# BIFURCATIONS AND EXACT TRAVELLING WAVE SOLUTIONS OF M-N-WANG EQUATION 

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#### Abstract

By using the method of dynamical systems to Mikhailov-NovikovWang Equation, through qualitative analysis, we obtain bifurcations of phase portraits of the traveling system of the derivative $\phi(\xi)$ of the wave function $\psi(\xi)$. Under different parameter conditions, for $\phi(\xi)$, exact explicit solitary wave solutions, periodic peakon and anti-peakon solutions are obtained. By integrating known $\phi(\xi)$, nine exact explicit traveling wave solutions of $\psi(\xi)$ are given.


Keywords Solitary wave solution, periodic peakon, anti-peakon, Mikhailov-Novikov-Wang integrable equation.
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## 1. Introduction

Mikhailov, et al. [12, p11] considered the classification problem for integrable ( $1+$ 1 )-dimensional scalar partial differential equations (PDEs) that are second order in time. In the course of performing the classification, the equation

$$
\begin{equation*}
w_{t t}=w_{x x x t}+8 w_{x} w_{x t}+4 w_{x x} w_{t}-2 w_{x} w_{x x x x}-4 w_{x x} w_{x x x}-24 w_{x}^{2} w_{x x} \tag{1.1}
\end{equation*}
$$

was found. This equation was obtained by means of the perturbative symmetry approach to the classification of integrable PDEs.

By rewriting equation (1.1) as the following system

$$
\begin{equation*}
u_{t}=u_{x x x}+6 u u_{x}+v_{x}, \quad v_{t}=4 u_{x} v+2 u v_{x} \tag{1.2}
\end{equation*}
$$

and applying the Wahlquist-Estabrook prolongation algebra method, Hone, et al. [4, p11] obtained the zero curvature representation of the equation, which leads to a Lax representation in terms of an energy-dependent Schrödinger spectral problem of the type studied by [1,2]. The solutions of system (1.2) and of its associated hierarchy of commuting flows display weak Painlevé behavior, i.e. they have algebraic branching. By considering the travelling wave solutions of the next flow in the hierarchy, they found an integrable perturbation of the case (ii) of Hénon-Heiles system which has the weak Painlevé property. They performed separation of variables for this generalized Hénon-Heiles system and described the corresponding solutions of the PDE (1.2). By taking the traveling wave reduction of system (1.2), Hone, et al. [5,

[^0]p11] showed that the integrable case (ii) of Hénon-Heiles system can be extended by adding an arbitrary number of non-polynomial (rational) terms to the potential.

We notice that to the best of our knowledge, the dynamical behavior and exact traveling wave solutions of equation (1.1) has not be studied before. It is different from [4, p11]. In this paper, we investigate straightly the exact traveling wave solutions of equation (1.1) by using the dynamical system approach.

Let $w(x, t)=\psi(x-c t)=\psi(\xi)$, where $c$ is the wave speed. Then, equation(1.1) becomes

$$
\begin{equation*}
c^{2} \psi^{\prime \prime}=-c \psi^{\prime \prime \prime \prime}-6 c\left(\left(\psi^{\prime}\right)^{2}\right)^{\prime}-2\left(\psi^{\prime} \psi^{\prime \prime \prime}\right)^{\prime}-\left(\left(\psi^{\prime \prime}\right)^{2}\right)^{\prime}-8\left(\left(\psi^{\prime}\right)^{3}\right)^{\prime} \tag{1.3}
\end{equation*}
$$

where "" stands for the derivative with respect to $\xi$. Integrating this equation once and setting the integration constant as $g$, it follows that

$$
\begin{equation*}
c^{2} \psi^{\prime}=-c \psi^{\prime \prime \prime}-6 c\left(\psi^{\prime}\right)^{2}-2 \psi^{\prime} \psi^{\prime \prime \prime}-\left(\psi^{\prime \prime}\right)^{2}-8\left(\psi^{\prime}\right)^{3}-g \tag{1.4}
\end{equation*}
$$

Letting $\phi=\psi^{\prime}$, equation (1.4) yields

$$
\begin{equation*}
(2 \phi+c) \phi^{\prime \prime}=-\left(\phi^{\prime}\right)^{2}-8 \phi^{3}-6 c \phi^{2}-c^{2} \phi-g . \tag{1.5}
\end{equation*}
$$

Equation (1.5) is equivalent to the following two-dimensional system:

$$
\begin{equation*}
\frac{d \phi}{d \xi}=y, \quad \frac{d y}{d \xi}=-\frac{y^{2}+8 \phi^{3}+6 c \phi^{2}+c^{2} \phi+g}{2 \phi+c} \tag{1.6}
\end{equation*}
$$

which has the following first integral:

$$
\begin{equation*}
H(\phi, y)=y^{2}(2 \phi+c)+4 \phi^{4}+4 c \phi^{3}+c^{2} \phi^{2}+2 g \phi=h \tag{1.7}
\end{equation*}
$$

When we solve $\phi=\phi(\xi)$ from system (1.6), we obtain

$$
\begin{equation*}
w(x, t)=\psi(\xi)=\int \phi(\xi) d \xi \tag{1.8}
\end{equation*}
$$

Without loss of generality, we assume that the wave speed $c$ is fixed. Obviously, system (1.6) is a planar dynamical system with one-parameter $g$. We shall investigate all possible phase portraits of system (1.6) in the $(\phi, y)$-phase plane as the parameter $g$ is varied.

We notice that the right hand of the second equation in system (1.6) is not continuous when $\phi=\phi_{s}=-\frac{c}{2}$. In other words, on this straight lines in the phase plane $(\phi, y), \phi_{\xi}^{\prime \prime}$ is not well-defined. This implies that the differential system (1.6) could have non-smooth traveling wave solutions. Such phenomenon has been studied by several authors $[7-9,13,14]$. The existence of singular lines for a traveling wave equation is the reason why there exist peakons, periodic peakons and compactons. More references can see $[6,10,11]$.

The main result of this paper is the following conclusion.
Theorem 1.1. (1) For a given parameter $c \neq 0$, when $g$ is varied, system (1.6) has different bifurcations of phase portraits given by Figure 1 and Figure 2.
(2) When $g \in\left(-g_{0}, g_{0}\right)$, where $g_{0}=\frac{\sqrt{3}}{36}|c|^{3}$, system (1.6) has three different equilibrium points. Corresponding to the homoclinic orbit, system (1.6) has solitary wave solutions given by (3.3), (3.5), (3.6), (4.2), (4.3), (4.4), (4.5) and (4.7).
(3) When $c>0$ and $g=-\frac{1}{27} c^{3}$, corresponding to the heteroclinic triangle of system (2.1), system (1.6) has an exact anti-peakon solution given by (4.6).
(4) Equation (1.1) has 9 exact traveling wave solutions given by (5.1)-(5.9).

The proof of this theorem can be seen in next sections. This paper is organized as follows. In section 2, we discuss the bifurcations of phase portraits of system (1.6) depending on the change of parameter $g$. In section 3 and 4 , we calculate the explicit parametric representations for the homoclinic orbits of system (1.6). In sections 5 , we compute the exact solutions of equation (1.1).

## 2. Bifurcations of phase portraits of system (1.6)

Imposing the transformation $d \xi=(2 \phi+c) d \zeta$ for $\phi \neq-\frac{c}{2}$ on system (1.6) leads to the following associated regular system:

$$
\begin{equation*}
\frac{d \phi}{d \zeta}=y(2 \phi+c), \quad \frac{d y}{d \zeta}=-\left(y^{2}+8 \phi^{3}+6 c \phi^{2}+c^{2} \phi+g\right) \tag{2.1}
\end{equation*}
$$

This system has the same first integral as (1.7) and the same phase orbits as system (1.6) except for the straight line $\phi=-\frac{c}{2}$. Apparently, the singular line $\phi=-\frac{c}{2}$ is an invariant straight line solution of system (2.1) but not a orbit of system (1.6). Near this straight line, the variable " $\zeta$ " is a fast variable while the variable " $\xi$ " is a slow variable in the sense of the geometric singular perturbation theory.

To see the equilibrium points of (2.1), we write that

$$
\begin{equation*}
f(\phi)=\phi^{3}+\frac{3}{4} c \phi^{2}+\frac{1}{8} c^{2} \phi+\frac{1}{8} g, f^{\prime}(\phi)=3 \phi^{2}+\frac{3}{2} c \phi+\frac{1}{8} c^{2}=3\left(\phi-\tilde{\phi}_{a}\right)\left(\phi-\tilde{\phi}_{b}\right), \tag{2.2}
\end{equation*}
$$

where $\tilde{\phi}_{b}=\frac{1}{12} c(-3+\sqrt{3}), \tilde{\phi}_{a}=-\frac{1}{12} c(3+\sqrt{3})$.
Let $q=-\frac{c^{2}}{48}, r=-\frac{g}{16}$. Then, the discriminant $S=q^{3}+r^{2}$ of the cubic polynomial $f(\phi)=0$ just is that $S=\frac{1}{256}\left(g^{2}-\frac{1}{432} c^{6}\right)$. Write that $g_{0}=\frac{\sqrt{3}}{36}|c|^{3}$, It is easy to see that for given $c$, when $g \in\left(-g_{0}, g_{0}\right)$, we have $S<0$. It follows that there exist three simple real roots $\phi_{j}(j=1,2,3)$ of $f(\phi)$. When $g= \pm g_{0}$ there exist a simple real root and a double real root of $f(\phi)$.

Obviously, system (2.1) has at most 3 equilibrium points at $E_{j}\left(\phi_{j}, 0\right), j=1,2,3$. in the $\phi$-axis. On the straight line $\phi=-\frac{c}{2}$, there exist two equilibrium points $S_{\mp}\left(-\frac{c}{2}, \mp \sqrt{-g}\right)$ of system (2.1) if $g \leq 0$.

Let $M\left(\phi_{j}, y_{j}\right)$ be the coefficient matrix of the linearized system of (2.1) at an equilibrium point $E_{j}\left(\phi_{j}, y_{j}\right)$. We have

$$
\begin{align*}
J\left(\phi_{j}, 0\right) & =\operatorname{det} M\left(\phi_{j}, 0\right)=8\left(2 \phi_{j}+c\right) f^{\prime}\left(\phi_{j}\right)  \tag{2.3}\\
J\left(-\frac{c}{2}, \mp \sqrt{-g}\right) & =\operatorname{det} M\left(-\frac{c}{2}, \mp \sqrt{-g}\right)=4 g<0, \text { for } g<0 \tag{2.4}
\end{align*}
$$

By the theory of planar dynamical systems, for an equilibrium point of a planar integrable system, if $J<0$, then the equilibrium point is a saddle point; If $J>0$ and $(\operatorname{trace} M)^{2}-4 J<0(>0)$, then it is a center point (a node point); if $J=0$ and the Poincare index of the equilibrium point is 0 , then this equilibrium point is cusped [6].

We see from (2.3) that the sign of $f^{\prime}\left(\phi_{j}\right)$ and the relative positions of the equilibrium points $E_{j}\left(\phi_{j}, 0\right)$ of (2.1) with respect to the singular line $\phi=-\frac{c}{2}$ can determine the types (saddle points or centers) of the equilibrium points $E_{j}\left(\phi_{j}, 0\right)$. And two equilibrium points $S_{\mp}\left(\frac{c}{2}, \mp \sqrt{-g}\right)$ are saddle points for $g<0$.

Let $h_{j}=H\left(\phi_{j}, 0\right)$ and $h_{s}=H\left(-\frac{c}{2}, \mp \sqrt{-g}\right)=-c g$, where $H$ is given by (1.7). Notice that for $c>0$, when $g=-\frac{c^{3}}{27}$, we have $h_{2}=H\left(-\frac{1}{3} c, 0\right)=h_{s}$, where $\phi_{2}=-\frac{1}{3} c$ is a solution of the algebraic equations $f(\phi)=0$.

For a fixed parameter $c<0$ and $c>0$ respectively, we take $g$ as a bifurcation parameter. Then, as $g$ increasing from $-\infty$ to $\infty$, we obtain different topological phase portraits of equation (2.1) shown in Figure 1 and Figure 2. The corresponding parameter conditions are also given.

(a) $-\infty<g<-g_{0}$

(b) $g=-g_{0}$

(c) $-g_{0}<g<0$

(d) $g=0$


Figure 1. The bifurcations of phase portraits of system (2.1) for $c<0$
We see from Figure 1 and Figure 2 that as $g$ is varied, the cases $c<0$ and $c>0$ have different bifurcation behaviors.

In next sections, we investigate all possible exact explicit parametric representations of the orbits of system (1.6).

## 3. Exact explicit parametric representations of the orbits of system (1.6) for $c<0$

From (1.7), we have $y^{2}=\frac{h-4 \phi^{4}-4 c \phi^{3}-c^{2} \phi^{2}-2 g \phi}{2 \phi+c}$, for $2 \phi+c \neq 0$. By using the first equation of system (1.6), we have

$$
\begin{equation*}
\xi=\int_{\phi_{0}}^{\phi} \frac{1}{y} d \phi \tag{3.1}
\end{equation*}
$$

From Figure 1(a)-(c), we know that for every $h \in\left(h_{3}, h_{s}\right)$, a branch of the level curves defined by $H(\phi, y)=h$ is a periodic orbit enclosing the equilibrium point $\left(\phi_{3}, 0\right)$ of system (1.6). Now, $H(\phi, y)=h$ can be written as $y^{2}=\frac{4\left(r_{1}-\phi\right)\left(\phi-r_{2}\right)\left[\left(\phi-b_{1}\right)^{2}+a_{1}^{2}\right]}{2 \phi+c}$, where $r_{1}, r_{2}$ are the roots of the equation $H(\phi, 0)=h$. From (3.1), we have

$$
\begin{equation*}
\xi=\int_{r_{2}}^{\phi} \frac{\sqrt{\phi+\frac{c}{2}} d \phi}{\sqrt{2\left(r_{1}-\phi\right)\left(\phi-r_{2}\right)\left[\left(\phi-b_{1}\right)^{2}+a_{1}^{2}\right]}} \tag{3.2}
\end{equation*}
$$



Figure 2. The bifurcations of phase portraits of system (2.1) for $c>0$

We can not obtain the exact explicit parametric representation of the periodic wave solution of system (1.6). By numerical method, considering the above periodic orbits closing to the singular line $\phi=-\frac{c}{2}$, we obtain the profile of periodic peakon solutions shown in Figure $4(\mathrm{a})$. As the limit curve of the periodic orbits, the level curve defined by $H(\phi, y)=h_{s}$ is a heteroclinic loop connecting two saddle points $S_{ \pm}$ of system (2.1) (see Figure 3 (c)). For the singular system (1.6), this loop also gives rise to a periodic peakon solution [6]. We have the similar parametric representation as (4.1).
3.1. When $-g_{0}<g<0$, we see from Figure 3(e) that corresponding to the level curves defined by $H(\phi, y)=h_{1}$, there exist a homoclinic orbit enclosing the center $E_{2}\left(\phi_{2}, 0\right)$ and an open orbit passing through the point $\left(\phi_{L}, 0\right)$ and tending to the straight line $\phi=\frac{|c|}{2}$ when $|y| \rightarrow \infty$ for which we have $y^{2}=$ $\frac{h_{1}-4 \phi^{4}-4 c \phi^{3}-c^{2} \phi^{2}-2 g \phi}{2 \phi+c}=\frac{2\left(\phi_{L}-\phi\right)\left(\phi_{M}-\phi\right)\left(\phi-\phi_{1}\right)^{2}}{\left(\frac{1}{2}|c|-\phi\right)}$ and $y^{2}=\frac{2\left(\phi_{L}-\phi\right)\left(\phi-\phi_{M}\right)\left(\phi-\phi_{1}\right)^{2}}{\left(\phi-\frac{1}{2}|c|\right)}$ respectively, where $\phi_{M}, \phi_{L}$ are the roots of the equation $H(\phi, 0)=h_{1}$ and satisfy
$0<\phi_{1}<\phi_{M}<\phi_{L}$. From (3.1), we have $\sqrt{2} \xi=\int_{\phi}^{\phi_{M}} \frac{\sqrt{\frac{|c|}{2}-\phi} d \phi}{\left(\phi-\phi_{1}\right) \sqrt{\left(\phi_{L}-\phi\right)\left(\phi_{M}-\phi\right)}}$ and $\sqrt{2} \xi=\int_{\phi_{L}}^{\phi} \frac{\sqrt{\phi-\frac{|c|}{2}} d \phi}{\left(\phi_{1}-\phi\right) \sqrt{\left(\phi_{L}-\phi\right)\left(\phi-\phi_{M}\right)}}$ respectively. The homoclinic orbit gives rise to a solitary wave solution of system (1.6) which has the parametric representation:

$$
\begin{align*}
\phi(\chi) & =\frac{\phi_{M}-\frac{|c|}{2} \operatorname{sn}^{2}(\chi, k)}{\operatorname{cn}^{2}(\chi, k)} \\
\xi(\chi) & =\frac{g\left(\frac{1}{2}|c|-\phi_{M}\right)}{\sqrt{2}\left(\phi_{M}-\phi_{1}\right)} \Pi\left(\arcsin (\operatorname{sn}(\chi, k)), \alpha_{1}^{2}, k\right) \tag{3.3}
\end{align*}
$$

where $g=\frac{2}{\sqrt{\phi_{L}-\phi_{M}}}, k^{2}=\frac{\phi_{L}-\frac{1}{2}|c|}{\phi_{L}-\phi_{M}}, \alpha_{1}^{2}=\frac{\frac{1}{2}|c|-\phi_{1}}{\phi_{M}-\phi_{1}}, \Pi\left(\cdot, \alpha^{2}, k\right)$ is the elliptic integral of the third kind, $\operatorname{sn}(u, k), \operatorname{cn}(u, k)$ are the Jacobian elliptic functions [3]. And the open orbit gives rise to a compacton solution of system (1.6) [6] which has the parametric representation:

$$
\begin{align*}
\phi(\chi) & =\phi_{L}-\left(\phi_{L}-\frac{1}{2}|c|\right) \operatorname{sn}^{2}(\chi, k) \\
\xi(\chi) & =\frac{g}{\sqrt{2}}\left[\chi+\frac{\phi_{1}-\frac{1}{2}|c|}{\phi_{L}-\phi_{1}} \Pi\left(\arcsin (\operatorname{sn}(\chi, k)), \hat{\alpha}_{1}^{2}, k\right)\right], \tag{3.4}
\end{align*}
$$

where $g=\frac{2}{\sqrt{\phi_{L}-\phi_{M}}}, k^{2}=\frac{\phi_{L}-\frac{1}{2}|c|}{\phi_{L}-\phi_{M}}, \hat{\alpha}_{1}^{2}=\frac{\phi_{L}-\frac{1}{2}|c|}{\phi_{L}-\phi_{1}}, \Pi\left(\cdot, \alpha^{2}, k\right)$ is the elliptic integral of the third kind, $\operatorname{sn}(u, k)$ is the Jacobian elliptic functions [3].


Figure 3. The changes of level curves defined by $H(\phi, y)=h$ in Fig.1(c)
3.2. When $g=0$, we see from Figure $1(\mathrm{~d})$ that corresponding to the homoclinic orbit defined by $H(\phi, y)=h_{1}=h_{s}=0$ to the origin $E_{1}(0,0)$, enclosing the center $E_{2}\left(\frac{1}{4}|c|, 0\right)$, we have $y^{2}=2 \phi^{2}\left(\frac{1}{2}|c|-\phi\right)$. It gives rise to the following parametric representation of the solitary wave solution of system (1.6):

$$
\begin{equation*}
\phi(\xi)=\frac{1}{2}|c| \operatorname{sech}^{2}\left(\frac{1}{2} \sqrt{|c|} \xi\right) \tag{3.5}
\end{equation*}
$$

3.3. When $0<g<g_{0}$, we see from Figure 1(e) that corresponding to the homoclinic orbit defined by $H(\phi, y)=h_{3}$, i.e., $y^{2}=\frac{h_{3}-4 \phi^{4}-4 c \phi^{3}-c^{2} \phi^{2}-2 g \phi}{2 \phi+c}=\frac{2\left(\phi-\phi_{l}\right)\left(\phi-\phi_{m}\right)\left(\phi_{3}-\phi\right)^{2}}{\left(\frac{1}{2}|c|-\phi\right)}$ enclosing the center $E_{2}\left(\phi_{2}, 0\right)$, we have $\sqrt{2} \xi=\int_{\phi_{m}}^{\phi} \frac{\left(\frac{1}{2}|c|-\phi\right) d \phi}{\left(\phi_{3}-\phi\right) \sqrt{\left(\frac{1}{2}|c|-\phi\right)\left(\phi-\phi_{l}\right)\left(\phi-\phi_{m}\right)}}$, where $\phi_{l}, \phi_{m}$ are the roots of the equation $H(\phi, 0)=h_{3}$ and satisfy $\phi_{l}<0<\phi_{m}<$ $\phi_{3}$. Thus, we obtain the parametric representation of the solitary wave solution of system (1.6) as follows:

$$
\begin{align*}
\phi(\chi)= & \phi_{l}+\frac{\phi_{m}-\phi_{l}}{\operatorname{dn}^{2}(\chi, k)} \\
\xi(\chi)= & \frac{1}{\phi_{3}-\phi_{l}} \sqrt{\frac{2}{\frac{1}{2}|c|-\phi_{l}}}  \tag{3.6}\\
& {\left[\left(\frac{1}{2}|c|-\phi_{l}\right) \chi+\frac{\left(\phi_{m}-\phi_{l}\right)\left(\frac{1}{2}|c|-\phi_{3}\right)}{\phi_{3}-\phi_{m}} \Pi\left(\arcsin (\operatorname{sn}(\chi, k)), \alpha_{2}^{2}, k\right)\right] }
\end{align*}
$$

where $k^{2}=\frac{\frac{1}{2}|c|-\phi_{m}}{\frac{1}{2}|c|-\phi_{l}}, \alpha_{2}^{2}=\frac{k^{2}\left(\phi_{3}-\phi_{l}\right)}{\phi_{3}-\phi_{m}}$.
Using (3.3),(3.5) to draw the figures, we obtain the following smooth bright solitary wave profiles shown in Figure 4(b).While Using (3.6) to draw the figure, we obtain the following smooth dark solitary wave profiles shown in Figure 4(c). Figure $4(\mathrm{~d})$ is given by (3.4).


Figure 4. Solitary wave ,compacton and periodic peakon solutions of system (1.6)

## 4. Exact explicit parametric representations of the orbits of system (1.6) for $\mathbf{c}>\mathbf{0}$

4.1. When $g=-g_{0}$, for $h \in\left(h_{1}, h_{s}\right)$, the level curves defined by $H(\phi, y)=h$ contain a family of periodic orbits enclosing the equilibrium points $E_{1}\left(\phi_{1}, 0\right)$ and $E_{3}\left(\phi_{3}, 0\right)$. When $\left|h-h_{s}\right| \ll 1$, these periodic orbits give rise to a family of periodic peakon solutions (see Figure 6(a)). As the limit curve of these periodic orbits, the level curve defined by $H(\phi, y)=h_{s}=\frac{\sqrt{3}}{36} c^{4}$ is a heteroclinic loop connecting two saddle points $S_{ \pm}$of system (2.1) (see Figure $2(\mathrm{~b})$ ). For the singular system (1.6), this loop also gives rise to a periodic peakon solution [6]. Now, we have $y^{2}=2\left(\phi_{M 1}-\phi\right)[(\phi-$ $\left.\left.b_{1}\right)^{2}+a_{1}^{2}\right]$, and $\sqrt{2} \xi=\int_{\phi}^{\phi_{M 1}} \frac{d \phi}{\sqrt{\left(\phi_{M 1}-\phi\right)\left[\left(\phi-b_{1}\right)^{2}+a_{1}^{2}\right]}}$, where $\phi_{M 1}=\frac{1}{6}\left(\frac{1}{2} \sigma+\frac{2}{\sigma}-2\right) c$, $b_{1}=-\frac{1}{6}\left(\frac{1}{4} \sigma+\frac{1}{\sigma}+2\right) c, a_{1}^{2}=\frac{1}{48}\left(\frac{1}{2} \sigma-\frac{2}{\sigma}\right)^{2} c^{2}, \sigma=(8+12 \sqrt{3}+4 \sqrt{27+12 \sqrt{3}})^{\frac{1}{3}}$.

Hence, we obtain the following periodic peakon solution of system (1.6):

$$
\begin{equation*}
\phi(\xi)=\frac{\left(A_{1}+\phi_{M 1}\right) \operatorname{cn}\left(\sqrt{2 A_{1}} \xi, k\right)-\left(A_{1}-\phi_{M 1}\right)}{1+\operatorname{cn}\left(\sqrt{2 A_{1}} \xi, k\right)} \tag{4.1}
\end{equation*}
$$

where $A_{1}^{2}=\left(b_{1}-\phi_{M 1}\right)^{2}+a_{1}^{2}, k^{2}=\frac{A_{1}-b_{1}+\phi_{M 1}}{2 A_{1}}$.
For $h=h_{1}=h_{2}=\left(\frac{1}{48}+\frac{\sqrt{3}}{72}\right) c^{4}$, a branch of the level curves defined by $H(\phi, y)=h_{1}$ is a homoclinic orbit to the double equilibrium point of system (1.6). We have $y^{2}=\frac{\left.2\left(\phi-\phi_{1}\right)^{3}\left(\phi_{M 2}-\phi\right)\right)}{\phi+\frac{1}{2} c}$ and $\sqrt{2} \xi=\int_{\phi}^{\phi_{M 2}} \frac{\left(\phi+\frac{1}{2} c\right) d \phi}{\left(\phi-\phi_{1}\right) \sqrt{\left(\phi_{M 2}-\phi\right)\left(\phi-\phi_{1}\right)\left(\phi+\frac{1}{2} c\right)}}$, where $\phi_{1}=\phi_{2}=-\frac{1}{4}\left(1+\frac{\sqrt{3}}{3}\right) c, \phi_{M 2}=\frac{1}{4}(-1+\sqrt{3}) c$. Thus, we obtain the following solitary wave solution of system (1.6) (see Figure 6(b)):

$$
\begin{align*}
\phi(\chi)= & \phi_{M 2}-\left(\phi_{M 2}-\phi_{1}\right) \mathrm{sn}^{2}(\chi, k)=\phi_{M 2}-\frac{\sqrt{3} c}{3} \operatorname{sn}^{2}(\chi, k) \\
\xi(\chi)= & \frac{\sqrt{2}\left(\phi_{1}+\frac{1}{2} c\right)}{\sqrt{\phi_{M 2}+\frac{1}{2} c}}\left(\frac{1}{\left(\phi_{1}+\frac{1}{2} c\right)}+\frac{\sqrt{3}}{c}\right) \chi  \tag{4.2}\\
& \frac{\sqrt{2}\left(\phi_{1}+\frac{1}{2} c\right)}{\sqrt{\phi_{M 2}+\frac{1}{2} c}} \frac{\sqrt{3}}{c \sqrt{1-k^{2}}}(\operatorname{dn}(\chi, k) \operatorname{tn}(\chi, k)-E(\arcsin (\operatorname{sn}(\chi, k)), k)),
\end{align*}
$$

where $k^{2}=\frac{\phi_{M 2}-\phi_{1}}{\phi_{M 2}+\frac{1}{2} c}, E(\cdot, k)$ is the normal elliptic integral of the second kind [3].
4.2. When $-g_{0}<g<-\frac{c^{3}}{27}$, for $h=h_{2}$, the level curves defined by $H(\phi, y)=$ $h_{2}$ are two homoclinic loops to the saddle points $E_{2}\left(\phi_{2}, 0\right)$ of system (1.6) (see Figure $2(\mathrm{c})$ ) for which we have $\sqrt{2} \xi=\int_{\phi}^{\phi_{M}} \frac{\left(\phi+\frac{1}{2} c\right) d \phi}{\left(\phi-\phi_{2}\right) \sqrt{\left(\phi_{M}-\phi\right)\left(\phi-\phi_{m}\right)\left(\phi+\frac{1}{2} c\right)}}$ and $\sqrt{2} \xi=$ $\int_{\phi_{m}}^{\phi} \frac{\left(\phi+\frac{1}{2} c\right) d \phi}{\left(\phi-\phi_{2}\right) \sqrt{\left(\phi_{M}-\phi\right)\left(\phi-\phi_{m}\right)\left(\phi+\frac{1}{2} c\right)}}$ respectively, where $\phi_{M}, \phi_{m}$ are the roots of the equation $H(\phi, 0)=h_{2}$ and satisfy $\phi_{m}<\phi_{2}<\phi_{M}<0$. Hence, we obtain the following two solitary wave solutions of system (1.6)(see Figure 6(b)-(c)):

$$
\begin{align*}
& \phi(\chi)=\phi_{M}-\left(\phi_{M}-\phi_{m}\right) \operatorname{sn}^{2}(\chi, k), \chi \in\left(-\mathrm{sn}^{-1} \sqrt{\frac{\phi_{M}-\phi_{2}}{\phi_{M}-\phi_{m}}}, \mathrm{sn}^{-1} \sqrt{\frac{\phi_{M}-\phi_{2}}{\phi_{M}-\phi_{m}}}\right), \\
& \xi(\chi)=\sqrt{\frac{2}{\phi_{M}+\frac{1}{2} c}}\left[\chi+\frac{\phi_{2}+\frac{1}{2} c}{\phi_{M}-\phi_{2}} \Pi\left(\arcsin (\operatorname{sn}(\chi, k)), \alpha_{3}^{2}, k\right)\right] \tag{4.3}
\end{align*}
$$

where $k^{2}=\frac{\phi_{M}-\phi_{m}}{\phi_{M}+\frac{1}{2} c}, \alpha_{3}^{2}=\frac{\phi_{M}-\phi_{m}}{\phi_{M}-\phi_{2}}$ and

$$
\begin{align*}
& \phi(\chi)=-\frac{1}{2} c+\frac{\phi_{m}+\frac{1}{2} c}{\operatorname{dn}^{2}(\chi, k)} \\
& \xi(\chi)=\left(\frac{\left.\phi_{m}+\frac{1}{2} c\right)}{\phi_{m}-\phi_{2}} \sqrt{\frac{2}{\phi_{M}+\frac{1}{2} c}}\right) \Pi\left(\arcsin (\operatorname{sn}(\chi, k)), \alpha_{4}^{2}, k\right), \tag{4.4}
\end{align*}
$$

where $k^{2}=\frac{\phi_{M}-\phi_{m}}{\phi_{M}+\frac{1}{2} c}, \alpha_{4}^{2}=\frac{k^{2}\left(\phi_{2}+\frac{1}{2} c\right)}{\phi_{2}-\phi_{m}}$.

When $h \in\left(h_{2}, h_{s}\right)$, we have the similar phenomenon as $h \in\left(h_{1}, h_{s}\right)$ in section 4.1.
4.3. When $g=-\frac{c^{3}}{27}$, for $h=h_{s}=h_{2}=\frac{1}{27} c^{4}$, the level curves defined by $H(\phi, y)=h_{s}$ contain a homoclinic orbit to the saddle point $E_{2}\left(\phi_{2}, 0\right)$ enclosing the equilibrium point $E_{3}\left(\phi_{3}, 0\right)$ and a heteroclinic triangle enclosing the equilibrium point $E_{1}\left(\phi_{1}, 0\right)$ of system $(2.1)$, where $\phi_{1}=-\frac{1}{24}(5+\sqrt{33}) c, \phi_{2}=-\frac{1}{3} c, \phi_{3}=\frac{1}{24}(-5+$ $\sqrt{33}) c$. Now, for the homoclinic orbit, we have $y^{2}=2\left(\frac{1}{6} c-\phi\right)\left(\phi+\frac{1}{3} c\right)^{2}$. Therefore, we obtain the following solitary wave solution (see Figure 6(b)):

$$
\begin{equation*}
\phi(\xi)=-\frac{c}{3}+\frac{c}{2} \operatorname{sech}^{2}\left(\frac{\sqrt{c}}{2} \xi\right) \tag{4.5}
\end{equation*}
$$

On the other hand, when $h$ increases from $h_{1}=\frac{1}{3456}(59+11 \sqrt{33}) c^{4}$ to $h_{s}$, the periodic orbit enclosing the equilibrium point $E_{1}\left(\phi_{1}, 0\right)$ defined by $H(\phi, y)=h, h \in$ $\left(h_{1}, h_{s}\right)$ approaches to the boundary triangle curves. As the limit of the periodic orbit family, the boundary curve gives rise to an anti-peakon solution of system (1.6) [6]. Let $\xi_{0}=\operatorname{ctnh}^{-1} \frac{2}{\sqrt{3}}$. Then, the anti-peakon solution (see Figure 6(e)) has the following parametric representation which is a limit solution of a periodic peakon family (see Figure 6(d)):

$$
\begin{equation*}
\phi(\xi)=\frac{c}{6}-\frac{c}{2} \operatorname{ctnh}^{2}\left(\frac{\sqrt{c}}{2} \xi-\xi_{0}\right) . \tag{4.6}
\end{equation*}
$$



Figure 5. The changes of level curves defined by $H(\phi, y)=h$ in Figure 2(d)
4.4. When $-\frac{c^{3}}{27}<g<g_{0}$, the level curves defined by $H(\phi, y)=h_{2}$,i.e., $y^{2}=$ $\frac{2\left(\phi_{M}-\phi\right)\left(\phi-\phi_{2}\right)^{2}\left(\phi-\phi_{l}\right)}{\phi+\frac{1}{2} c}$ contain a homoclinic orbit to the equilibrium point $E_{2}\left(\phi_{2}, 0\right)$ enclosing the equilibrium point $E_{2}\left(\phi_{3}, 0\right)$ (see Figure $2(\mathrm{e}),(\mathrm{f})$ and (g)). Now, we have $\sqrt{2} \xi=\int_{\phi}^{\phi_{M}} \frac{\left(\phi+\frac{1}{2} c\right) d \phi}{\left(\phi-\phi_{2}\right) \sqrt{\left(\phi_{M}-\phi\right)\left(\phi+\frac{1}{2} c\right)\left(\phi-\phi_{l}\right)}}$, where $\left(\phi_{M}, 0\right)$ and $\left(\phi_{l}, 0\right)$ are the points at which the curve $H(\phi, y)=h_{2}$ intersects the $\phi$ - axis. Hence, we obtain the following solitary wave solution:

$$
\begin{align*}
& \phi(\chi)=\phi_{M}-\left(\phi_{M}+\frac{1}{2} c\right) \operatorname{sn}^{2}(\chi, k), c h i \in\left(-\mathrm{sn}^{-1} \sqrt{\frac{\phi_{M}-\phi_{2}}{\phi_{M}+\frac{1}{2} c}}, \mathrm{sn}^{-1} \sqrt{\frac{\phi_{M}-\phi_{2}}{\phi_{M}+\frac{1}{2} c}}\right) \\
& \xi(\chi)=\sqrt{\frac{2}{\phi_{M}-\phi_{l}}}\left[\chi+\frac{\phi_{2}+\frac{1}{2} c}{\phi_{M}-\phi_{2}} \Pi\left(\arcsin (\operatorname{sn}(\chi, k)), \alpha_{5}^{2}, k\right)\right] \tag{4.7}
\end{align*}
$$

where $k^{2}=\frac{\phi_{M}+\frac{1}{2} c}{\phi_{M}-\phi_{l}}, \alpha_{5}^{2}=\frac{\phi_{M}+\frac{1}{2} c}{\phi_{M}-\phi_{2}}$.


Figure 6. Solitary wave and Anti-peakon solutions of system (1.6)

## 5. Exact traveling wave solutions of equation (1.1)

In this section, we apply the solutions of system (1.6) given by section 3 and 4 to obtain some exact traveling wave solutions of equation (1.1).

First, by using solutions (3.5), (4.5) and (4.6), formula (1.8) follows that, for $c<0$,

$$
\begin{equation*}
w(x, t)=\psi(x-c t)=\int_{0}^{\xi} \frac{1}{2}|c| \operatorname{sech}^{2}\left(\frac{1}{2} \sqrt{|c|} \xi\right) d \xi=\sqrt{|c|} \tanh \left(\frac{1}{2} \sqrt{|c|} \xi\right) \tag{5.1}
\end{equation*}
$$

and for $c>0$,

$$
\begin{align*}
& w(x, t)=\int_{0}^{\xi}\left(-\frac{c}{3}+\frac{c}{2} \operatorname{sech}^{2}\left(\frac{\sqrt{c}}{2} \xi\right)\right) d \xi=-\frac{1}{3} c \xi+\sqrt{c} \tanh \left(\frac{1}{2} \sqrt{c} \xi\right)  \tag{5.2}\\
& w(x, t)=\int_{0}^{\xi}\left(\frac{c}{6}-\frac{c}{2} \operatorname{ctnh}^{2}\left(\frac{\sqrt{c}}{2} \xi-\xi_{0}\right)\right) d \xi  \tag{5.3}\\
&=-\frac{1}{3} c \xi+\sqrt{c} \operatorname{ctnh}\left(\frac{\sqrt{c}}{2} \xi-\xi_{0}\right)+\frac{2 c}{\sqrt{3}}
\end{align*}
$$

Second, we see from $(3.3),(3.6),(4.2),(4.3),(4.4)$ and (4.7) that by using formula (1.8), we obtain the following exact solutions [3] of $w(x, t)=w(\xi)$ :

$$
\begin{align*}
w(\chi) & =\int_{0}^{\chi}\left(\frac{\phi_{M}-\frac{|c|}{2} \operatorname{sn}^{2}(\chi, k)}{\operatorname{cn}^{2}(\chi, k)}\right) d \chi \\
& =\phi_{M} \chi+\frac{\phi_{M}-\frac{1}{2}|c|}{1-k^{2}}[\operatorname{dn}(\chi, k) \operatorname{tn}(\chi, k)-E(\arcsin (\operatorname{sn}(\chi, k)), k)]  \tag{5.4}\\
\xi(\chi) & =\frac{\tilde{g}_{1}\left(\frac{1}{2}|c|-\phi_{M}\right)}{\sqrt{2}\left(\phi_{M}-\phi_{1}\right)} \Pi\left(\arcsin (\operatorname{sn}(\chi, k)), \alpha_{1}^{2}, k\right)
\end{align*}
$$

where $\tilde{g}_{1}=\frac{2}{\sqrt{\phi_{L}-\phi_{M}}}, k^{2}=\frac{\phi_{L}-\frac{|c|}{2}}{\phi_{L}-\phi_{M}}, \alpha_{1}^{2}=\frac{\frac{1}{2}|c|-\phi_{1}}{\phi_{M}-\phi_{1}}$.

$$
\begin{align*}
w(\chi)= & \int_{0}^{\chi}\left(\phi_{l}+\frac{\phi_{m}-\phi_{l}}{\operatorname{dn}^{2}(\chi, k)}\right) d \chi \\
= & \phi_{l} \chi-\frac{\phi_{M}-\phi_{l}}{1-k^{2}}\left[k^{2} \operatorname{sn}(\chi, k) \operatorname{cd}(\chi, k)-E(\arcsin (\operatorname{sn}(\chi, k)), k)\right] \\
\xi(\chi)= & -\sqrt{\frac{2}{\left(\frac{1}{2}|c|-\phi_{l}\right)\left(\phi_{3}-\phi_{l}\right)}}  \tag{5.5}\\
& {\left[\left(\frac{1}{2}|c|-\phi_{l}\right) \chi+\frac{\left(\phi_{m}-\phi_{l}\right)\left(\frac{1}{2}|c|-\phi_{3}\right)}{\phi_{3}-\phi_{m}} \Pi\left(\arcsin (\operatorname{sn}(\chi, k)), \alpha_{2}^{2}, k\right)\right] }
\end{align*}
$$

where $k^{2}=\frac{\frac{1}{2}|c|-\phi_{m}}{\frac{1}{2}|c|-\phi_{l}}, \alpha_{2}^{2}=\frac{k^{2}\left(\phi_{3}-\phi_{l}\right)}{\phi_{3}-\phi_{m}}$.

$$
\begin{align*}
w(\chi)= & \int_{0}^{\chi}\left(\phi_{M 2}-\frac{\sqrt{3} c}{3} \operatorname{sn}^{2}(\chi, k)\right) d \chi \\
= & \left(\phi_{M 2}-\frac{\sqrt{3} c}{3 k^{2}}\right) \chi+\frac{\sqrt{3} c}{3 k^{2}} E(\arcsin (\operatorname{sn}(\chi, k)), k) \\
\xi(\chi)= & \frac{\sqrt{2}\left(\phi_{1}+\frac{1}{2} c\right)}{\sqrt{\phi_{M 2}+\frac{1}{2} c}}\left(\frac{1}{\left(\phi_{1}+\frac{1}{2} c\right)}+\frac{\sqrt{3}}{c}\right) \chi  \tag{5.6}\\
& +\frac{\sqrt{2}\left(\phi_{1}+\frac{1}{2} c\right)}{\sqrt{\phi_{M 2}+\frac{1}{2} c}} \frac{\sqrt{3}}{c \sqrt{1-k^{2}}}(\operatorname{dn}(\chi, k) \operatorname{tn}(\chi, k)-E(\arcsin (\operatorname{sn}(\chi, k)), k))
\end{align*}
$$

where $k^{2}=\frac{\phi_{1}+\frac{1}{2} c}{\phi_{M 2}+\frac{1}{2} c}$.

$$
\begin{align*}
w(\chi) & =\int_{0}^{\chi}\left(\phi_{M}-\left(\phi_{M}-\phi_{m}\right) \operatorname{sn}^{2}(\chi, k)\right) d \chi \\
& =\left(\phi_{M}-\frac{\phi_{M}-\phi_{m}}{k^{2}}\right) \chi+\left(\frac{\phi_{M}-\phi_{m}}{k^{2}}\right) E(\arcsin (\operatorname{sn}(\chi, k)), k) \\
\chi & \in\left(-\operatorname{sn}^{-1} \sqrt{\frac{\phi_{M}-\phi_{2}}{\phi_{M}-\phi_{m}}}, \mathrm{sn}^{-1} \sqrt{\frac{\phi_{M}-\phi_{2}}{\phi_{M}-\phi_{m}}}\right)  \tag{5.7}\\
\xi(\chi) & =\sqrt{\frac{2}{\phi_{M}+\frac{1}{2} c}}\left[\chi+\frac{\phi_{2}+\frac{1}{2} c}{\phi_{M}-\phi_{2}} \Pi\left(\arcsin (\operatorname{sn}(\chi, k)), \alpha_{3}^{2}, k\right)\right]
\end{align*}
$$

where $k^{2}=\frac{\phi_{M}-\phi_{m}}{\phi_{M}+\frac{1}{2} c}, \alpha_{3}^{2}=\frac{\phi_{M}-\phi_{m}}{\phi_{M}-\phi_{2}}$.

$$
\begin{align*}
w(\chi) & =\int_{0}^{\chi}\left(-\frac{1}{2} c+\frac{\phi_{m}+\frac{1}{2} c}{\operatorname{dn}^{2}(\chi, k)}\right) d \chi \\
& =-\frac{1}{2} c \chi+\frac{\phi_{m}+\frac{1}{2} c}{1-k^{2}}\left[E(\arcsin (\operatorname{sn}(\chi, k)), k)-k^{2} \operatorname{sn}(\chi, k) \operatorname{cd}(\chi, k)\right]  \tag{5.8}\\
\xi(\chi) & =\left(\frac{\sqrt{2\left(\phi_{M}+\frac{1}{2} c\right)}}{\phi_{2}-\phi_{m}}\right) \Pi\left(\arcsin (\operatorname{sn}(\chi, k)), \alpha_{4}^{2}, k\right)
\end{align*}
$$

where $k^{2}=\frac{\phi_{M}-\phi_{m}}{\phi_{M}+\frac{1}{2} c}, \alpha_{4}^{2}=\frac{k^{2}\left(\phi_{M}+\frac{1}{2} c\right)}{\phi_{2}-\phi_{m}}$.

$$
\begin{align*}
w(\chi) & =\int_{0}^{\chi}\left(\phi_{M}-\left(\phi_{M}+\frac{1}{2} c\right) \operatorname{sn}^{2}(\chi, k)\right) d \chi \\
& =\left(\phi_{M}-\frac{\phi_{M}+\frac{1}{2} c}{k^{2}}\right) \chi+\left(\frac{\phi_{M}+\frac{1}{2} c}{k^{2}}\right) E(\arcsin (\operatorname{sn}(\chi, k)), k) \\
\chi & \in\left(-\mathrm{sn}^{-1} \sqrt{\frac{\phi_{M}-\phi_{2}}{\phi_{M}+\frac{1}{2} c}}, \mathrm{sn}^{-1} \sqrt{\frac{\phi_{M}-\phi_{2}}{\phi_{M}+\frac{1}{2} c}}\right)  \tag{5.9}\\
\xi(\chi) & =\sqrt{\frac{2}{\phi_{M}-\phi_{l}}}\left[\chi+\frac{\phi_{2}+\frac{1}{2} c}{\phi_{M}-\phi_{2}} \Pi\left(\arcsin (\operatorname{sn}(\chi, k)), \alpha_{5}^{2}, k\right)\right]
\end{align*}
$$

where $k^{2}=\frac{\phi_{M}+\frac{1}{2} c}{\phi_{M}-\phi_{l}}, \alpha_{5}^{2}=\frac{\phi_{M}+\frac{1}{2} c}{\phi_{M}-\phi_{2}}$.

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