,t) dx = 0, \tag{4.18}$$

$$\int_0^L \phi(x)\psi(x) q(x,t) dx = 0. \tag{4.19}$$

The orthogonality conditions (4.18) and (4.19) appear as crucial ingredients in the proof of stability.

Then, applying that the functions $E(U)$, $Q_1(U)$ and $Q_2(U)$ defined in (4.5), (4.10) and (4.11) are invariant by translations and rotations, the representation (4.17), the classical embedding $H^1_{per}([0,L]) \hookrightarrow L^\rho_{per}([0,L])$ for every $\rho \geq 2$, and the fact that ϕ satisfies (2.3), we have the following variation for the conserved continuous functional $F(\varepsilon,n) \equiv E(\varepsilon,n) + cQ_1(\varepsilon,n) + \omega Q_2(\varepsilon,n)$

$$\begin{aligned}\Delta F(\varepsilon,n) \equiv & F(\varepsilon,n) - F(\widetilde{\phi},\psi)\\ =&\frac{1}{2} < L_1 p, p > + \frac{1}{2} < L_2 q, q > + \int_0^L \big(\frac{c}{4}\gamma^2 + \frac{1}{2}\gamma(p^2+q^2) + \gamma\phi p + \frac{\phi^2 p^2}{c}\big) dx\\ &+ \int_0^L \big(\frac{\alpha}{4}(p^2+q^2)^2 + \alpha p^3\phi + \alpha\phi p q^2\big) dx,\end{aligned} \tag{4.20}$$

where L_1 and L_2 are the self-adjoint operators defined in (3.1). After some algebra for (4.20), we get that

$$\begin{aligned}\Delta F(\varepsilon,n) =& \frac{1}{2} < L_1 p, p > + \frac{1}{2} < L_2 q, q > + \int_0^L \frac{1}{4}\big(c\gamma^2 + 2\gamma(p^2+q^2) + 4\gamma\phi p\\ &+ \frac{(p^2+q^2)^2}{c} + \frac{4\phi^2 p^2}{c} + \frac{4\phi p(p^2+q^2)}{c}\big) dx - \frac{1-\alpha c}{4c}\int_0^L (p^2+q^2)^2 dx\\ &+ \int_0^L \alpha p^3\phi dx + \int_0^L \alpha\phi p q^2 dx - \int_0^L \frac{\phi p(p^2+q^2)}{c} dx\\ =& \frac{1}{2} < L_1 p, p > + \frac{1}{2} < L_2 q, q > + \int_0^L \frac{1}{4}(\sqrt{c}\gamma + \frac{p^2+q^2}{\sqrt{c}} + \frac{2\phi p}{\sqrt{c}})^2 dx\\ &- \frac{1-\alpha c}{4c}\int_0^L (p^2+q^2)^2 dx + \int_0^L \alpha p^3\phi dx + \int_0^L \alpha\phi p q^2 dx - \int_0^L \frac{\phi p(p^2+q^2)}{c} dx\\ \geq& \frac{1}{2} < L_1 p, p > + \frac{1}{2} < L_2 q, q > + \int_0^L \frac{1}{4}(\sqrt{c}\gamma + \frac{p^2+q^2}{\sqrt{c}} + \frac{2\phi p}{\sqrt{c}})^2 dx\\ &- C_1\|\xi\|^3_{H^1_{per}} - C_2\|\xi\|^4_{H^1_{per}}\\ \geq& \frac{1}{2} < L_1 p, p > + \frac{1}{2} < L_2 q, q > - C_1\|\xi\|^3_{H^1_{per}} - C_2\|\xi\|^4_{H^1_{per}},\end{aligned} \tag{4.21}$$

where C_1 and C_2 are positive constants. For the proof of orbital stability, we have to estimate the term $(L_1 p, p)$ and $(L_2 q, q)$, where $p(x,t)$, $q(x,t)$ satisfy (4.18), (4.19), respectively. We first estimate $(L_1 p, p)$. Consider the normalization $\|\varepsilon_0\|_{L^2_{per}} = \|\phi\|_{L^2_{per}}$ for every $t \in [0,T]$. By (4.17), we have

$$\|\varepsilon(t)\|^2_{L^2_{per}} = \|e^{i\theta}e^{-\frac{ic}{2}(x-ct)}\varepsilon(t)\|^2_{L^2_{per}} = \|\xi(t) + \phi(t)\|^2_{L^2_{per}},$$

that is,

$$\int_0^L \big((p+\phi)^2+q^2\big)dx = \int_0^L \phi^2 dx \Rightarrow \int_0^L \big(p^2+q^2\big)dx = -2\int_0^L p\phi dx.$$

Then, we have $-2(p,\phi) = \|\xi\|^2_{L^2_{per}}$ for all $t \geq 0$. Without loss of generality, we assume that $\|\phi\|_{L^2_{per}} = 1$. Define $p_\parallel$ and $p_\perp$ to be $p_\parallel = (p,\phi)\phi = -\frac{1}{2}[\|p\|^2+\|q\|^2]\phi$ and $p_\perp = p - p_\parallel$. Thus, $p_\perp \perp \phi$ and $p_\perp \perp (\phi\psi)'$. Therefore, from Theorem 3.3(ii), we have that

$$(L_1 p_\perp, p_\perp) \geq C_0\|p_\perp\|^2_{H^1_{per}} \geq C_0\|p\|^2_{H^1_{per}} - C_3\|\xi\|^3_{H^1_{per}} - C_4\|\xi\|^4_{H^1_{per}}.$$

Since $(L_1\phi,\phi) < 0$, we obtain that $(L_1 p_\parallel, p_\parallel) \geq -\widetilde{C_4}\|\xi\|^4_{H^1_{per}}$. From the Cauchy-Schwarz inequality and the definition of L_1, we have

$$(L_1 p_\perp, p_\parallel) = (p_\perp, L_1 p_\parallel) = (p,\phi)(p_\perp, 2(1-\alpha c)\psi\phi) \geq -\widetilde{C_3}\|\xi\|^3_{H^1_{per}}.$$

Hence, we obtain that

$$(L_1 p, p) \geq D_1\|p\|^2_{H^1_{per}} - D_2\|\xi\|^3_{H^1_{per}} - D_3\|\xi\|^4_{H^1_{per}}, \tag{4.22}$$

where $D_j > 0$, for $j = 1,2,3$. Moreover, using Theorem 3.4 and the definition of L_2, we have that there exists $\widetilde{C_0}$ such that

$$(L_2 q, q) \geq \widetilde{C_0}\|q\|^2_{H^1_{per}}. \tag{4.23}$$

Therefore, combining (4.21), (4.22) and (4.23), we get that

$$\begin{aligned}\Delta\mathcal{F}(\varepsilon,n) &\geq \widetilde{D_1}\|\xi\|^2_{H^1_{per}} - \widetilde{D_2}\|\xi\|^3_{H^1_{per}} - \widetilde{D_3}\|\xi\|^4_{H^1_{per}} \\ &\geq d_1\|\xi\|^2_{1,\nu} - d_2\|\xi\|^3_{1,\nu} - d_3\|\xi\|^4_{1,\nu},\end{aligned} \tag{4.24}$$

where $d_j > 0$, for $j = 1,2,3$, and $\|f\|^2_{1,\nu} \equiv \|f'\|^2_{L^2_{per}} + \nu\|f\|^2_{L^2_{per}}$.

Hence, from (4.16) and (4.24), we have that for $t \in [0,T]$

$$\Delta\mathcal{F}(\varepsilon(t),n(t)) \geq d_1\|\xi\|^2_{1,\nu}(1 - \frac{d_2}{d_1}\|\xi\|_{1,\nu} - \frac{d_3}{d_1}\|\xi\|^2_{1,\nu}). \tag{4.25}$$

For $\|\xi\|_{1,\nu}$ small, we have $d_1\|\xi\|^2_{1,\nu}(1-\frac{d_2}{d_1}\|\xi\|_{1,\nu}-\frac{d_3}{d_1}\|\xi\|^2_{1,\nu}) > 0$. From (4.25), we can immediately obtain the stability result. For convenience, we make $g(x) = d_1x^2 - d_2x^3 - d_3x^4$. Let $\epsilon > 0$. Since $\mathcal{F}(\varepsilon,n)$ is continuous on $B = \{\varepsilon_0 \in H^1_{per}([0,L]), n_0 \in L^2_{per}([0,L]) : \|\varepsilon_0\|_{L^2_{per}} = \|\phi\|_{L^2_{per}}\}$, we obtain $\Delta\mathcal{F}(\varepsilon,n)$ is continuous in time and the function $t \to \rho_\nu(\varepsilon(t),\Theta_\Phi)$ is continuous. Then, we have that there exists $\delta(\epsilon)$ such that if

$$\|\varepsilon_0 - \widetilde{\phi}\|_{H^1_{per}} < \delta, \quad \text{and} \quad \|n_0 - \psi\|_{L^2_{per}} < \delta,$$

then for $t \in [0,T]$,

$$\begin{aligned} g(\rho_\nu(\varepsilon(t),\Theta_\Psi)) \leq \Delta\mathcal{F}(\varepsilon(t),n(t)) &= \Delta\mathcal{F}(\varepsilon(0),n(0)) \leq g(\epsilon) \\ &\Rightarrow \rho_\nu(\varepsilon(t),\Theta_\Psi) \leq \epsilon. \end{aligned} \tag{4.26}$$

Since the mapping $t \to \inf_{(y,\theta)\in[0,L]\times[0,2\pi)} \Omega_t(y,\theta)$ is continuous, it follows from (4.26) that we have

$$\|\xi\|_{1,\nu}^2 \leq \epsilon. \tag{4.27}$$

Similar to the proof of 6.1 [4], we can obtain that the inequality (4.27) is still true for all $t > 0$. Finally, from (4.21) and the analysis made above for ξ, we arrive at

$$\int_0^L \frac{1}{4}(\sqrt{c}\gamma + \frac{p^2+q^2}{\sqrt{c}} + \frac{2\phi p}{\sqrt{c}})^2 dx \leq \epsilon. \tag{4.28}$$

Then, using the Cauchy-Schwarz inequality and (4.27), we arrive at $\|\gamma\|_{L^2_{per}} \leq \epsilon$. Hence, combining the above result and (4.27), we obtain (4.15). Next, using the fact that the mapping $\nu \in (\frac{2\pi^2}{L^2}, +\infty) \to (\phi, \psi)$ is continuous and the orbital stability Definition 4.1, we show that $(\widetilde{\phi}, \psi)$ is orbitally stable in X relative to small perturbations that preserve the L^2_{per}-norm of $\widetilde{\phi}$. This completes the proof of Theorem 1.2. □

5. Conclusion

In this article, we are interested in studying the stability of periodic traveling wave solutions for the generalized Long-Short equations (1.3). By using the classical method proposed by Benjamin, Bona et.al, and an extension of the Lyapunov techniques in Weinstein, we prove that periodic traveling wave solutions for the system (1.3) are orbital stability. In the sense of limit, we obtain the orbital stability results of solitary wave solutions with zero asymptotic value for the generalized Long-Short equations [16, 17]. In particular, as $\alpha = 0$, we can also obtain the orbital stability results of periodic wave solution and solitary wave solution for the long-short wave resonance equations. Our orbital stability approach for the periodic orbit can be easily modified for giving other proof of the stability of solitary wave solutions for the long-short wave resonance equations. The results in the present paper improve and extend the previous stability results of long-shore wave equations and its extension equations.

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