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GROUP-INVARIANT SOLUTIONS, NON-GROUP-INVARIANT SOLUTIONS AND CONSERVATION LAWS OF QIAO EQUATION*

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Abstract This paper considers a completely integrable nonlinear wave equation which is called Qiao equation. The equation is reduced via Lie symmetry analysis. Two classes of new exact group-invariant solutions are obtained by solving the reduced equations. Specially, a novel technique is proposed for constructing group-invariant solutions and non-group-invariant solutions based on travelling wave solutions. The obtained exact solutions include a set of traveling wave-like solutions with variable amplitude, variable velocity or both. Nonlocal conservation laws of Qiao equation are also obtained with the corresponding infinitesimal generators.

Keywords Qiao equation, group-invariant solution, non-group-invariant solution, traveling wave-like solution, variable amplitude, variable velocity, conservation law.

MSC(2010) 70G65, 35L65.

1. Introduction

In applied mathematics and physics, nonlinear wave equations play an important role in the study of nonlinear dynamics. Many researchers tried their best to find analytical solutions to a wide class of nonlinear wave equations in order to give a better description of dynamical properties to these equations. So far, there are many powerful methods which are used to find exact solutions. For example, inverse scattering method [8], Darboux transformation method [9], sine-cosine method [23], double exp-function method [11], dynamical system method [13], and so on. As a classic method, Lie symmetry analysis is an important tool to seek analytical solutions to nonlinear wave equations [3]. However, the method was oblivious as it is too complicated to manual computation for many equations. Until the fifties of last century, Lie group analysis method was applied to study some partial differential equations in fluid mechanics by Girkhoff et al [10]. A lot of valuable work has been done during the past seventy years by Lie symmetry method. With the development of computer algebra tools, Lie symmetry analysis has become a powerful tool for finding analytical solutions of PDEs.

Variable amplitude solutions and variable velocity solutions are common in nonlinear wave problems. Solutions with those physical characteristics describe the

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physical phenomena more exactly. Many significative results have been obtained. For instance, Yamashita investigated the multisolitons and soliton lattices in Sine-Gordon system with variable amplitude [24]. Latyshev and Bedrikova obtained analytical solutions of the second Stokes problem with variable amplitude [2]. Some asymptotic solutions of 2D wave equations with variable velocity were obtained by Dobrokhotov et al [7]. Allilueva and Shafarevich got localized asymptotic solutions of a wave equation with variable velocity [1].

This paper considers a completely integrable nonlinear wave equation:

$$F = u_t - u_{txx} + 3u^2 u_x - {u_x}^3 - (4u - 2u_{xx})u_x u_{xx} + ({u_x}^2 - u^2)u_{xxx} = 0, \quad (1.1)$$

where u is the fluid velocity and subscripts denote the partial derivatives. Eq. (1.1) was derived from two dimensional Euler equation by Qiao [21], and was named Qiao equation subsequently [22, 25]. Qiao proved that Eq. (1.1) has Lax pair and bi-Hamiltonian structures, and found a new kind of soliton solutions which were called as W/M-shape-peaks solitons [21]. Li and Dai distinguished two classes of singular nonlinear traveling wave equations in their book [14]. Qiao equation belongs to the second class. By using dynamical system approach, Li, Zhao and Chen not only found W/M-shape-peaks solitons consisting of three breaking wave solutions, but also investigated the dynamical behaviors of Qiao equation in details [15]. Recent advances on the study of travelling wave solutions via dynamical system approach, we refer to [16–20].

However, to our best knowledge, there are few papers discussing traveling wave-like solutions with variable amplitude or variable velocity of Qiao equation (1.1). In this paper, Lie symmetry analysis method is used to study Qiao equation. We derive the classic Lie symmetries admitted by Qiao equation, and obtain some reduced equations. Some new exact group-invariant solutions and non-group-invariant solutions are obtained by solving reduced equations. These solutions are quite different from those obtained by Li et al [15]. The exact expressions of these solutions indicate some physical characters of variable amplitude, variable wave velocity or both. In particular, a novel technique is proposed for constructing group-invariant solutions and non-group-invariant solutions based on travelling wave solutions. Moreover, we also get nonlocal conservation laws with the corresponding symmetries.

This paper is organized as follows. In Section 2, a few of necessary preliminaries are stated. In Section 3, by using Lie symmetries, we deal with the similarity reductions to Eq. (1.1) and obtain some new exact group-invariant solutions and non-group-invariant solutions. Some figures are given to illustrate distinguishing characteristics of those solutions. In Section 4, nonlocal conservation laws are obtained with the corresponding symmetries. A comprehensive conclusion is given in Section 5.

2. Preliminaries

In this section, some basic concepts and results are introduced briefly. For more details, we refer to [12].

Consider sth-order system of k partial differential equations with n independent variables $x = (x^1, x^2, \dots, x^n)$ and m dependent variables $u = (u^1, u^2, \dots, u^m)$, reads

$$F_\sigma(x, u, u_{(1)}, u_{(2)}, \dots, u_{(s)}) = 0, \quad \sigma = 1, \dots, k, \quad (2.1)$$

where $u_{(i)}$ is the collection of ith order of partial derivatives, namely, $u_{(1)} = \{u_i^\alpha\}, u_{(2)} = \{u_{ij}^\alpha\}, \ldots$, and $u_i^\alpha(x) = \frac{\partial u^\alpha(x)}{\partial x^i}, u_{ij}^\alpha(x) = \frac{\partial^2 u^\alpha(x)}{\partial x^i \partial x^j}, \ldots$.

The total differential operator with respect to x^i is defined as

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \quad i, j = 1, \ldots, n, \ \alpha = 1, \ldots, m. \tag{2.2}$$

So, $u_i^\alpha = D_i(u^\alpha), u_{ij}^\alpha = D_j D_i(u^\alpha), \ldots$.

The Lie symmetry operator

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} \tag{2.3}$$

is considered in the following, where ξ^i, η^α are described in [12].

Definition 2.1 ([12]). A vector field $C(x, u, u_{(1)}, ...)$ with n components,

$$C = (C^1, C^2, ..., C^n) \tag{2.4}$$

is called a conserved vector if it satisfies the equation

$$divC = D_1(C^1) + D_2(C^2) + \ldots + D_n(C^n) = 0 \tag{2.5}$$

on each solution $u = u(x)$ of Eqs. (2.1). Eq. (2.5) is termed a conservation equation, or a conservation law for Eqs. (2.1).

Remark 2.1 ([12]). Note that if

$$C^1|_{(2.1)} = \tilde{C}^1 + D_2(h^2) + \ldots + D_n(h^n), \tag{2.6}$$

the conservation equation (2.5) can be equivalently rewritten in the form

$$D_1(\tilde{C}^1) + D_2(\tilde{C}^2) + \ldots + D_n(\tilde{C}^n) = 0 \tag{2.7}$$

with

$$\tilde{C}^2 = C^2 + D_1(h^2), \ldots, \tilde{C}^n = C^n + D_1(h^n).$$

Definition 2.2 ([12]). The adjoint system to a system (2.1) is given by

$$F_\sigma^*(x, u, v, \ldots, u_{(s)}, v_{(s)}) = 0, \quad \sigma = 1, \ldots, k, \tag{2.8}$$

where F_σ^* is defined by

$$F_\sigma^*(x, u, v, \ldots, u_{(s)}, v_{(s)}) = \frac{\delta(v^\beta F_\beta)}{\delta u^\alpha}, \tag{2.9}$$

and

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 \cdots i_s}} \tag{2.10}$$

is the Euler-Lagrange operator.

Definition 2.3 ([12]). The system (2.1) is said to be self-adjoint if the system obtained by substituting $v = u$ in the adjoint system (2.8) is equivalent to the original system.

Theorem 2.1 ([12]). *For any system (2.1) admitting an operator (2.3), the following quantities provide a nonlocal conserved vector:*

$$C^i = N^i(L), i = 1, \ldots, n. \tag{2.11}$$

Here $N^i (i = 1, 2, \ldots, n)$ are infinite-order operators defined by

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u_i^\alpha} + \sum_{s=1}^{\infty} D_{i_1} \ldots D_{i_s}(W^\alpha) \frac{\delta}{\delta u_{ii_1\ldots i_s}^\alpha}, \; i = 1, \ldots, n, \; \alpha = 1, \ldots, m. \tag{2.12}$$

where $L = v^\beta F_\beta$ is called the formal Lagrangian for the system (2.1), and $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$ is the Lie characteristic function.

3. Symmetry reductions, exact group-invariant solutions and non-group-invariant solutions

In this section, we deal with the symmetry reductions, exact group-invariant solutions and non-group-invariant solutions for Eq. (1.1) based on the symmetry analysis.

It is easy to obtain the following linearly independent infinitesimal symmetries using Soft Package GEM v.032.02 developed by Cheviakov [4–6]:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}. \tag{3.1}$$

Moreover, Eq. (1.1) has a 3-dimensional Lie algebra with the basis (3.1).

3.1. Symmetry reductions and group-invariant solutions

In this subsection, we consider symmetry reductions and seek exact group-invariant solutions via linear combinations X of infinitesimal symmetries X_1, X_2 and X_3.

Case (I). For the generator $X_1 + cX_2 = \frac{\partial}{\partial t} + c\frac{\partial}{\partial x}$, where c is an arbitrary constant, integrating the characteristic system

$$\frac{dt}{1} = \frac{dx}{c} = \frac{du}{0}, \tag{3.2}$$

we obtain the following similarity variables and group-invariant solution:

$$\xi = x - ct, \quad \omega = u(t, x) = f(\xi). \tag{3.3}$$

Substituting (3.3) into Eq. (1.1), we obtain the following reduction equation:

$$\begin{aligned} &- cf'(\xi) + cf'''(\xi) + 3f(\xi)^2 f'(\xi) - f'(\xi)^3 + (2f''(\xi) \\ &- 4f(\xi))f'(\xi)f''(\xi) + (f'(\xi)^2 - f(\xi)^2)f'''(\xi) = 0. \end{aligned} \tag{3.4}$$

Integrating Eq. (3.4), we can obtain:

$$f(\xi) = c_1 \exp(\xi) + c_2 \exp(-\xi), \tag{3.5}$$

where c_1 and c_2 are arbitrary constants.

So, Eq. (1.1) has the following traveling wave solutions

$$u(t,x) = c_1 \exp(x - ct) + c_2 \exp(-(x - ct)), \tag{3.6}$$

where c, c_1 and c_2 are arbitrary constants.

Case (II). For the generator

$$\alpha X_1 + \beta X_2 + \gamma X_3 = (2\gamma t + \alpha)\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial x} - \gamma u\frac{\partial}{\partial u},$$

where α, γ and β are arbitrary constants, integrating the characteristic system

$$\frac{dt}{2\gamma t + \alpha} = \frac{dx}{\beta} = \frac{du}{-\gamma u}, \tag{3.7}$$

we obtain the following similarity variables:

$$\xi = x - \frac{\beta \ln(2\gamma t + \alpha)}{2\gamma}, \quad \omega = u\sqrt{2\gamma t + \alpha}, \tag{3.8}$$

and the group-invariant solution is $\omega = f(\xi)$, that is

$$u(t,x) = \frac{f(x - \frac{\beta \ln(2\gamma t+\alpha)}{2\gamma})}{\sqrt{2\gamma t + \alpha}}. \tag{3.9}$$

Substituting (3.9) into Eq. (1.1), we obtain the following reduction equation:

$$\begin{aligned}&- f'(\xi)^3 + f'(\xi)^2 f'''(\xi) + 3f(\xi)^2 f'(\xi) - 4f'(\xi)f''(\xi)f(\xi) + 2f'(\xi)f''(\xi)^2\\ &- f(\xi)^2 f'''(\xi) - \beta f'(\xi) - \gamma f(\xi) + \beta f'''(\xi) + \gamma f''(\xi) = 0.\end{aligned} \tag{3.10}$$

It should be noted that the Eq. (3.10) is a higher-order nonlinear ordinary differential equation and difficult to be solved. However, Eq. (3.10) can be rewritten as the following form

$$(f''(\xi) - f(\xi))(2f'(\xi)f(\xi) - 2f'(\xi)f''(\xi) - \gamma) - (\beta + f'(\xi)^2 - f(\xi)^2)(f'''(\xi) - f'(\xi)) = 0. \tag{3.11}$$

It is easy to see that $f(\xi) = f''(\xi)$ satisfies Eq. (3.11). By solving the equation $f(\xi) = f''(\xi)$, we obtain

$$f(\xi) = c_1 \exp(\xi) + c_2 \exp(-\xi). \tag{3.12}$$

So, Eq. (1.1) has the group-invariant solutions of the form

$$u(t,x) = \frac{c_1 \exp(x - \frac{\beta \ln(2\gamma t+\alpha)}{2\gamma})) + c_2 \exp(-x + \frac{\beta \ln(2\gamma t+\alpha)}{2\gamma}))}{\sqrt{2\gamma t + \alpha}}, \tag{3.13}$$

where α, β, γ, c_1 and c_2 are arbitrary constants.

3.2. Constructing group-invariant solutions and non-group-invariant solutions via travelling wave solutions

In this subsection, we present a novel technique of constructing group-invariant solutions and non-group-invariant solutions based on travelling wave solutions. The steps of this technique is as follows:

Step 1. Construct a class of solutions by transforming constant amplitude and constant velocity in the analytic expression of given travelling wave solution into variable amplitude and variable velocity respectively.

Step 2. Find appropriate variable amplitude and variable velocity based on invariance condition so as to identify which solutions constructed in Step 1 are group-invariant.

We give an example to illustrate this technique. Based on travelling wave solution (3.6), we construct solutions of the form

$$\hat{u}(t,x) = g(t)(c_1 \exp(x - c(t)t) + c_2 \exp(-x + c(t)t)), \tag{3.14}$$

where $g(t), c(t)$ are to be determined.

By a direct calculation, it is easy to verify that for arbitrary derivable functions $g(t), c(t)$, and arbitrary constants c_1, c_2, (3.14) is a solution of Eq. (1.1).

Next, we seek conditions which guarantee (3.14) is group-invariant.

Set

$$G(t,x,u) = -u + g(t)(c_1 \exp(x - c(t)t) + c_2 \exp(-x + c(t)t)).$$

It follows from Theorem 5.4 in [12] that (3.14) is a group-invariant solution of Eq. (1.1) if and only if for some λ_1, λ_2 and λ_3, the following invariance condition is satisfied:

$$XG(t,x,u)\ |_{u=g(t)(c_1\exp(x-c(t)t)+c_2\exp(-x+c(t)t))} = 0, \tag{3.15}$$

where

$$X = \lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 = (2\lambda_3 t + \lambda_1)\frac{\partial}{\partial t} + \lambda_2 \frac{\partial}{\partial x} - \lambda_3 u \frac{\partial}{\partial u}.$$

By a simple calculation, it is easy to see that (3.15) is equivalent to the following conditions

$$\begin{aligned} &-2g(t)\frac{dc(t)}{dt}\lambda_3 t^2 - g(t)\frac{dc(t)}{dt}\lambda_1 t - 2g(t)c(t)\lambda_3 t + 2\frac{dg(t)}{dt}\lambda_3 t \\ &- g(t)c(t)\lambda_1 + \frac{dg(t)}{dt}\lambda_1 + g(t)\lambda_2 + g(t)\lambda_3 = 0, \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} &2g(t)\frac{dc(t)}{dt}\lambda_3 t^2 + g(t)\frac{dc(t)}{dt}\lambda_1 t + 2g(t)c(t)\lambda_3 t + 2\frac{dg(t)}{dt}\lambda_3 t \\ &+ g(t)c(t)\lambda_1 + \frac{dg(t)}{dt}\lambda_1 - g(t)\lambda_2 + g(t)\lambda_3 = 0. \end{aligned} \tag{3.17}$$

Therefore, if $g(t), c(t)$ satisfy Eqs. (3.16) and (3.17) for some λ_1, λ_2 and λ_3, then (3.14) is a group-invariant solution of Eq. (1.1). Otherwise, (3.14) is not invariant under any transformation group admitted by Eq. (1.1).

Finally, we find $g(t), c(t)$ which satisfy Eqs. (3.16) and (3.17).

Eq. (3.16) plus Eq. (3.17) gives

$$(2\lambda_3 t + \lambda_1)\frac{dg(t)}{dt} + \lambda_3 g(t) = 0. \tag{3.18}$$

By solving Eq. (3.18), we obtain

$$g(t) = \frac{C_1}{\sqrt{2\lambda_3 t + \lambda_1}}, \tag{3.19}$$

where C_1 is an arbitrary constant.

Eq. (3.16) minus Eq. (3.17) gives

$$g(t)(-2\frac{dc(t)}{dt}\lambda_3 t^2 - \frac{dc(t)}{dt}\lambda_1 t - 2c(t)t\lambda_3 - c(t)\lambda_1 + \lambda_2) = 0. \tag{3.20}$$

For any $g(t) \neq 0$, by solving Eq. (3.20), we obtain

$$c(t) = \begin{cases} \frac{\lambda_2 \ln(2\lambda_3 t+\lambda_1)+2\lambda_3 C_2}{2\lambda_3 t}, \text{ for } \lambda_3 \neq 0, \\ \frac{\lambda_2}{\lambda_1} + \frac{C_2}{t}, \text{ for } \lambda_3 = 0, \end{cases} \tag{3.21}$$

where C_2 is an arbitrary constant.

It is easy to verify that (3.19) and (3.21) satisfy Eqs. (3.16) and (3.17).

According to above analysis, we have the following result.

Theorem 3.1. *(i) Qiao equation (1.1) has group-invariant solutions of the form (3.14), where $g(t)$ is given in (3.19), and $c(t)$ is given in (3.21).*

(ii) Qiao equation (1.1) has non-group-invariant solutions of the form (3.14) with arbitrary nonconstant derivable functions $g(t)$ and $c(t)$ satisfying

$$g(t) \neq \frac{C_1}{\sqrt{2\lambda_3 t + \lambda_1}}, \tag{3.22}$$

or

$$c(t) \neq \begin{cases} \frac{\lambda_2 \ln(2\lambda_3 t+\lambda_1)+2\lambda_3 C_2}{2\lambda_3 t}, \textit{ for } \lambda_3 \neq 0, \\ \frac{\lambda_2}{\lambda_1} + \frac{C_2}{t}, \textit{ for } \lambda_3 = 0, \end{cases} \tag{3.23}$$

where $\lambda_i (i = 1, 2, 3)$, C_1 and C_2 are arbitrary constants.

Remark 3.1. Exact solutions of the form (3.14) are different from those solutions obtained in [14, 15, 21, 22, 25]. Non-group-invariant solutions of the form (3.14) can be seen as traveling wave-like solutions with variable amplitude, variable velocity or both.

Remark 3.2. Based on traveling wave solutions (3.6), we not only obtain group-invariant solutions (3.13), but also obtain a class of non-group-invariant solutions.

Finally, we give some figures to demonstrate the above exact solutions. Figure 1 depicts a group-invariant solution. Figure 2 depicts traveling wave-like solution with unbounded variable amplitude or bounded variable amplitude respectively. Figure 3 depicts traveling wave-like solution with variable velocity or both of variable amplitude and variable velocity respectively. Figure 4 depicts two traveling wave-like solutions with variable amplitude and variable velocity.

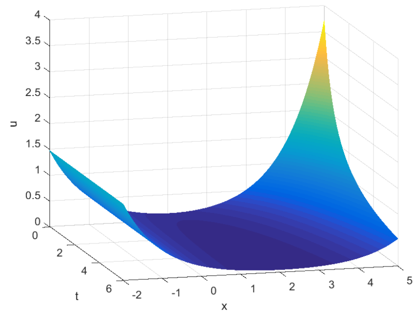
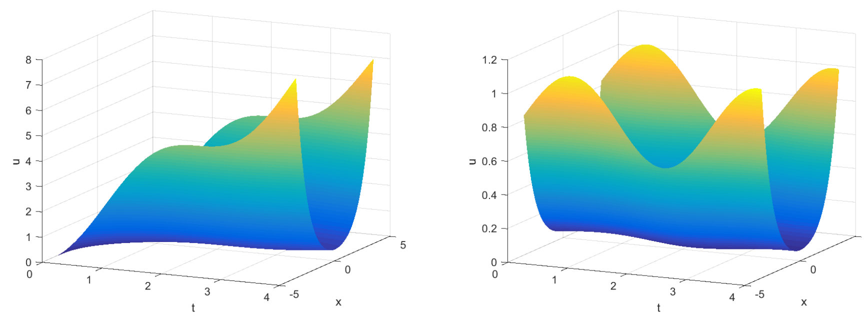
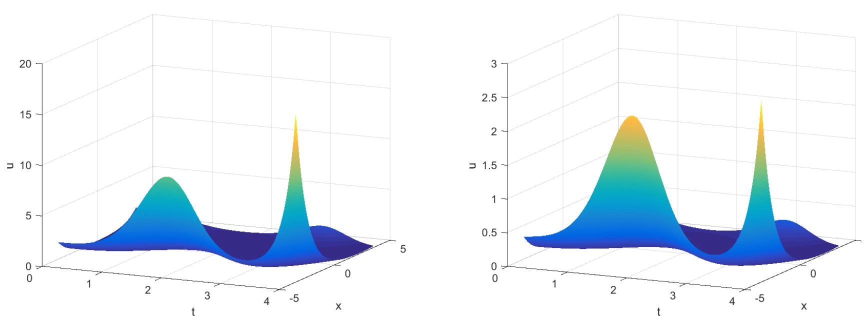


Figure 1. A group-invariant solution $u(t,x) = \frac{c_1 \exp(x - \frac{\beta \ln(2\gamma t+\alpha)}{2\gamma}) + c_2 \exp(-x + \frac{\beta \ln(2\gamma t+\alpha)}{2\gamma})}{\sqrt{2\gamma t+\alpha}}$ with $c_1 = 0.05, c_2 = 0.2, \alpha = 2, \beta = 1, \gamma = 1$.



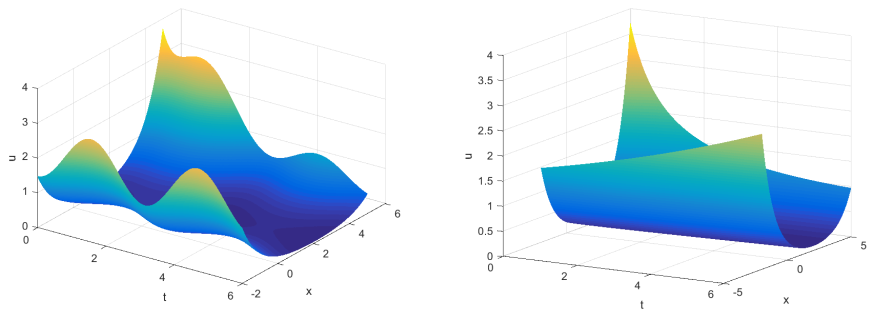
(a) (b)

Figure 2. Two traveling wave-like solutions with variable amplitude $\hat{u}(t,x) = g(t)(c_1 \exp(x - ct) + c_2 \exp(-x + ct))$ with $c = 0.01, c_1 = c_2 = 0.05$, and $g(t) = t + \sin^2 t$ in (a), $g(t) = 0.5 + 0.15 \sin(2t)$ in (b).



(a) (b)

Figure 3. Two traveling wave-like solutions with variable velocity $\hat{u}(t,x) = g(t)(c_1 \exp(x - c(t)t) + c_2 \exp(-x + c(t)t))$ with $c_1 = c_2 = 0.05, c(t) = 0.001 + \sin^2 t$, and $g(t) = 1$ in (a), $g(t) = 0.2 + 0.05 \sin t$ in (b).



(a) (b)

Figure 4. Two traveling wave-like solutions with variable amplitude and variable velocity $\hat{u}(t,x) = \frac{h(t)}{\sqrt{2\gamma t+\alpha}}(c_1 \exp(x - \frac{\beta\ln(2\gamma t+\alpha)}{2\gamma}) + c_2 \exp(-x + \frac{\beta\ln(2\gamma t+\alpha)}{2\gamma}))$ with $c_1 = 0.05$, $c_2 = 0.2$, $\alpha = 2$, $\beta = 1$, $\gamma = 1$, and $h(t) = 1 + \sin^2 t$ in (a), $h(t) = \exp(\frac{t}{10})$ in (b).

4. Conservation laws

In this section, we consider nonlocal conservation laws to Qiao equation.

According to Theorem 2.1, we have the formulae of the nonlocal conserved vector to the aforementioned infinitesimal symmetry of Eq. (1.1):

$$\begin{aligned} C^x =& \xi^x L + W\left[\frac{\partial L}{\partial u_x} - D_x\left(\frac{\partial L}{\partial u_{xx}}\right) + D_x D_x\left(\frac{\partial L}{\partial u_{xxx}}\right) + D_x D_t\left(\frac{\partial L}{\partial u_{xxt}}\right)\right. \\ &\left. + D_t D_x\left(\frac{\partial L}{\partial u_{xtx}}\right)\right] + D_x(W)\left[\frac{\partial L}{\partial u_{xx}} - D_x\left(\frac{\partial L}{\partial u_{xxx}}\right) - D_t\left(\frac{\partial L}{\partial u_{xxt}}\right)\right] \\ &+ D_t(W)\left[-D_x\left(\frac{\partial L}{\partial u_{xtx}}\right)\right] + D_x D_x(W)\left(\frac{\partial L}{\partial u_{xxx}}\right) \\ &+ D_x D_t(W)\left(\frac{\partial L}{\partial u_{xxt}}\right) + D_t D_x(W)\left(\frac{\partial L}{\partial u_{xtx}}\right), \end{aligned} \tag{4.1}$$

$$\begin{aligned} C^t =& \xi^t L + W\left[\frac{\partial L}{\partial u_t} + D_x D_x\left(\frac{\partial L}{\partial u_{xxt}}\right)\right] + D_x(W)\left[-D_x\left(\frac{\partial L}{\partial u_{txx}}\right)\right] \\ &+ D_x D_x(W)\left(\frac{\partial L}{\partial u_{txx}}\right), \end{aligned} \tag{4.2}$$

where $W = \eta - \xi^j u_j$, and ξ^j refers to the coefficients of infinitesimal symmetry:

$$X = \xi^x \frac{\partial}{\partial x} + \xi^t \frac{\partial}{\partial t} + \eta^u \frac{\partial}{\partial u}. \tag{4.3}$$

For Eq. (1.1), the Lagrangian is

$$L = vF = v(u_t - u_{xxt} + 3u^2 u_x - u_x^3 + (2u_{xx} - 4u)u_x u_{xx} + (u_x^2 - u^2)u_{xxx}), \tag{4.4}$$

and the adjoint equation has the form

$$\begin{aligned} F^* = \frac{\delta(L)}{\delta u} =& - v_t + v_{xxt} + v_x u_x^2 - 3v_x u^2 - 2v_{xx}u_x u_{xx} \\ &+ 2v_{xx}uu_x + 2v_x uu_{xx} - v_{xxx}u_x^2 + v_{xxx}u^2, \end{aligned} \tag{4.5}$$

where v is a new dependent variable and the Euler-Lagrange operator is defined by the formal sum

$$\begin{aligned}\frac{\delta}{\delta u}=&\frac{\partial}{\partial u}-D_x\frac{\partial}{\partial u_x}-D_t\frac{\partial}{\partial u_t}+D_x^2\frac{\partial}{\partial u_{xx}}+D_xD_t\frac{\partial}{\partial u_{xt}}+D_t^2\frac{\partial}{\partial u_{tt}}\\&+-D_x^3\frac{\partial}{\partial u_{xxx}}-D_x^2D_t\frac{\partial}{\partial u_{xxt}}-D_xD_t^2\frac{\partial}{\partial u_{xtt}}-D_t^3\frac{\partial}{\partial u_{ttt}}+\cdots.\end{aligned}\tag{4.6}$$

Substitute $v=u$ into Eq. (4.5), it is easy to verify that Eq. (1.1) is self-adjoint.

Next, we calculate the conservation laws of Eq. (1.1). It is noted that the Lagrangian should be rewritten as following in calculations:

$$\begin{aligned}L=vF=&v(u_t-\frac{1}{3}u_{xxt}-\frac{1}{3}u_{xtx}-\frac{1}{3}u_{txx}+3u^2u_x-u_x^3\\&+(2u_{xx}-4u)u_xu_{xx}+(u_x^2-u^2)u_{xxx}).\end{aligned}\tag{4.7}$$

There are three cases to consider.

Case (I). For the generator $X_1=\frac{\partial}{\partial t}$, we have

$$\xi^x=0,\quad \xi^t=1,\quad \eta=0,\quad W=\eta-\xi^xu_x-\xi^tu_t=-u_t.\tag{4.8}$$

From (4.1) and (4.2), we can obtain the nonlocal conserved vector of (1.1):

$$\begin{aligned}C_1^x=&-u_t\left(3vu^2-vu_x^2+v_{xx}u_x^2-v_{xx}u^2-2vuu_{xx}-\frac{2}{3}v_{xt}\right)\\&-u_{xt}\left(2vu_xu_{xx}-2vuu_x-v_xu_x^2+v_xu^2+\frac{1}{3}v_t\right)\\&-\frac{1}{3}v_xu_{tt}-u_{xxt}(vu_x^2-vu^2)+\frac{2}{3}vu_{xtt},\end{aligned}\tag{4.9}$$

$$\begin{aligned}C_1^t=&vu_{xxx}u_x^2-vu_{xxx}u^2-vu_x^3+2vu_xu_{xx}^2-4vuu_xu_{xx}\\&+3vu_xu^2+\frac{1}{3}v_{xx}u_t-\frac{1}{3}v_xu_{xt}-\frac{2}{3}vu_{xxt}.\end{aligned}\tag{4.10}$$

Substituting $v=u$ into Eqs. (4.9) and (4.10), and transferring the terms of the conserved vector according to Remark 2.1 in Sec. 2, we obtain

$$\begin{aligned}C_1^x&=-3u^3u_t+u^2u_xu_{xt}+uu_x^2u_t+u_x^3u_{xt}\\&\quad+D_t\left(-u_{xx}uu_x^2+u^3u_{xx}+\frac{2}{3}uu_{xt}-\frac{1}{3}u_tu_x\right)\\&=\tilde{C_1^x}+D_t(h_1),\end{aligned}\tag{4.11}$$

where

$$\begin{aligned}\tilde{C_1^x}&=-3u^3u_t+u^2u_xu_{xt}+uu_tu_x^2+u_x^3u_{xt},\\h_1&=-u_{xx}uu_x^2+u^3u_{xx}+\frac{2}{3}uu_{xt}-\frac{1}{3}u_tu_x.\end{aligned}\tag{4.12}$$

So we have

$$\tilde{C_1^t}=C_1^t+D_x(h_1)=-u_x^3u-u^2u_xu_{xx}+3u^3u_x-u_{xx}u_x^3.\tag{4.13}$$

It is easy to verify that

$$D_x(\tilde{C_1^x}) + D_t(\tilde{C_1^t}) = 0. \tag{4.14}$$

Therefore, (4.14) provides a nonlocal conservation law of Eq. (1.1).

Case (II). For the generator $X_2 = \frac{\partial}{\partial x}$, we have

$$\xi^x = 1, \quad \xi^t = 0, \quad \eta = 0, \quad W = \eta - \xi^x u_x - \xi^t u_t = -u_x. \tag{4.15}$$

From (4.1) and (4.2), we can obtain the nonlocal conserved vector of (1.1):

$$\begin{aligned} C_2^x =& - v_{xx}u_x^3 + v_x u_x^2 u_{xx} + v_{xx}u_x u^2 - v_x u_{xx} u^2 + v u_t \\ &+ \frac{2}{3}v_{xt}u_x - \frac{1}{3}v_t u_{xx} - \frac{1}{3}v_x u_{xt} - \frac{1}{3}v u_{xxt}, \end{aligned} \tag{4.16}$$

$$C_2^t = -v u_x + \frac{1}{3}v_{xx}u_x - \frac{1}{3}v_x u_{xx} + \frac{1}{3}v u_{xxx}. \tag{4.17}$$

Substituting $v = u$ into Eqs. (4.16) and (4.17), and transferring the terms of the conserved vector according to Remark 2.1 in Sec. 2, we obtain

$$C_2^x = \frac{1}{3}u_x u_{xt} + D_t\left(\frac{1}{2}u^2 - \frac{1}{3}u u_{xx}\right) = \tilde{C_2^x} + D_t(h_2), \tag{4.18}$$

where

$$\tilde{C_2^x} = \frac{1}{3}u_x u_{xt}, \quad h_2 = \frac{1}{2}u^2 - \frac{1}{3}u u_{xx}. \tag{4.19}$$

So we have

$$\tilde{C_2^t} = C_2^t + D_x(h_2) = -u_x u + \frac{1}{3}u_{xxx}u + D_x\left(\frac{1}{2}u^2 - \frac{1}{3}u u_{xx}\right) = -\frac{1}{3}u_x u_{xx}. \tag{4.20}$$

It is easy to verify that

$$D_x(\tilde{C_2^x}) + D_t(\tilde{C_2^t}) = 0. \tag{4.21}$$

Therefore, (4.21) provides a nonlocal conservation law of Eq. (1.1).

Case (III). For the generator $X_3 = 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}$, we have

$$\xi^1 = 0, \quad \xi^2 = 2t, \quad \eta = -u, \quad W = \eta - \xi^1 u_x - \xi^2 u_t = -u - 2tu_t. \tag{4.22}$$

From (4.1) and (4.2), we can obtain the nonlocal conserved vector of (1.1):

$$\begin{aligned} C_3^x =&(-u - 2tu_t)(3vu^2 - vu_x^2 + v_{xx}u_x^2 - v_{xx}u^2 - 2vuu_{xx} - \frac{2}{3}v_{xt}) \\ &- (u_x + 2tu_{xt})(2vu_x u_{xx} - 2vuu_x - v_x u_x^2 + v_x u^2 + \frac{1}{3}v_t) \\ &- \frac{1}{3}(3u_t + 2tu_{tt})v_x - (u_{xx} + 2tu_{xxt})(vu_x^2 - vu^2) + \frac{2}{3}(3u_{xt} + 2tu_{xtt})v, \end{aligned} \tag{4.23}$$

$$\begin{aligned} C_3^t =&2tv(u_t - u_{xxt} + 3u^2u_x - u_x^3 + (2u_{xx} - 4u)u_x u_{xx} + (u_x^2 - u^2)u_{xxx}) \\ &- (u + 2tu_t)\left(v - \frac{1}{3}v_{xx}\right) - \frac{1}{3}(u_x + 2tu_{xt})v_x + \frac{1}{3}(u_{xx} + 2tu_{xxt})v. \end{aligned} \tag{4.24}$$

Substituting $v = u$ into Eqs. (4.23) and (4.24), and transferring the terms of the conserved vector according to Remark 2.1 in Sec. 2, we obtain

$$C_3^x = u^2u_x^2 - 2uu_{xx}u_x^2 + 2u^3u_{xx} - \frac{3}{2}u^4 + \frac{1}{2}u_x^4 - 2u_t u_x$$

$$+D_t\left(tu^2u_x^2 - 2tuu_{xx}u_x^2 + 2tu^3u_{xx} - \frac{3}{2}tu^4 + \frac{1}{2}tu_x^4 \right.$$
$$\left. -\frac{2}{3}tu_tu_x + \frac{4}{3}uu_x + \frac{4}{3}tuu_{xt}\right) = \tilde{C_3^x} + D_t(h_3), \tag{4.25}$$

where

$$\tilde{C_3^x} = u^2u_x^2 - 2uu_{xx}u_x^2 + 2u^3u_{xx} - \frac{3}{2}u^4 + \frac{1}{2}u_x^4 - 2u_tu_x,$$
$$h_3 = tu^2u_x^2 - 2tuu_{xx}u_x^2 + 2tu^3u_{xx} - \frac{3}{2}tu^4 + \frac{1}{2}tu_x^4 - \frac{2}{3}tu_tu_x + \frac{4}{3}uu_x + \frac{4}{3}tuu_{xt}. \tag{4.26}$$

So we have

$$\tilde{C_3^t} = C_3^t + D_x(h_3) = -u^2 + 2uu_{xx} + u_x^2. \tag{4.27}$$

Under the condition that

$$u_t = u_{xxt} - 3u^2u_x + u_x^3 + 4uu_xu_{xx} - 2u_{xx}^2u_x - u_x^2u_{xxx} + u^2u_{xxx},$$

it is easy to verify that

$$D_x(\tilde{C_3^x}) + D_t(\tilde{C_3^t}) = 0. \tag{4.28}$$

Therefore, (4.28) provides a nonlocal conservation law of Eq. (1.1).

By above analysis and computation, we get the following result.

Theorem 4.1. *(i) For the generator $X_1 = \frac{\partial}{\partial t}$, Qiao equation has a nonlocal conversation law (4.14) with conversation vector $(\tilde{C_1^x}, \tilde{C_1^t})$.*

(ii) For the generator $X_2 = \frac{\partial}{\partial x}$, Qiao equation has a nonlocal conversation law (4.21) with conversation vector $(\tilde{C_2^x}, \tilde{C_2^t})$.

(iii) For the generator $X_3 = 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}$, Qiao equation has a nonlocal conversation law (4.28) with conversation vector $(\tilde{C_3^x}, \tilde{C_3^t})$.

5. Conclusions

In this paper, we use the infinitesimal symmetry to reduce Qiao equation. A few of new exact group-invariant solutions are obtained by solving the reduced equations. The group-invariant solutions are different and more succinct than those obtained in previous literature. A novel technique is also proposed for constructing group-invariant solutions and non-group-invariant solutions based on travelling wave solutions. The obtained non-group-invariant solutions can be seen as traveling wave-like solutions with variable-amplitude, variable-velocity or both. Some figures are given to illustrate the distinction of the obtained solutions by choosing some exact expressions. Nonlocal conservation vectors of Qiao equation are also obtained with the corresponding infinitesimal generators.

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