Journal of Applied Analysis and Computation
Volume 10, Number 2, April 2020, 713–728
Website:http://jaac-online.com/
DOI:10.11948/20190115

# UNIQUENESS OF MEROMORPHIC FUNCTIONS CONCERNING SHARING TWO SMALL FUNCTIONS WITH THEIR DERIVATIVES*

Linke Ma[1], Dan Liu[1] and Mingliang Fang[1,†]

**Abstract** In this paper, we study the uniqueness of meromorphic functions that share two small functions with their derivatives. We prove the following result: Let $f$ be a nonconstant meromorphic function such that $\overline{\lim\limits_{r\to\infty}} \frac{\bar{N}(r,f)}{T(r,f)} < \frac{3}{128}$, and let $a$, $b$ be two distinct small functions of $f$ with $a \not\equiv \infty$ and $b \not\equiv \infty$. If $f$ and $f'$ share $a$ and $b$ IM, then $f \equiv f'$.

**Keywords** Meromorphic functions, shared small functions, derivatives.

**MSC(2010)** 30D35.

## 1. Introduction and Main Results

Throughout this paper, we assume that the reader is familiar with the basic notions of Nevanlinna value distribution theory, see [5–7, 9, 18, 19]. In the following, a meromorphic function always means meromorphic in the whole complex plane. By $S(r,f)$, we denote any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$ outside of an exceptional set of finite linear measure. Let $f(z)$ and $a$ be meromorphic functions and $a$ is said a small function of $f$ if and only if $T(r,a) = S(r,f)$.

Let $a$ be both a small function of $f$ and a small function of $g$. We say that two nonconstant meromorphic functions $f$ and $g$ share the small function $a$ IM(CM) if $f - a$ and $g - a$ have the same zeros ignoring multiplicities (counting multiplicities) [14, 16]. If $a$ is a constant, then we say that $f$ and $g$ share the value $a$ IM(CM) [15]. Moreover, we introduce the following denotations: $S_{(m,n)}(a) = \{z|z$ is a common zero of $f - a$ and $f' - a$ with multiplicities $m$ and $n$ respectively$\}$. $\bar{N}_{(m,n)}(r, \frac{1}{f-a})$ denotes the counting function of $f$ with respect to the set $S_{(m,n)}(a)$. $\bar{N}_{n)}(r, \frac{1}{f-a})$ denotes the counting function of all zeros of $f - a$ with multiplicities $n$ at most. $\bar{N}_{(n}(r, \frac{1}{f-a})$ denotes the counting function of all zeros of $f - a$ with multiplicities $n$ at least. $N_n(r, \frac{1}{f-a})$ denotes the counting function of all zeros of $f - a$ with multiplicities $n$.

In 1977, Rubel and Yang [13] proved

†the corresponding author. Email address:mlfang@scau.edu.cn(M. Fang)
[1]Institute of Applied Mathematics, South China Agricultural University, Guangzhou, 510642, China
*The authors were supported by National Natural Science Foundation of China (No. 11701188).

**Theorem 1.1.** *Let $f$ be a nonconstant entire function, and let $a$ and $b$ be two distinct finite values. If $f$ and $f'$ share $a$, $b$ CM, then $f \equiv f'$.*

Since then, there were a lot of researches on this topic, such as [1,2,4,10–12,20].

In 1979, Mues and Steinmetz [10] improved Theorem 1.1 and obtained

**Theorem 1.2.** *Let $f$ be a nonconstant entire function, and let $a$ and $b$ be two distinct finite values. If $f$ and $f'$ share $a$, $b$ IM, then $f \equiv f'$.*

In 1983, Gundersen [4] and Mues and Steinmetz [11] improved Theorem 1.1 and obtained

**Theorem 1.3.** *Let $f$ be a nonconstant meromorphic function, and let $a$ and $b$ be two distinct finite values. If $f$ and $f'$ share $a$, $b$ CM, then $f \equiv f'$.*

In 1992, Zheng and Wang [20] considered the case of sharing small functions and proved

**Theorem 1.4.** *Let $f$ be a nonconstant entire function, and let $a$ and $b$ be two distinct small functions of $f$ with $a \not\equiv \infty$ and $b \not\equiv \infty$. If $f$ and $f'$ share $a$ and $b$ CM, then $f \equiv f'$.*

In 2000, Qiu [12] improved Theorem 1.2, Theorem 1.4 and proved

**Theorem 1.5.** *Let $f$ be a nonconstant entire function, and let $a$ and $b$ be two distinct small functions of $f$ with $a \not\equiv \infty$ and $b \not\equiv \infty$. If $f$ and $f'$ share $a$ and $b$ IM, then $f \equiv f'$.*

**Example 1.1** ( [8]). Let $f = \beta + (\beta - \alpha)/(h-1)$, where

$$\alpha = -\frac{1}{3}e^{-2z} - \frac{1}{2}e^{-z}, \qquad \beta = -\frac{1}{3}e^{-2z} + \frac{1}{2}e^{-z}, \qquad h = e^{-e^{z}}.$$

Set $a = \beta', b = \alpha'$. Then $T(r,a) = S(r,f)$ and $T(r,b) = S(r,f)$. It is easy to verify that

$$f' - a = e^{2z}(f-a)(f-\beta), \quad f' - b = e^{2z}(f-b)(f-\alpha).$$

Thus $f$ and $f'$ share $a$ and $b$ IM, but $f \not\equiv f'$. This example shows that the conclusion in Theorem 1.5 is not valid for meromorphic functions.

**Example 1.2.** Let $f = \frac{2}{1-e^{-2z}}$. Then $f$ and $f'$ share 0 CM and share 1 IM, but $f \not\equiv f'$.
This example shows that for meromorphic functions, the conclusion doesn't hold any more when they share one constant CM and another constant IM.

In this paper, using the methods different from [12], we improve Theorem 1.5 and prove the following result.

**Theorem 1.6.** *Let $f$ be a nonconstant meromorphic function such that $\overline{\lim\limits_{r\to\infty}} \frac{\bar{N}(r,f)}{T(r,f)} < \frac{3}{128}$, and let $a$, $b$ be two distinct small functions of $f$ with $a \not\equiv \infty$ and $b \not\equiv \infty$. If $f$ and $f'$ share $a$ and $b$ IM, then $f \equiv f'$.*

## 2. Some Lemmas

**Lemma 2.1** ( [17]). *Let $f$ be a nonconstant meromorphic function, and let $P = a_0 f^n + \cdots + a_{n-1} f + a_n (a_0 \neq 0)$, where $a_n (n = 0, 1, 2, \cdots)$ are constants. Then*

$$T(r, P) = nT(r, f) + S(r, f).$$

**Lemma 2.2** ( [18]). *Let $f$ be a nonconstant meromorphic function, and let $\psi = a_0 f + a_1 f' + \cdots + a_k f^{(k)} (a_k \neq 0,\ k \geqslant 1)$, where $a_k (k = 0, 1, 2, \cdots)$ are constants. Then*

$$m(r, \frac{\psi}{f}) = S(r, f),$$

*and*

$$T(r, \psi) \leqslant T(r, f) + k\bar{N}(r, f) + S(r, f).$$

**Lemma 2.3** ( [3]). *Let $f$ be a nonconstant meromorphic function, and let $f^n P(f) \equiv Q(f)$, where $P(f)$ and $Q(f)$ are differential polynomials in $f$ and the degree of $Q(f)$ is at most $n$. Then*

$$m(r, P(f)) = S(r, f).$$

**Lemma 2.4** ( [12]). *Let $f$ be a nonconstant meromorphic function, and let $a$ and $b$ be two distinct small functions of $f$ with $a \not\equiv \infty$ and $b \not\equiv \infty$. Set*

$$\Delta(f) = \begin{vmatrix} f - a & a - b \\ f' - a' & a' - b' \end{vmatrix} = \begin{vmatrix} f - b & a - b \\ f' - b' & a' - b' \end{vmatrix}.$$

*Then*

$$\begin{aligned}
&(1) \quad \Delta(f) \not\equiv 0, \\
&(2) \quad m(r, \frac{\Delta(f)}{f - a}) = S(r, f),\ m(r, \frac{\Delta(f)}{f - b}) = S(r, f), \\
&(3) \quad m(r, \frac{\Delta(f)}{(f - a)(f - b)}) = S(r, f), \\
&(4) \quad m(r, \frac{\Delta(f)(f - \beta)}{(f - a)(f - b)}) = S(r, f),
\end{aligned}$$

*where $\beta$ is a small function of $f$.*

**Lemma 2.5.** *Let $f$ be a nonconstant meromorphic function, and let $a$ and $b$ be two distinct small functions of $f$ with $a \not\equiv \infty$ and $b \not\equiv \infty$. Assume that $c_k = a + k(a - b)$, $(k = 1, 2, \cdots, n)$. Then*

$$(n+1)T(r, f') \leqslant \bar{N}(r, \frac{1}{f' - a}) + \bar{N}(r, \frac{1}{f' - b}) + \sum_{k=1}^{n} \bar{N}(r, \frac{1}{f' - c_k}) + \bar{N}(r, f') + S(r, f').$$

**Proof.** It is easy to see from $c_k = a + k(a - b)$ that $c_k \not\equiv a$, $c_k \not\equiv b$ and $c_k$ $(k = 1, 2, \cdots, n)$ are distinct small functions of $f$. Let

$$F = \frac{f' - a}{b - a},$$

then

$$T(r,F)=T(r,f')+S(r,f').$$

By the Second Fundamental Theorem, we have

$$(n+1)T(r,F)\leqslant \bar{N}(r,F)+\bar{N}(r,\frac{1}{F})+\bar{N}(r,\frac{1}{F-1})+\sum_{k=1}^{n}\bar{N}(r,\frac{1}{F+k})+S(r,F)$$

$$\leqslant \bar{N}(r,\frac{1}{f'-a})+\bar{N}(r,\frac{1}{f'-b})+\sum_{k=1}^{n}\bar{N}(r,\frac{1}{f'-c_k})+\bar{N}(r,f')+S(r,f').$$

Thus

$$(n+1)T(r,f')\leqslant \bar{N}(r,\frac{1}{f'-a})+\bar{N}(r,\frac{1}{f'-b})+\sum_{k=1}^{n}\bar{N}(r,\frac{1}{f'-c_k})+\bar{N}(r,f')+S(r,f').$$

This completes the proof of Lemma 2.5.

## 3. The proof of Theorem 1.6

We prove Theorem 1.6 by contradiction.

Assume that $f\not\equiv f'$. Since $f$ and $f'$ share $a$ and $b$ IM, then by the Second Fundamental Theorem, we get

$$\begin{aligned}T(r,f)&\leqslant \bar{N}(r,\frac{1}{f-a})+\bar{N}(r,\frac{1}{f-b})+\bar{N}(r,f)+S(r,f)\\&=\bar{N}(r,\frac{1}{f'-a})+\bar{N}(r,\frac{1}{f'-b})+\bar{N}(r,f')+S(r,f)\\&\leqslant 3T(r,f')+S(r,f),\end{aligned}$$

and

$$\begin{aligned}T(r,f')&\leqslant \bar{N}(r,\frac{1}{f'-a})+\bar{N}(r,\frac{1}{f'-b})+\bar{N}(r,f')+S(r,f')\\&=\bar{N}(r,\frac{1}{f-a})+\bar{N}(r,\frac{1}{f-b})+\bar{N}(r,f)+S(r,f')\\&\leqslant 3T(r,f)+S(r,f').\end{aligned}$$

Therefore

$$S(r,f)=S(r,f'). \tag{3.1}$$

Set

$$\varphi=\frac{\Delta(f)(f-f')}{(f-a)(f-b)}, \tag{3.2}$$

where $\Delta(f)$ is defined by Lemma 2.4.

Obviously, $\varphi\not\equiv 0$. It follows from the fact that $f$ and $f'$ share $a$ and $b$ IM and (3.2) that $N(r,\varphi)=2\bar{N}(r,f)+S(r,f)$. By Lemma 2.2 and Lemma 2.4, we have

$$m(r,\varphi)\leqslant m(r,\frac{\Delta(f)f}{(f-a)(f-b)})+m(r,1-\frac{f'}{f})=S(r,f).$$

Thus

$$T(r,\varphi) \leqslant 2\bar{N}(r,f) + S(r,f).$$

Since $f$ and $f'$ share $a$ and $b$ IM, we get

$$\begin{aligned}
&\bar{N}(r,\frac{1}{f-a}) + \bar{N}(r,\frac{1}{f-b})\\
&\leqslant N(r,\frac{1}{f-f'}) + S(r,f)\\
&\leqslant T(r,f-f') + S(r,f)\\
&= N(r,f-f') + m(r,f-f') + S(r,f)\\
&= N(r,f') + m(r,f(1-\frac{f}{f'})) + S(r,f)\\
&\leqslant N(r,f) + \bar{N}(r,f) + m(r,f) + S(r,f)\\
&= T(r,f) + \bar{N}(r,f) + S(r,f). \qquad (3.3)
\end{aligned}$$

By (3.2), we have

$$\varphi(f-a)(f-b) = \Delta(f)(f-f').$$

From the definition of $\Delta(f)$, we get

$$[\varphi - (a'-b')]f^2 = b_1 f + b_2 f' + b_3 f f' + b_4 (f')^2 + b_5, \qquad (3.4)$$

where $b_1 = ab' - ba' + (a+b)\varphi,\ b_2 = ba' - ab',\ b_3 = b + b' - a - a',\ b_4 = a - b,\ b_5 = -ab\varphi$. Next, we discuss the following two cases.

**Case 1.** $\varphi - (a'-b') \not\equiv 0$. By (3.4), we have

$$\begin{aligned}
2m(r,f) &\leqslant m(r,\frac{1}{\varphi-a'+b'}) + m(r,f(b_1+b_2\frac{f'}{f}+b_3 f'+b_4 f'\cdot\frac{f'}{f})) + m(r,b_5) + S(r,f)\\
&\leqslant T(r,\varphi) + m(r,f) + m(r,f'(b_3+b_4\frac{f'}{f})) + S(r,f)\\
&\leqslant 2\bar{N}(r,f) + m(r,f) + m(r,f') + S(r,f).
\end{aligned}$$

It follows that

$$m(r,f) \leqslant 2\bar{N}(r,f) + m(r,f') + S(r,f).$$

Since $N(r,f') = N(r,f) + \bar{N}(r,f)$, then we get

$$T(r,f) \leqslant T(r,f') + \bar{N}(r,f) + S(r,f). \qquad (3.5)$$

Let $c = a + k(a-b)$ $(k \neq 0, -1)$. By the First Fundamental Theorem, Lemma 2.5, (3.3) and (3.5), we have

$$\begin{aligned}
2T(r,f') &\leqslant \bar{N}(r,\frac{1}{f'-a}) + \bar{N}(r,\frac{1}{f'-b}) + \bar{N}(r,\frac{1}{f'-c}) + \bar{N}(r,f') + S(r,f')\\
&\leqslant \bar{N}(r,\frac{1}{f-a}) + \bar{N}(r,\frac{1}{f-b}) + N(r,\frac{1}{f'-c}) + \bar{N}(r,f) + S(r,f)\\
&\leqslant T(r,f) + \bar{N}(r,f) + T(r,f') - m(r,\frac{1}{f'-c}) + \bar{N}(r,f) + S(r,f)\\
&\leqslant 2T(r,f') + 3\bar{N}(r,f) - m(r,\frac{1}{f'-c}) + S(r,f).
\end{aligned}$$

Thus

$$m(r,\frac{1}{f'-c})\leqslant 3\bar{N}(r,f)+S(r,f). \tag{3.6}$$

On the other hand, it follows from the First Fundamental Theorem, Lemma 2.2, (3.5) and (3.6) that we get

$$\begin{aligned}
m(r,\frac{f-c}{f'-c}) &= T(r,\frac{f'-c}{f-c})-N(r,\frac{f-c}{f'-c})+O(1)\\
&= m(r,\frac{(f'-c')+(c'-c)}{f-c})+N(r,\frac{f'-c}{f-c})-N(r,\frac{f-c}{f'-c})+O(1)\\
&\leqslant m(r,\frac{f'-c'}{f-c})+m(r,\frac{1}{f-c})+N(r,\frac{1}{f-c})+N(r,f'-c)\\
&- N(r,f-c)-N(r,\frac{1}{f'-c})+S(r,f)\\
&\leqslant T(r,\frac{1}{f-c})+\bar{N}(r,f)-N(r,\frac{1}{f'-c})+S(r,f)\\
&= T(r,\frac{1}{f-c})+\bar{N}(r,f)-T(r,\frac{1}{f'-c})+m(r,\frac{1}{f'-c})+S(r,f)\\
&\leqslant T(r,f)-T(r,f')+4\bar{N}(r,f)+S(r,f)\\
&\leqslant 5\bar{N}(r,f)+S(r,f).
\end{aligned} \tag{3.7}$$

Set

$$\chi=\frac{\Delta(f')(f-f')}{(f'-a)(f'-b)}, \tag{3.8}$$

where $\Delta(f')$ is defined by Lemma 2.4.

Obviously, $\chi\not\equiv 0$. It is easy to see from (3.8) that $N(r,\chi)=\bar{N}(r,f)+S(r,f)$. From (3.8), we have

$$\chi=(\frac{a-c}{a-b}\cdot\frac{\Delta(f')}{f'-a}-\frac{b-c}{a-b}\cdot\frac{\Delta(f')}{f'-b})(\frac{f-c}{f'-c}-1).$$

By Lemma 2.4 and (3.7), we get

$$m(r,\chi)\leqslant 5\bar{N}(r,f)+S(r,f).$$

Thus

$$T(r,\chi)\leqslant 6\bar{N}(r,f)+S(r,f).$$

Let

$$H_{n,m}=n\varphi-m\chi,$$

where $n$ and $m$ are positive integers. Next, we consider two cases.

**Case 1.1** $n\varphi-m\chi\equiv 0$. By (3.2) and (3.8), we have

$$n(\frac{f'-a'}{f-a}-\frac{f'-b'}{f-b})\equiv m(\frac{f''-a'}{f'-a}-\frac{f''-b'}{f'-b}).$$

Solving the above equation, which means

$$(\frac{f-a}{f-b})^n\equiv C_1(\frac{f'-a}{f'-b})^m, \tag{3.9}$$

where $C_1$ is a nonzero constant. In the following, we discuss three subcases.
**Case 1.1.1** $n = m$. By (3.9), we get

$$\frac{f-a}{f-b} \equiv C_2 \frac{f'-a}{f'-b}, \tag{3.10}$$

where $C_2$ is a nonzero constant. Since $f \not\equiv f'$, thus $C_2 \neq 1$. So by (3.10) we have

$$T(r,f) = T(r,f') + S(r,f), \tag{3.11}$$

and

$$f[(1-C_2)f' + C_2a - b] \equiv (a - C_2b)f' - (1-C_2)ab. \tag{3.12}$$

**Case 1.1.1.1** $(1-C_2)f' + C_2a - b \not\equiv 0$. It follows from (3.12) that

$$N(r,f) = S(r,f). \tag{3.13}$$

By Lemma 2.3 and (3.12), we get

$$m(r,(1-C_2)f' + C_2a - b) = S(r,f),$$

which means $m(r,f') = S(r,f)$. Combining this with (3.11) and (3.13), we have

$$\begin{aligned} T(r,f) &= T(r,f') + S(r,f) = N(r,f') + S(r,f) \\ &= N(r,f) + \bar{N}(r,f) + S(r,f) \\ &= S(r,f), \end{aligned}$$

a contradiction.
**Case 1.1.1.2** $(1-C_2)f' + C_2a - b \equiv 0$. Then $f' \equiv \frac{b-C_2a}{1-C_2}$. So by (3.1) and (3.11) we get

$$T(r,f) = T(r,f') + S(r,f) = S(r,f') + S(r,f) = S(r,f),$$

a contradiction.
**Case 1.1.2** $n > m$. Set $n = l_1m + l_2$, where $l_1 \geqslant 1$, $0 \leqslant l_2 < m$, and $l_1$, $l_2$ are integers.
(1) There exists a simple zero $z_0$ of $f' - a$, and $a(z_0) - b(z_0) \neq 0$. Then $z_0$ is a zero of both sides of (3.9), but the multiplicity of $z - z_0$ are different. This is a contradiction.
(2) There exists a zero $z_0$ of $f' - a$ with multiplicity two, and $a(z_0) - b(z_0) \neq 0$. Then from (3.9) we have $n = 2m$. By Lemma 2.2, we get

$$2mT(r,f) = mT(r,f') + S(r,f) \leqslant mT(r,f) + m\bar{N}(r,f) + S(r,f),$$

that is

$$T(r,f) \leqslant \bar{N}(r,f) + S(r,f).$$

Thus $1 = \lim\limits_{r\to\infty} \frac{T(r,f)}{T(r,f)} \leqslant \overline{\lim\limits_{r\to\infty}} \frac{\bar{N}(r,f)}{T(r,f)} + \lim\limits_{r\to\infty} \frac{S(r,f)}{T(r,f)} < \frac{3}{128}$, a contradiction.
(3) There exists a simple zero $z_0$ of $f' - b$, and $a(z_0) - b(z_0) \neq 0$. Then $z_0$ is a pole of both sides of (3.9), but the multiplicity of $z - z_0$ are different. This is a contradiction.
(4) There exists a zero $z_0$ of $f' - b$ with multiplicity two, and $a(z_0) - b(z_0) \neq 0$. Then from (3.9) we have $n = 2m$. As mentioned above, this is a contradiction.

(5) All simple zeros and zeros with multiplicities two of $f'-a$ and $f'-b$ are the zeros of $(a-b)$. Then

$$N_{1)}(r,\frac{1}{f'-a})\leqslant N(r,\frac{1}{a-b})=S(r,f),\ N_2(r,\frac{1}{f'-a})\leqslant 2N(r,\frac{1}{a-b})=S(r,f),$$

and

$$N_{1)}(r,\frac{1}{f'-b})\leqslant N(r,\frac{1}{a-b})=S(r,f),\ N_2(r,\frac{1}{f'-b})\leqslant 2N(r,\frac{1}{a-b})=S(r,f).$$

By the Second Fundamental Theorem, we get

$$\begin{aligned}T(r,f') &\leqslant \bar{N}(r,\frac{1}{f'-a})+\bar{N}(r,\frac{1}{f'-b})+\bar{N}(r,f')+S(r,f')\\ &\leqslant \frac{1}{3}N(r,\frac{1}{f'-a})+\frac{1}{3}N(r,\frac{1}{f'-b})+\bar{N}(r,f)+S(r,f')\\ &\leqslant \frac{1}{3}T(r,f')+\frac{1}{3}T(r,f')+\bar{N}(r,f)+S(r,f')\\ &\leqslant \frac{2}{3}T(r,f')+\bar{N}(r,f)+S(r,f').\end{aligned}$$

When $r$ is sufficiently large, we have $\frac{\overline{N}(r,f)}{T(r,f)}<\frac{4}{128}$ by $\overline{\lim\limits_{r\to\infty}}\frac{\bar{N}(r,f)}{T(r,f)}<\frac{3}{128}$. Then

$$\begin{aligned}T(r,f') &\leqslant \frac{2}{3}T(r,f')+\bar{N}(r,f)+S(r,f')\\ &< \frac{2}{3}T(r,f')+\frac{4}{128}T(r,f)+S(r,f')\\ &< \frac{2}{3}T(r,f')+\frac{4}{128}T(r,f')+S(r,f')\\ &= \frac{67}{96}T(r,f')+S(r,f').\end{aligned}$$

Thus $T(r,f')=S(r,f')$, a contradiction.

**Case 1.1.3** $n<m$. We use the argument similar to Case 1.1.2 and can also get a contradiction.

**Case 1.2** $n\varphi-m\chi\not\equiv 0$, for all $n$ and $m$.

Let $z_0\in S_{(m,n)}(a)\bigcup S_{(m,n)}(b)$, then $z_0$ is a common zero of $f-a$ (or $f-b$) and $f'-a$ (or $f'-b$) with multiplicities $m$ and $n$ respectively. Suppose $a(z_0)\neq\infty$, $b(z_0)\neq\infty$, and $a(z_0)-b(z_0)\neq 0$. It is easy to see from

$$n\varphi-m\chi=(f-f')[(n\frac{f'-b'}{f-b}-m\frac{f''-b'}{f'-b})-(n\frac{f'-a'}{f-a}-m\frac{f''-a'}{f'-a})],$$

that $n\varphi(z_0)-m\chi(z_0)=0$. Then

$$\begin{aligned}&\bar{N}_{(m,n)}(r,\frac{1}{f-a})+\bar{N}_{(m,n)}(r,\frac{1}{f-b})\\ &\leqslant N(r,\frac{1}{n\varphi-m\chi})+N(r,a)+N(r,b)+N(r,\frac{1}{a-b})\\ &\leqslant T(r,n\varphi-m\chi)+S(r,f)\\ &\leqslant T(r,\varphi)+T(r,\chi)+S(r,f)\\ &\leqslant 8\bar{N}(r,f)+S(r,f).\end{aligned}\tag{3.14}$$

**Case 1.2.1** $a \equiv a'$ and $b \equiv b'$. Let $z_0$ be a zero of $f-a$ (or $f-b$) with multiplicity $l$, then $z_0$ is a zero of $f'-a'$ (or $f'-b'$) with multiplicity $l-1$. Since $f$ and $f'$ share $a$ and $b$ IM, then $z_0$ is also a zero of $f'-a$ (or $f'-b$) with multiplicity $l-1$. Obviously, there exists no simple zero of $f-a$ (or $f-b$).
By the Second Fundamental Theorem, Lemma 2.2 and (3.14), we have

$$\begin{aligned}
&T(r,f)\\
\leqslant\ &\bar{N}(r,\frac{1}{f-a})+\bar{N}(r,\frac{1}{f-b})+\bar{N}(r,f)+S(r,f)\\
=\ &\bar{N}_{1)}(r,\frac{1}{f-a})+\sum_{m=2}^{3}\bar{N}_m(r,\frac{1}{f-a})+\bar{N}_{(4}(r,\frac{1}{f-a})\\
&+\bar{N}_{1)}(r,\frac{1}{f-b})+\sum_{m=2}^{3}\bar{N}_m(r,\frac{1}{f-b})+\bar{N}_{(4}(r,\frac{1}{f-b})+\bar{N}(r,f)+S(r,f)\\
\leqslant\ &\sum_{m=2}^{3}\bar{N}_m(r,\frac{1}{f-a})+\frac{1}{4}[N(r,\frac{1}{f-a})-\sum_{m=2}^{3}m\bar{N}_m(r,\frac{1}{f-a})]\\
&+\sum_{m=2}^{3}\bar{N}_m(r,\frac{1}{f-b})+\frac{1}{4}[N(r,\frac{1}{f-b})-\sum_{m=2}^{3}m\bar{N}_m(r,\frac{1}{f-b})]+\bar{N}(r,f)+S(r,f)\\
\leqslant\ &\sum_{m=2}^{3}\sum_{n=1}^{3}(1-\frac{m}{4})\bar{N}_{(m,n)}(r,\frac{1}{f-a})+\sum_{m=2}^{3}(1-\frac{m}{4})\bar{N}_{(m,n\geqslant4)}(r,\frac{1}{f-a})\\
&+\sum_{m=2}^{3}\sum_{n=1}^{3}(1-\frac{m}{4})\bar{N}_{(m,n)}(r,\frac{1}{f-b})+\sum_{m=2}^{3}(1-\frac{m}{4})\bar{N}_{(m,n\geqslant4)}(r,\frac{1}{f-b})\\
&+\frac{1}{4}N(r,\frac{1}{f-a})+\frac{1}{4}N(r,\frac{1}{f-b})+\bar{N}(r,f)+S(r,f).
\end{aligned}\tag{3.15}$$

Noticing that

$$\begin{aligned}
&\sum_{m=2}^{3}(1-\frac{m}{4})\bar{N}_{(m,n\geqslant4)}(r,\frac{1}{f-a})\\
=\ &\frac{2}{4}\bar{N}_{(2,n\geqslant4)}(r,\frac{1}{f-a})+\frac{1}{4}\bar{N}_{(3,n\geqslant4)}(r,\frac{1}{f-a})\\
=\ &\frac{1}{4}[\bar{N}_{(2,n\geqslant4)}(r,\frac{1}{f-a})+\bar{N}_{(3,n\geqslant4)}(r,\frac{1}{f-a})]+\frac{1}{4}\bar{N}_{(2,n\geqslant4)}(r,\frac{1}{f-a})\\
\leqslant\ &\frac{1}{4}\times\frac{1}{4}\{N(r,\frac{1}{f'-a})-[\bar{N}_{(2,1)}(r,\frac{1}{f-a})+\bar{N}_{(3,1)}(r,\frac{1}{f-a})]-2[\bar{N}_{(2,2)}(r,\frac{1}{f-a})\\
&+\bar{N}_{(3,2)}(r,\frac{1}{f-a})]-3[\bar{N}_{(2,3)}(r,\frac{1}{f-a})+\bar{N}_{(3,3)}(r,\frac{1}{f-a})]\}\\
&+\frac{1}{4}\times\frac{1}{4}\{N(r,\frac{1}{f'-a})-[\bar{N}_{(2,1)}(r,\frac{1}{f-a})+\bar{N}_{(3,1)}(r,\frac{1}{f-a})]-2[\bar{N}_{(2,2)}(r,\frac{1}{f-a})\\
&+\bar{N}_{(3,2)}(r,\frac{1}{f-a})]-3[\bar{N}_{(2,3)}(r,\frac{1}{f-a})+\bar{N}_{(3,3)}(r,\frac{1}{f-a})]\}\\
=\ &\frac{1}{4}\times\frac{2}{4}[N(r,\frac{1}{f'-a})-\sum_{m=2}^{3}\sum_{n=1}^{3}n\bar{N}_{(m,n)}(r,\frac{1}{f-a})].
\end{aligned}\tag{3.16}$$

The same inequality can also be obtained for $b$.
Thus by (3.15) and (3.16), we get

$$\begin{aligned}
T(r,f) \leqslant & \sum_{m=2}^{3}\sum_{n=1}^{3}(1-\frac{m}{4})\bar{N}_{(m,n)}(r,\frac{1}{f-a}) + \frac{1}{4}\times\frac{2}{4}[N(r,\frac{1}{f'-a}) \\
& - \sum_{m=2}^{3}\sum_{n=1}^{3} n\bar{N}_{(m,n)}(r,\frac{1}{f-a})] + \sum_{m=2}^{3}\sum_{n=1}^{3}(1-\frac{m}{4})\bar{N}_{(m,n)}(r,\frac{1}{f-b}) \\
& + \frac{1}{4}\times\frac{2}{4}[N(r,\frac{1}{f'-b}) - \sum_{m=2}^{3}\sum_{n=1}^{3} n\bar{N}_{(m,n)}(r,\frac{1}{f-b})] \\
& + \frac{1}{4}N(r,\frac{1}{f-a}) + \frac{1}{4}N(r,\frac{1}{f-b}) + \bar{N}(r,f) + S(r,f) \\
\leqslant & \frac{6}{16}\bar{N}_{(2,1)}(r,\frac{1}{f-a}) + \frac{6}{16}\bar{N}_{(2,1)}(r,\frac{1}{f-b}) + \frac{2}{16}N(r,\frac{1}{f'-a}) + \frac{2}{16}N(r,\frac{1}{f'-b}) \\
& + \frac{1}{4}N(r,\frac{1}{f-a}) + \frac{1}{4}N(r,\frac{1}{f-b}) + \bar{N}(r,f) + S(r,f) \\
\leqslant & 4\bar{N}(r,f) + \frac{1}{4}T(r,f') + \frac{1}{2}T(r,f) + S(r,f). \qquad (3.17)
\end{aligned}$$

Since $f$ and $f'$ share $a$ and $b$ IM, then by the Second Fundamental Theorem, (3.1) and (3.17), we have

$$\begin{aligned}
T(r,f') &\leqslant \bar{N}(r,\frac{1}{f'-a}) + \bar{N}(r,\frac{1}{f'-b}) + \bar{N}(r,f') + S(r,f') \\
&= \bar{N}(r,\frac{1}{f-a}) + \bar{N}(r,\frac{1}{f-b}) + \bar{N}(r,f) + S(r,f) \\
&\leqslant 4\bar{N}(r,f) + \frac{1}{4}T(r,f') + \frac{1}{2}T(r,f) + S(r,f).
\end{aligned}$$

That is

$$T(r,f') \leqslant \frac{16}{3}\bar{N}(r,f) + \frac{2}{3}T(r,f) + S(r,f). \qquad (3.18)$$

By (3.17) and (3.18), we get

$$\begin{aligned}
T(r,f) &\leqslant 4\bar{N}(r,f) + \frac{1}{4}T(r,f') + \frac{1}{2}T(r,f) + S(r,f) \\
&\leqslant 4\bar{N}(r,f) + \frac{4}{3}\bar{N}(r,f) + \frac{1}{6}T(r,f) + \frac{1}{2}T(r,f) + S(r,f) \\
&= \frac{16}{3}\bar{N}(r,f) + \frac{2}{3}T(r,f) + S(r,f),
\end{aligned}$$

which means

$$T(r,f) \leqslant 16\overline{N}(r,f) + S(r,f).$$

Thus $1 = \lim_{r\to\infty}\frac{T(r,f)}{T(r,f)} \leqslant 16\,\overline{\lim_{r\to\infty}}\frac{\bar{N}(r,f)}{T(r,f)} + \lim_{r\to\infty}\frac{S(r,f)}{T(r,f)} < \frac{3}{8}$, a contradiction.

**Case 1.2.2** $a \equiv a'$, $b \not\equiv b'$ or $a \not\equiv a'$, $b \equiv b'$. Without loss of generality, we can assume that $a \equiv a'$ and $b \not\equiv b'$. Let $z_0$ be a zero of $f-a$ with multiplicity $l$, then $z_0$ is a zero of $f'-a'$ with multiplicity $l-1$. Since $f$ and $f'$ share $a$ IM, then $z_0$ is also a zero of $f'-a$ with multiplicity $l-1$. Obviously, there exists no simple zero of $f-a$.

Let $z_1$ be a zero of $f-b$ with multiplicity $l$ ($l \geqslant 2$). Then by $f$ and $f'$ share $b$ IM, we obtain

$$f(z_1)-b(z_1)=0,\ f'(z_1)-b(z_1)=0,\ f'(z_1)-b'(z_1)=0.$$

From this, we have $b(z_1)-b'(z_1)=0$. Thus

$$\bar{N}_{(2}(r,\frac{1}{f-b}) \leqslant N(r,\frac{1}{b-b'})+S(r,f)=S(r,f).$$

By the Second Fundamental Theorem, Lemma 2.2, (3.14) and (3.16), we get

$$\begin{aligned}
T(r,f) &\leqslant \bar{N}(r,\frac{1}{f-a})+\bar{N}(r,\frac{1}{f-b})+\bar{N}(r,f)+S(r,f)\\
&= \bar{N}_{1)}(r,\frac{1}{f-a})+\sum_{m=2}^{3}\bar{N}_m(r,\frac{1}{f-a})+\bar{N}_{(4}(r,\frac{1}{f-a})\\
&\quad+\bar{N}_{1)}(r,\frac{1}{f-b})+\bar{N}_{(2}(r,\frac{1}{f-b})+\bar{N}(r,f)+S(r,f)\\
&\leqslant \sum_{m=2}^{3}\bar{N}_m(r,\frac{1}{f-a})+\frac{1}{4}[N(r,\frac{1}{f-a})-\sum_{m=2}^{3}m\bar{N}_m(r,\frac{1}{f-a})]\\
&\quad+\bar{N}_{1)}(r,\frac{1}{f-b})+\bar{N}(r,f)+S(r,f)\\
&\leqslant \sum_{m=2}^{3}\sum_{n=1}^{3}(1-\frac{m}{4})\bar{N}_{(m,n)}(r,\frac{1}{f-a})+\sum_{m=2}^{3}(1-\frac{m}{4})\bar{N}_{(m,n\geqslant 4)}(r,\frac{1}{f-a})\\
&\quad+\sum_{n=1}^{3}\bar{N}_{(1,n)}(r,\frac{1}{f-b})+\bar{N}_{(1,n\geqslant 4)}(r,\frac{1}{f-b})+\frac{1}{4}N(r,\frac{1}{f-a})+\bar{N}(r,f)+S(r,f)\\
&\leqslant \sum_{m=2}^{3}\sum_{n=1}^{3}(1-\frac{m}{4})\bar{N}_{(m,n)}(r,\frac{1}{f-a})+\frac{1}{4}\times\frac{2}{4}[N(r,\frac{1}{f'-a})-\sum_{m=2}^{3}\sum_{n=1}^{3}n\bar{N}_{(m,n)}(r,\frac{1}{f-a})]\\
&\quad+\sum_{n=1}^{3}\bar{N}_{(1,n)}(r,\frac{1}{f-b})+\frac{1}{4}[N(r,\frac{1}{f'-b})-\sum_{n=1}^{3}n\bar{N}_{(1,n)}(r,\frac{1}{f-b})]\\
&\quad+\frac{1}{4}N(r,\frac{1}{f-a})+\bar{N}(r,f)+S(r,f)\\
&\leqslant \frac{6}{16}\bar{N}_{(2,1)}(r,\frac{1}{f-a})+\frac{3}{4}\bar{N}_{(1,1)}(r,\frac{1}{f-b})+\frac{2}{4}\bar{N}_{(1,2)}(r,\frac{1}{f-b})+\frac{1}{4}\bar{N}_{(1,3)}(r,\frac{1}{f-b})\\
&\quad+\frac{2}{16}N(r,\frac{1}{f'-a})+\frac{1}{4}N(r,\frac{1}{f'-b})+\frac{1}{4}N(r,\frac{1}{f-a})+\bar{N}(r,f)+S(r,f)\\
&\leqslant 16\bar{N}(r,f)+\frac{3}{8}T(r,f')+\frac{1}{4}T(r,f)+S(r,f).
\end{aligned} \tag{3.19}$$

Since $f$ and $f'$ share $a$ and $b$ IM, then by the Second Fundamental Theorem, (3.1) and (3.19), we have

$$\begin{aligned}T(r,f') &\leqslant \bar{N}(r,\frac{1}{f'-a}) + \bar{N}(r,\frac{1}{f'-b}) + \bar{N}(r,f') + S(r,f')\\ &= \bar{N}(r,\frac{1}{f-a}) + \bar{N}(r,\frac{1}{f-b}) + \bar{N}(r,f) + S(r,f)\\ &\leqslant 16\bar{N}(r,f) + \frac{3}{8}T(r,f') + \frac{1}{4}T(r,f) + S(r,f).\end{aligned}$$

That is

$$T(r,f') \leqslant \frac{128}{5}\bar{N}(r,f) + \frac{2}{5}T(r,f) + S(r,f). \tag{3.20}$$

By (3.19) and (3.20), we get

$$\begin{aligned}T(r,f) &\leqslant 16\bar{N}(r,f) + \frac{3}{8}T(r,f') + \frac{1}{4}T(r,f) + S(r,f)\\ &\leqslant 16\bar{N}(r,f) + \frac{48}{5}\bar{N}(r,f) + \frac{3}{20}T(r,f) + \frac{1}{4}T(r,f) + S(r,f)\\ &= \frac{128}{5}\bar{N}(r,f) + \frac{2}{5}T(r,f) + S(r,f),\end{aligned}$$

which means

$$T(r,f) \leqslant \frac{128}{3}\overline{N}(r,f) + S(r,f).$$

Thus $1 = \varliminf_{r\to\infty} \frac{T(r,f)}{T(r,f)} \leqslant \frac{128}{3} \varlimsup_{r\to\infty} \frac{\bar{N}(r,f)}{T(r,f)} + \varliminf_{r\to\infty} \frac{S(r,f)}{T(r,f)} < 1$, a contradiction.

**Case 1.2.3** $a \not\equiv a'$ and $b \not\equiv b'$. Let $z_0$ be a zero of $f - a$ with multiplicity $l_1$ ($l_1 \geqslant 2$) and $z_1$ be a zero of $f - b$ with multiplicity $l_2$ ($l_2 \geqslant 2$). Then by $f$ and $f'$ share $a$ and $b$ IM, we obtain

$$f(z_0) - a(z_0) = 0,\ f'(z_0) - a(z_0) = 0,\ f'(z_0) - a'(z_0) = 0,$$

and

$$f(z_1) - b(z_1) = 0,\ f'(z_1) - b(z_1) = 0,\ f'(z_1) - b'(z_1) = 0.$$

From this, we have $a(z_0) - a'(z_0) = 0,\ b(z_1) - b'(z_1) = 0$. Thus

$$\bar{N}_{(2}(r,\frac{1}{f-a}) \leqslant N(r,\frac{1}{a-a'}) + S(r,f) = S(r,f),$$

and

$$\bar{N}_{(2}(r,\frac{1}{f-b}) \leqslant N(r,\frac{1}{b-b'}) + S(r,f) = S(r,f).$$

By the Second Fundamental Theorem, Lemma 2.2 and (3.14), we get

$$
\begin{aligned}
T(r,f) &\leqslant \bar{N}(r,\frac{1}{f-a}) + \bar{N}(r,\frac{1}{f-b}) + \bar{N}(r,f) + S(r,f) \\
&= \bar{N}_{1)}(r,\frac{1}{f-a}) + \bar{N}_{(2}(r,\frac{1}{f-a}) + \bar{N}_{1)}(r,\frac{1}{f-b}) + \bar{N}_{(2}(r,\frac{1}{f-b}) + \bar{N}(r,f) + S(r,f) \\
&\leqslant \bar{N}_{1)}(r,\frac{1}{f-a}) + \bar{N}_{1)}(r,\frac{1}{f-b}) + \bar{N}(r,f) + S(r,f) \\
&= \sum_{n=1}^{3} \bar{N}_{(1,n)}(r,\frac{1}{f-a}) + \bar{N}_{(1,n\geqslant 4)}(r,\frac{1}{f-a}) + \sum_{n=1}^{3} \bar{N}_{(1,n)}(r,\frac{1}{f-b}) \\
&\quad + \bar{N}_{(1,n\geqslant 4)}(r,\frac{1}{f-b}) + \bar{N}(r,f) + S(r,f) \\
&\leqslant \sum_{n=1}^{3} \bar{N}_{(1,n)}(r,\frac{1}{f-a}) + \frac{1}{4}[N(r,\frac{1}{f'-a}) - \sum_{n=1}^{3} n\bar{N}_{(1,n)}(r,\frac{1}{f-a})] \\
&\quad + \sum_{n=1}^{3} \bar{N}_{(1,n)}(r,\frac{1}{f-b}) + \frac{1}{4}[N(r,\frac{1}{f'-b}) - \sum_{n=1}^{3} n\bar{N}_{(1,n)}(r,\frac{1}{f-b})] \\
&\quad + \bar{N}(r,f) + S(r,f) \\
&\leqslant \frac{3}{4}\bar{N}_{(1,1)}(r,\frac{1}{f-a}) + \frac{2}{4}\bar{N}_{(1,2)}(r,\frac{1}{f-a}) + \frac{1}{4}\bar{N}_{(1,3)}(r,\frac{1}{f-a}) \\
&\quad + \frac{3}{4}\bar{N}_{(1,1)}(r,\frac{1}{f-b}) + \frac{2}{4}\bar{N}_{(1,2)}(r,\frac{1}{f-b}) + \frac{1}{4}\bar{N}_{(1,3)}(r,\frac{1}{f-b}) \\
&\quad + \frac{1}{4}N(r,\frac{1}{f'-a}) + \frac{1}{4}N(r,\frac{1}{f'-b}) + \bar{N}(r,f) + S(r,f) \\
&\leqslant 13\bar{N}(r,f) + \frac{1}{2}T(r,f') + S(r,f). \qquad (3.21)
\end{aligned}
$$

Since $f$ and $f'$ share $a$ and $b$ IM, then by the Second Fundamental Theorem, (3.1) and (3.21), we have

$$
\begin{aligned}
T(r,f') &\leqslant \bar{N}(r,\frac{1}{f'-a}) + \bar{N}(r,\frac{1}{f'-b}) + \bar{N}(r,f') + S(r,f') \\
&= \bar{N}(r,\frac{1}{f-a}) + \bar{N}(r,\frac{1}{f-b}) + \bar{N}(r,f) + S(r,f) \\
&\leqslant 13\bar{N}(r,f) + \frac{1}{2}T(r,f') + S(r,f).
\end{aligned}
$$

That is

$$T(r,f') \leqslant 26\bar{N}(r,f) + S(r,f). \qquad (3.22)$$

By (3.21) and (3.22), we get

$$
\begin{aligned}
T(r,f) &\leqslant 13\bar{N}(r,f) + \frac{1}{2}T(r,f') + S(r,f) \\
&\leqslant 13\bar{N}(r,f) + 13\bar{N}(r,f) + S(r,f) \\
&= 26\bar{N}(r,f) + S(r,f).
\end{aligned}
$$

Thus $1 = \underline{\lim}_{r\to\infty} \frac{T(r,f)}{T(r,f)} \leqslant 26\, \overline{\lim}_{r\to\infty} \frac{\bar{N}(r,f)}{T(r,f)} + \underline{\lim}_{r\to\infty} \frac{S(r,f)}{T(r,f)} < \frac{39}{64}$, a contradiction.

**Case 2.** $\varphi - (a' - b') \equiv 0$. That is

$$\varphi \equiv a' - b'. \tag{3.23}$$

Next, we discuss three subcases.
**Case 2.1** $a \equiv a'$ and $b \equiv b'$. Then

$$a - b \equiv a' - b' \equiv A_1 e^z, \tag{3.24}$$

where $A_1$ is a nonzero constant. Taking (3.2) and (3.24) into (3.23), we get

$$(f' - a)^2 \equiv [2(f' - a) + A_1 e^z](f - a). \tag{3.25}$$

It follows from (3.25) that $2(f'(z) - a) + A_1 e^z \not\equiv 0$ and $f$ is an entire function. By Theorem 1.5, we have $f \equiv f'$, a contradiction.
**Case 2.2** $a \equiv a'$ and $b \not\equiv b'$ or $a \not\equiv a'$ and $b \equiv b'$. Without loss of generality, we can assume that $a \equiv a'$ and $b \not\equiv b'$. Then by (3.2) and (3.23), we get

$$(a - b)(f' - b)(f' - b') \equiv [(a - b + a' - b')(f' - b') - (a' - b')^2](f - b), \tag{3.26}$$

Obviously, $a' \not\equiv b'$. From (3.26), we have

$$\begin{aligned}\bar{N}(r, \frac{1}{f' - b'}) &\leqslant \bar{N}_{(2}(r, \frac{1}{f - b}) + \bar{N}(r, \frac{1}{a' - b'}) + O[\bar{N}(r, a) + \bar{N}(r, b)] \\ &= \bar{N}_{(2}(r, \frac{1}{f - b}) + S(r, f).\end{aligned} \tag{3.27}$$

Let $z_0$ be a zero of $f - b$ with multiplicity $l$ ($l \geqslant 2$). Then by $f$ and $f'$ share $b$ IM, we obtain

$$f(z_0) - b(z_0) = 0,\ f'(z_0) - b(z_0) = 0,\ f'(z_0) - b'(z_0) = 0.$$

From this, we have $b(z_0) - b'(z_0) = 0$. Thus

$$\bar{N}_{(2}(r, \frac{1}{f - b}) \leqslant N(r, \frac{1}{b - b'}) + S(r, f) = S(r, f). \tag{3.28}$$

By (3.27) and (3.28), we get

$$\bar{N}(r, \frac{1}{f' - b'}) = S(r, f). \tag{3.29}$$

In addition, from (3.26), we have

$$(f' - a')^2 - [(1 + \frac{a' - b'}{a - b})(f' - a') + (a' - b')](f - a) \equiv 0. \tag{3.30}$$

If $1 + \frac{a' - b'}{a - b} \equiv 0$, then $a' - b' \equiv -(a - b) \equiv A_2 e^{-z}$, where $A_2$ is a nonzero constant.
From (3.30), we get

$$f - a \equiv \frac{(f' - a')^2}{-(a - b)}. \tag{3.31}$$

Differentiating both sides of (3.31), we have

$$2(f'' - a'') + (f' - a') \equiv A_2 e^{-z}. \tag{3.32}$$

By solving this linear differential equation, we get

$$f - a \equiv C_1 + C_2 e^{-\frac{z}{2}} + A_2 e^{-z}, \tag{3.33}$$

where $C_1$ and $C_2$ are constants.

From (3.33), we have $T(r,f) = S(r,f)$, which is a contradiction.

Hence $1 + \frac{a'-b'}{a-b} \not\equiv 0$, it is easy to see from (3.30) that

$$\bar{N}(r, \frac{1}{f' - (a' - \frac{a'-b'}{1+\frac{a'-b'}{a-b}})}) \leqslant \bar{N}(r, \frac{1}{a'-b'}) + \bar{N}(r, \frac{1}{a-b}) + \bar{N}(r,b) = S(r,f). \tag{3.34}$$

Since $a' - b' \not\equiv 0$, then $a' - \frac{a'-b'}{1+\frac{a'-b'}{a-b}} \not\equiv b'$. By the Second Fundamental Theorem, (3.1), (3.29) and (3.34), we get

$$\begin{aligned} T(r,f') &\leqslant \bar{N}(r, \frac{1}{f' - (a' - \frac{a'-b'}{1+\frac{a'-b'}{a-b}})}) + \bar{N}(r, \frac{1}{f'-b'}) + \bar{N}(r,f') + S(r,f') \\ &\leqslant \bar{N}(r,f) + S(r,f') \leqslant \frac{1}{2} N(r,f') + S(r,f') \\ &\leqslant \frac{1}{2} T(r,f') + S(r,f'). \end{aligned}$$

Thus $T(r,f') = S(r,f')$, a contradiction.

**Case 2.3** $a \not\equiv a'$ and $b \not\equiv b'$. Then (3.26) still holds. As same as the discussion of Case 2.2, (3.29) still holds. Similarly, we have

$$\bar{N}(r, \frac{1}{f'-a'}) = S(r,f). \tag{3.35}$$

Since $a' \not\equiv b'$, by the Second Fundamental Theorem, (3.1), (3.29) and (3.35), we get

$$\begin{aligned} T(r,f') &\leqslant \bar{N}(r, \frac{1}{f'-a'}) + \bar{N}(r, \frac{1}{f'-b'}) + \bar{N}(r,f') + S(r,f') \\ &\leqslant \bar{N}(r,f) + S(r,f') \leqslant \frac{1}{2} N(r,f') + S(r,f') \\ &\leqslant \frac{1}{2} T(r,f') + S(r,f'). \end{aligned}$$

Thus $T(r,f') = S(r,f')$, a contradiction.

According to all the above discussions, we obtain $f \equiv f'$.

This completes the proof of Theorem 1.6. □

# References


[1] A. Banerjee and B. Chakraborty, *A note on uniqueness of meromorphic functions and their derivatives sharing two sets*, Appl. Math. E-Notes, 2017, 17, 164–176.

[2] J. Chen, *A note on entire functions that share two small functions*, Abstr. Appl. Anal., 2014, Art. ID 601507, 9.

[3] J. Clunie, *On integral and meromorphic functions*, J. London Math. Soc., 1962, s1-37, 17–27.

[4] G. G. Gundersen, *Meromorphic functions that share two finite values with their derivative*, Pacific J. Math., 1983, 105(2), 299–309.

[5] W. K. Hayman, *Meromorphic functions*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1964.

[6] Z. Huang and R. Zhang, *Uniqueness of the differences of meromorphic functions*, Anal. Math., 2018, 44(4), 461–473.

[7] I. Laine, *Nevanlinna theory and complex differential equations*, Walter de Gruyter & Co., Berlin, 1993.

[8] P. Li, *Unicity of meromorphic functions and their derivatives*, J. Math. Anal. Appl., 2003, 285(2), 651–665.

[9] S. Majumder, *Uniqueness and value distribution of differences of meromorphic functions*, Appl. Math. E-Notes, 2017, 17, 114–123.

[10] E. Mues and N. Steinmetz, *Meromorphe Funktionen, die mit ihrer Ableitung Werte teilen*, Manuscripta Math., 1979, 29(2-4), 195–206.

[11] E. Mues and N. Steinmetz, *Meromorphe Funktionen, die mit ihrer Ableitung zwei Werte teilen*, Results Math., 1983, 6(1), 48–55.

[12] G. Qiu, *Uniqueness of entire functions that share some small functions*, Kodai Math. J., 2000, 23(1), 1–11.

[13] L. A. Rubel and C. Yang, *Values shared by an entire function and its derivative*, Complex analysis, 1977, 599, 101–103.

[14] P. Sahoo and G. Biswas, *Meromorphic function that share a set of small functions with its derivative*, Mat. Stud., 2016, 46(2), 130–136.

[15] H. P. Waghamore and S. Anand, *Uniqueness of meromorphic functions with their derivatives sharing a small function*, Tbilisi Math. J., 2017, 10(2), 145–155.

[16] P. Wang and L. Liao, *On entire and meromorphic functions that share small functions with their derivatives*, Int. J. Appl. Math. Anal. Appl., 2016, 11(1), 97–106.

[17] C. Yang, *On deficiencies of differential polynomials. II*, Math. Z., 1972, 125, 107–112.

[18] C. Yang and H. Yi, *Uniqueness theory of meromorphic functions*, Kluwer Academic Publishers Group, Dordrecht, 2003.

[19] L. Yang, *Value distribution theory*, Springer-Verlag, Berlin, 1983.

[20] J. Zheng and S. Wang, *Unicity of meromorphic functions and their derivatives*, Adv. in Math. (China), 1992, 21(3), 334–341.