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DYNAMICAL PROPERTIES OF A STOCHASTIC PREDATOR-PREY MODEL WITH FUNCTIONAL RESPONSE

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Abstract A stochastic prey-predator model with functional response is investigated in this paper. A complete threshold analysis of coexistence and extinction is obtained. Moreover, we point out that the stochastic predator-prey model undergoes a stochastic Hopf bifurcation from the viewpoint of numerical simulations. Some numerical simulations are carried out to support our results.

Keywords Strong stochastic persistence, stochastic Hopf bifurcation, extinction, Crowley-Martin functional response.

MSC(2010) 60H10, 37A99, 65P30.

1. Introduction

Predator-prey dynamics is one of the dominant fields in both theoretical and applied ecology, which has encouraged numerous researchers to develop various mathematical models to better understand it over the last few decades [2,22,25]. In population dynamics, the functional response is one of the nonlinear components in biological systems, which describes the feeding rate of prey consumption by predators, and plays a key role in understanding the dynamical complexity of the systems [16,18].

In fact, there are many works based on functional responses, see [1,5,6,11,18,33] and the references cited therein. Especially, *Crowley, Martin* [5] introduced the Crowley-Martin functional response: $p(x,y) = \frac{fx(t)y(t)}{1+\alpha_1 x(t)+\alpha_2 y(t)+\alpha_3 x(t)y(t)}$. It is a modified form of the Holling type and Beddington-DeAngelis functional responses. Obviously, the Crowley-Martin functional response becomes a Holling type I functional response if $\alpha_1 = \alpha_2 = \alpha_3 = 0$, the functional response is simplified to a Holling type II functional response when $\alpha_2 = \alpha_3 = 0$, and it is a Beddington-DeAngelis functional response when $\alpha_3 = 0$.

A predator-prey model with the Crowley-Martin functional response is described as follows:

$$\begin{cases} \frac{dx}{dt} = \left(r - ax - \frac{\omega y}{1+\alpha_1 x+\alpha_2 y+\alpha_3 xy}\right) x, \\ \frac{dy}{dt} = \left(c - by + \frac{fx}{1+\alpha_1 x+\alpha_2 y+\alpha_3 xy}\right) y, \end{cases} \tag{1.1}$$

where x, y designate the population densities of prey and predator. The parameters r, a, b, ω, f are positive constants and $\alpha_1, \alpha_2, \alpha_3$ are non-negative constants, r is the

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growth rate of prey, c represents the growth rate of predator when it's positive and the death rate when it's negative. f stands for the conversion rate of nutrients into predator production, while a, b measure the competition strength among individuals of prey and predator respectively. In recent years, there were some relevant predator-prey models with this type of functional response [3, 20, 24, 26, 32].

As a matter of fact, environmental noises play an inevitable role in population dynamics and always contribute to random fluctuations on parameters appearing in ecosystems [9, 10, 21]. Therefore, we take the influence of randomly fluctuating environment into account. After incorporating white noise into the system (1.1), we consider the following stochastic system:

$$\begin{cases} dx = \left(r - ax - \frac{\omega y}{1+\alpha_1 x+\alpha_2 y+\alpha_3 xy}\right) xdt + \sigma_1 x dB_1(t),\ x(0) > 0, \\ dy = \left(c - by + \frac{fx}{1+\alpha_1 x+\alpha_2 y+\alpha_3 xy}\right) ydt + \sigma_2 y dB_2(t),\ y(0) > 0, \end{cases} \tag{1.2}$$

where $B_1(t)$, $B_2(t)$ are mutually independent *Brownian* motions, σ_1^2 and σ_2^2 represent the intensities of white noise.

As this kind of stochastic model accommodates interference among predators and preys and is a better fit to the experimental data, we believe it deserves further attention. Some literatures used the corresponding stochastic model to describe the dynamic properties [18, 19, 27, 28]. *Liu* et al [18] studied stochastic boundedness, stochastic permanence and extinction for a corresponding stochastic system with Crowley-Martin functional response. *Zhang* et al [28] showed the existence, boundedness and uniform continuity of the positive solution for a stochastic population system with this kind of functional response.

The threshold analysis of strong stochastic persistence and extinction is given for some stochastic population models [29–31]. However, to the best of our knowledge, literatures on the threshold analysis of coexistence and extinction, stochastic Hopf bifurcation for the stochastic predator-prey system (1.2) have not yet appeared. The Crowley-Martin functional response is a generalization of Holling type and Beddington-DeAngelis functional responses. And the parameter c may be positive or negative. If $c > 0$, the species y has extra source of food except x, however, if $c < 0$, the species y has no extra source of food except x. Both the two cases are considered in this paper. The aim of this paper is to investigate these issues for the system (1.2).

In Section 2, we obtain a complete threshold analysis of coexistence and extinction. Section 3 considers stochastic Hopf bifurcation of the stochastic predator-prey model (1.2) from the viewpoint of numerical simulations. A final discussion concludes the paper in Section 4.

2. Coexistence and extinction

2.1. Threshold analysis of persistence and extinction

Motivated by [12, 13], we will consider strong stochastic persistence and extinction of the stochastic system (1.2). To characterize these properties, we introduce the notation

$$\tilde{\Pi}_t(\cdot) := \frac{1}{t}\int_0^t \mathbf{1}_{\{(x(s),y(s))\in\cdot\}} ds, t > 0,$$

Table 1. The threshold analysis of coexistence and extinction of the solution when $c < 0$

$\lambda_1(\delta^*)$	$\mu_x(\cdot)$	$\lambda_2(\mu_x)$	distributions
< 0	$\nexists$	-	$\tilde{\Pi}_t(\cdot) \to \delta^*$
> 0	$\exists$	< 0	$\tilde{\Pi}_t(\cdot) \to \mu_x(\cdot)$
		> 0	$\tilde{\Pi}_t(\cdot) \to \pi(\cdot)$

to denote a random normalized occupation measure [12].

For the following equation

$$dx = x\left(a_1 - b_1 x\right)dt + \sigma_1 x dB_1(t). \tag{2.1}$$

For the system (2.1), [8] implies that the system (2.1) has a unique stationary distribution $\mu_x(\cdot)$ with the density function

$$\rho^*(x) = \frac{A^q x^{q-1} e^{-Ax}}{\Gamma(q)},\ x > 0,\ A = \frac{2b_1}{\sigma_1^2} > 0,\ q = \frac{2a_1}{\sigma_1^2} - 1 > 0.$$

It's easy to verify that Assumptions 1.1 and 1.4 presented in [12] are satisfied if we choose $c_1 = 1$, $c_2 = \frac{\omega}{f}$. Besides, we can obtain that $\lambda_1(\delta^*) = r - \frac{\sigma_1^2}{2}$, $\lambda_2(\delta^*) = c - \frac{\sigma_2^2}{2}$.

Furthermore, In view of Theorems 1.1, 1.2, 1.3 in [12], we discuss the following cases.

Case A: If $c < 0$, i.e. y has no extra source of food except x, then $\lambda_2(\delta^*) < 0$. Under this assumption, we have the results as follows.

(A1). If $\lambda_1(\delta^*) = r - \frac{\sigma_1^2}{2} < 0$, then the random normalized occupation measure $\tilde{\Pi}_t(\cdot)$ converges to δ^* for any initial value $(x_0, y_0) \in R_+^2$ almost surely, which implies $x(t)$ converges to 0 and $y(t)$ converges to 0 almost surely.

If $\lambda_1(\delta^*) = r - \frac{\sigma_1^2}{2} > 0$, there exists a unique invariant probability measure $\mu_x(\cdot)$ on $R_1^+ = \{(x,0), x > 0\}$, such that

$$\lambda_1(\mu_x) = \int_0^{+\infty} (r - \frac{\sigma_1^2}{2} - ax)\mu_x(dx) = r - \frac{\sigma_1^2}{2} - a\int_0^{+\infty} x\mu_x(dx) = 0,$$

that is, $\int_0^{+\infty} x\mu_x(dx) = \frac{1}{a}(r - \frac{\sigma_1^2}{2})$. Hence

$$\lambda_2(\mu_x) = \int_0^{+\infty} (c - \frac{\sigma_2^2}{2} + \frac{fx}{1+\alpha_1 x})\mu_x(dx) = c - \frac{\sigma_2^2}{2} + \int_0^{+\infty} \frac{fx}{1+\alpha_1 x}\mu_x(dx).$$

(A2). If $\lambda_2(\mu_x) < 0$, then $\tilde{\Pi}_t(\cdot)$ converges weakly to $\mu_x(\cdot)$ for any initial value $(x_0, y_0) \in R_+^2$ almost surely, which implies $y(t)$ converges to 0 almost surely.

(A3). If $\lambda_2(\mu_x) > 0$, then there exists a uniquely ergodic stationary distribution $\pi(\cdot)$ in the interior of the first quadrant.

To illustrate the asymptotic behaviors of the sample paths of the solution discussed above clearly, we show them in Table 1.

Case B: If $c > 0$, i.e. y has extra source of food besides x, situations under this condition are more complicated. Recall that $\lambda_1(\delta^*) = r - \frac{\sigma_1^2}{2}$, $\lambda_2(\delta^*) = c - \frac{\sigma_2^2}{2}$, we have the properties as follows.

(B1). If $\lambda_1(\delta^*) < 0$, $\lambda_2(\delta^*) < 0$, then the random normalized occupation measure $\tilde{\Pi}_t(\cdot)$ converges to δ^* for any initial value $(x_0, y_0) \in R_+^2$ almost surely, which implies $x(t)$ and $y(t)$ both converge to 0 almost surely.

(B2). If $\lambda_1(\delta^*) > 0$, $\lambda_2(\delta^*) < 0$, $\mu_x(\cdot)$ exists, then the discussions are similar to those appearing in (A2) and (A3) of **Case A**.

(B3). If $\lambda_1(\delta^*) < 0$, $\lambda_2(\delta^*) > 0$, there is a unique invariant probability measure $\mu_y(\cdot)$ on $R_2^+ = \{(0, y), y > 0\}$, such that

$$\lambda_2(\mu_y) = c - \frac{\sigma_2^2}{2} - b\int_0^{+\infty} y\mu_y(dy) = 0,$$

then

$$\int_0^{+\infty} y\mu_y(dy) = \frac{1}{b}(c - \frac{\sigma_2^2}{2}).$$

In addition,

$$\lambda_1(\mu_y) = r - \frac{\sigma_1^2}{2} - \int_0^{+\infty} \frac{\omega y}{1 + \alpha_2 y}\mu_y(dy).$$

Under the above assumptions, it follows from $\lambda_1(\delta^*) = r - \frac{\sigma_1^2}{2} < 0$ that $\lambda_1(\mu_y) < 0$, then the random normalized occupation measure $\tilde{\Pi}_t(\cdot)$ converges weakly to $\mu_y(\cdot)$ for any initial value $(x_0, y_0) \in R_+^2$ almost surely, and $x(t)$ converges to 0.

(B4). If $\lambda_1(\delta^*) > 0$, $\lambda_2(\delta^*) > 0$, both $\mu_x(\cdot)$ and $\mu_y(\cdot)$ exist, obviously, we have $\lambda_1(\mu_x) = 0, \lambda_2(\mu_y) = 0$. Under the assumption $\lambda_2(\delta^*) = c - \frac{\sigma_2^2}{2} > 0$, we obtain that

$$\lambda_2(\mu_x) = c - \frac{\sigma_2^2}{2} + \int_0^{+\infty} \frac{fx}{1 + \alpha_1 x}\mu_x(dx) > 0.$$

We only need to discuss signs of $\lambda_1(\mu_y)$.

If $\lambda_1(\mu_y) > 0$, then there exists a uniquely ergodic stationary distribution $\pi(\cdot)$ in the interior of the first quadrant.

If $\lambda_1(\mu_y) < 0$, then $\tilde{\Pi}_t(\cdot)$ converges to $\mu_y(\cdot)$ for any initial value $(x_0, y_0) \in R_+^2$ almost surely, and $x(t)$ converges to 0 almost surely.

Similarly, we show the discussions in Table 2 (Blue parts stand for those who have been deduced by the previous conditions).

Remark 2.1. It is necessary to point out that there exist mistakes in our proofs of Lemma 3 $\Big[$Xiaoling Zou, Jingliang Lv, A new idea on almost sure permanence and uniform boundedness for a stochastic predator-prey model, Journal of the Franklin Institute, 354 (2017) 6119-6137$\Big]^{[J1]}$ and Lemma 4.1.2 $\Big[$Jingliang Lv, Xiaoling Zou, Luhua Tian, A geometric method for asymptotic properties of the stochastic Lotka-Volterra model, Communications in Nonlinear Science and Numerical Simulation, 67 (2019) 449-459$\Big]^{[J2]}$. This paper considers the strong stochastic persistence of the system (1.2) and the complete threshold analysis of coexistence and extinction. The method used in this paper improves the method of the references [J1],[J2]. For the detailed revisions of the stochastic Lotka-Volterra model [J2], the readers may refer to the reference [12]. And we give the subsequent corrections in our next work for the revisions of the stochastic predator-prey model with response function [J1].

Table 2. The threshold analysis of coexistence and extinction of the solution when $c > 0$

$\lambda_1(\delta^*)$	$\lambda_2(\delta^*)$	$\mu_x(\cdot)$	$\mu_y(\cdot)$	$\lambda_2(\mu_x)$	$\lambda_1(\mu_y)$	distributions
< 0	< 0	$\nexists$	$\nexists$	-	-	$\tilde{\Pi}_t(\cdot) \to \delta^*$
> 0	< 0	$\exists$	$\nexists$	< 0	-	$\tilde{\Pi}_t(\cdot) \to \mu_x(\cdot)$
				> 0	-	$\tilde{\Pi}_t(\cdot) \to \pi(\cdot)$
< 0	> 0	$\nexists$	$\exists$	-	< 0	$\tilde{\Pi}_t(\cdot) \to \mu_y(\cdot)$
> 0	> 0	$\exists$	$\exists$	> 0	< 0	$\tilde{\Pi}_t(\cdot) \to \mu_y(\cdot)$
				> 0	> 0	$\tilde{\Pi}_t(\cdot) \to \pi(\cdot)$

2.2. Simulations of persistence and extinction

Three examples are introduced to illustrate Table 1:

Example 2.1. We choose $r = 0.2, a = 0.9, \omega = 0.9, \alpha_1 = 0.04, \alpha_2 = 0.5, \alpha_3 = 0.001, c = -0.4, f = 0.4, \sigma_1 = 0.9, \sigma_2 = 0.9, b = 0.4, Tmax = 100$, we deduce that $\lambda_1(\delta^*) = r - \frac{\sigma_1^2}{2} = -0.205 < 0$, $\lambda_2(\delta^*) = c - \frac{\sigma_2^2}{2} = -0.805 < 0$, then $\tilde{\Pi}_t(\cdot) \to \delta^*$. Figure 1 shows that both x and y go extinct.

Example 2.2. Let $r = 0.5, a = 0.9, \omega = 0.9, \alpha_1 = 0.04, \alpha_2 = 0.5, \alpha_3 = 0.05, c = -0.9, f = 0.4, \sigma_1 = 0.1, \sigma_2 = 0.8, b = 0.4, Tmax = 500$, we have $A = \frac{2a}{\sigma_1^2} = 18$, $q = \frac{2r}{\sigma_1^2} - 1 = 99$. we imply that $\lambda_1(\delta^*) = r - \frac{\sigma_1^2}{2} = 0.495 > 0$, $\lambda_2(\delta^*) = c - \frac{\sigma_2^2}{2} = -1.22 < 0$,

$$\begin{aligned}\lambda_2(\mu_x) &= c - \frac{\sigma_2^2}{2} + \int_0^{+\infty} \frac{fx}{1+\alpha_1 x}\mu_x(dx)\\ &= -0.9 - \frac{0.8^2}{2} + \int_0^{+\infty} \frac{0.4x}{1+0.04x}\frac{18^{99}x^{98}e^{-18x}}{\Gamma(99)}dx.\end{aligned}$$

Mathematical software can compute $\lambda_2(\mu_x) < 0$, thus $\tilde{\Pi}_t(\cdot) \to \mu_x(\cdot)$. Figure 2 stands for that x is persistent and y go extinct.

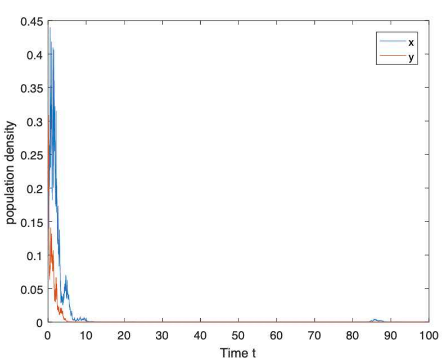

Figure 1. both x and y go extinct

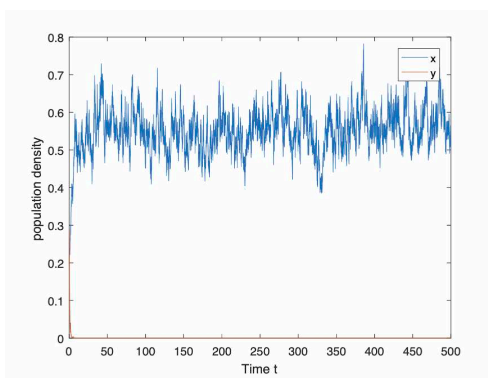

Figure 2. x is persistent and y goes extinct

Example 2.3. Let $r = 0.7, a = 0.4, \omega = 0.7, \alpha_1 = 0.04, \alpha_2 = 0.5, \alpha_3 = 0.05, c = -0.5, f = 0.9, \sigma_1 = 0.4, \sigma_2 = 0.04, b = 0.02, Tmax = 500$, then $A = \frac{2a}{\sigma_1^2} = 5,\ q = \frac{2r}{\sigma_1^2} - 1 = 7.75$. And we obtain that

$$\begin{aligned}\lambda_1(\delta^*) &= r - \frac{\sigma_1^2}{2} = 0.54 > 0,\ \lambda_2(\delta^*) = c - \frac{\sigma_2^2}{2} = -0.5016 < 0,\\ \lambda_2(\mu_x) &= c - \frac{\sigma_2^2}{2} + \int_0^{+\infty} \frac{fx}{1+\alpha_1 x}\mu_x(dx)\\ &= -0.5 - \frac{0.04^2}{2} + \int_0^{+\infty} \frac{0.9x}{1+0.04x}\frac{5^{7.75}x^{6.75}e^{-5x}}{\Gamma(7.75)}dx.\end{aligned}$$

By virtue of mathematical software, we can compute that $\lambda_2(\mu_x) > 0$, hence $\tilde{\Pi}_t(\cdot) \to \pi(\cdot)$. It is verified by Figure 3.

And six examples are listed to demonstrate Table 2:

Example 2.4. We choose $r = 0.3, a = 0.8, \omega = 0.7, \alpha_1 = 0.5, \alpha_2 = 0.05, \alpha_3 = 0.001, c = 0.2, f = 0.5, \sigma_1 = 0.9, \sigma_2 = 0.8, b = 0.8, Tmax = 100$, thus

$$\lambda_1(\delta^*) = r - \frac{\sigma_1^2}{2} = -0.105 < 0,\ \lambda_2(\delta^*) = c - \frac{\sigma_2^2}{2} = -0.12 < 0.$$

Under the conditions, both x and y go extinct, see Figure 4.

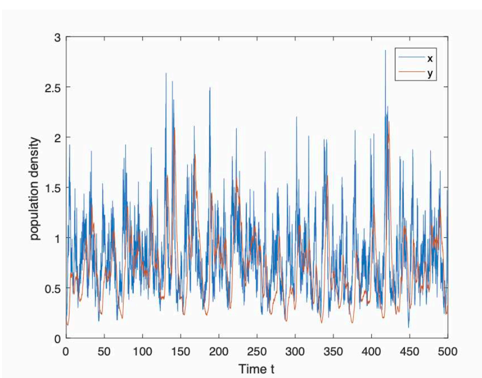

Figure 3. both x and y are persistent

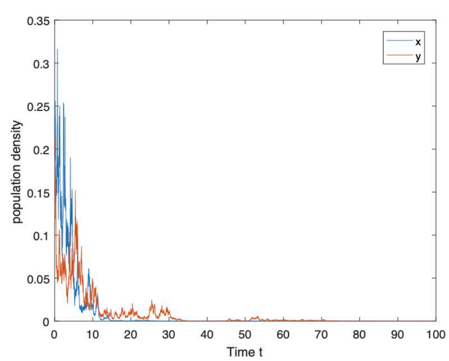

Figure 4. both x and y go extinct

Example 2.5. Let $r = 0.9, a = 0.9, \omega = 0.7, \alpha_1 = 0.4, \alpha_2 = 0.5, \alpha_3 = 0.05, c = 0.01, f = 0.4, \sigma_1 = 0.5, \sigma_2 = 0.9, b = 0.8, Tmax = 500$, we can compute

$$\lambda_1(\delta^*) = r - \frac{\sigma_1^2}{2} = 0.65 > 0,\ \lambda_2(\delta^*) = c - \frac{\sigma_2^2}{2} = -0.8 < 0.$$

Meanwhile,

$$\begin{aligned}\lambda_2(\mu_x) &= c - \frac{\sigma_2^2}{2} + \int_0^{+\infty} \frac{fx}{1+\alpha_1 x}\mu_x(dx)\\ &= 0.01 - \frac{0.9^2}{2} + \int_0^{+\infty} \frac{0.4x}{1+0.4x}\frac{7.2^{6.2}x^{5.2}e^{-7.2x}}{\Gamma(6.2)}dx.\end{aligned}$$

In view of mathematical software, we compute that $\lambda_2(\mu_x) < 0$, hence $\tilde{\Pi}_t(\cdot) \to \mu_x(\cdot)$. Figure 5 supports the result.

Example 2.6. Choose $r = 0.9, a = 0.9, \omega = 0.7, \alpha_1 = 0.04, \alpha_2 = 0.5, \alpha_3 = 0.05, c = 0.01, f = 0.5, \sigma_1 = 0.5, \sigma_2 = 0.5, b = 0.7, Tmax = 500$, we obtain that

$$\lambda_1(\delta^*) = r - \frac{\sigma_1^2}{2} = 0.775 > 0,\ \ \lambda_2(\delta^*) = c - \frac{\sigma_2^2}{2} = -0.115 < 0.$$

Meanwhile,

$$\begin{aligned}\lambda_2(\mu_x) &= c - \frac{\sigma_2^2}{2} + \int_0^{+\infty} \frac{fx}{1+\alpha_1 x}\mu_x(dx)\\ &= 0.01 - \frac{0.5^2}{2} + \int_0^{+\infty} \frac{0.5x}{1+0.04x}\frac{7.2^{6.2}x^{5.2}e^{-7.2x}}{\Gamma(6.2)}dx.\end{aligned}$$

Mathematical software can compute that $\lambda_2(\mu_x) > 0$, hence $\tilde{\Pi}_t(\cdot) \to \pi(\cdot)$. Figure 6 shows that both x and y are persistent.

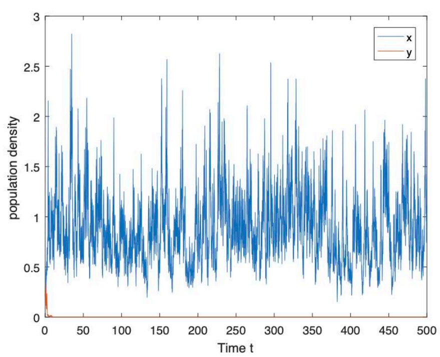

Figure 5. x is persistent and y goes extinct

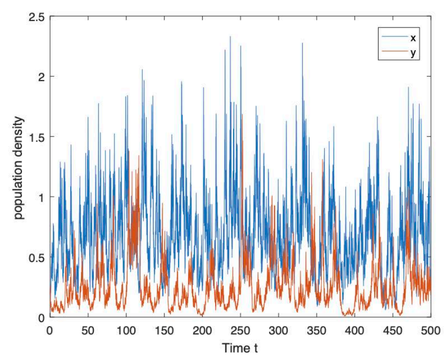

Figure 6. both x and y are persistent

Example 2.7. Let $r = 0.2, a = 0.7, \omega = 0.5, \alpha_1 = 2.1, \alpha_2 = 0.7, \alpha_3 = 0.1, c = 0.8, f = 0.9, \sigma_1 = 0.9, \sigma_2 = 0.7, b = 0.1, Tmax = 500$, we obtain that

$$\lambda_1(\delta^*) = r - \frac{\sigma_1^2}{2} = -0.205 < 0,\ \ \lambda_2(\delta^*) = c - \frac{\sigma_2^2}{2} = 0.555 > 0$$

and

$$\lambda_1(\mu_y) = r - \frac{\sigma_1^2}{2} - \int_0^{+\infty} \frac{\omega y}{1+\alpha_2 y}\mu_y(dy) = 0.2 - \frac{0.9^2}{2} - \int_0^{+\infty} \frac{0.5y}{1+0.7y}\mu_y(dy).$$

It is obvious that $\lambda_1(\mu_y) < 0$, hence $\tilde{\Pi}_t(\cdot) \to \mu_y(\cdot)$. We can see from Figure 7 that the prey x goes extinct, however the predator y is persistent. This support the point that y has extra source of food besides x.

Example 2.8. Choose $r = 0.5, a = 0.9, \omega = 0.9, \alpha_1 = 0.04, \alpha_2 = 0.1, \alpha_3 = 0.05, c = 0.9, f = 0.9, \sigma_1 = 0.5, \sigma_2 = 0.8, b = 0.8, Tmax = 500$, we get

$$\lambda_1(\delta^*) = r - \frac{\sigma_1^2}{2} = 0.375 > 0,\ \ \lambda_2(\delta^*) = c - \frac{\sigma_2^2}{2} = 0.58 > 0,$$

$$\lambda_2(\mu_x) = c - \frac{\sigma_2^2}{2} + \int_0^{+\infty} \frac{fx}{1+\alpha_1 x}\mu_x(dx) = 0.58 + \int_0^{+\infty} \frac{0.9x}{1+0.04x}\mu_x(dx) > 0.$$

And

$$\begin{aligned}\lambda_1(\mu_y) &= r - \frac{\sigma_1^2}{2} - \int_0^{+\infty} \frac{\omega y}{1+\alpha_2 y}\mu_y(dy) \\ &= 0.375 - \int_0^{+\infty} \frac{0.9y}{1+0.1y}\frac{2.5^{2.81}y^{1.81}e^{-2.5y}}{\Gamma(2.81)}dy.\end{aligned}$$

By mathematical software, we compute that $\lambda_1(\mu_y) < 0$, hence $\tilde{\Pi}_t(\cdot) \to \mu_y(\cdot)$. The prey x goes extinct, however the predator y is persistent, this is, y also has extra source of food besides x. See Figure 8.

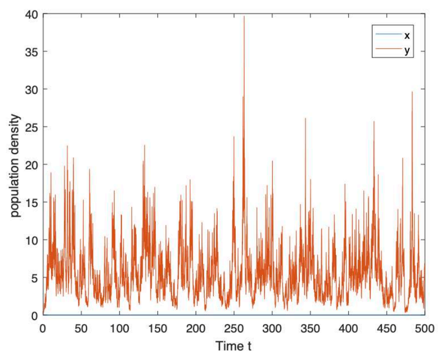

Figure 7. x goes extinct and y is persistent

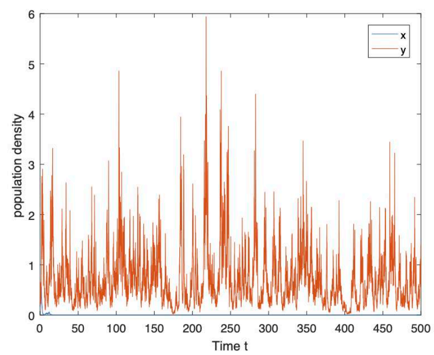

Figure 8. x goes extinct and y is persistent

Example 2.9. Let $r = 0.9, a = 0.9, \omega = 0.9, \alpha_1 = 0.04, \alpha_2 = 0.5, \alpha_3 = 0.05, c = 0.5, f = 0.7, \sigma_1 = 0.5, \sigma_2 = 0.8, b = 0.9, Tmax = 500$, we get

$$\lambda_1(\delta^*) = r - \frac{\sigma_1^2}{2} = 0.775 > 0, \;\; \lambda_2(\delta^*) = c - \frac{\sigma_2^2}{2} = 0.18 > 0,$$

$$\lambda_2(\mu_x) = c - \frac{\sigma_2^2}{2} + \int_0^{+\infty} \frac{fx}{1+\alpha_1 x}\mu_x(dx) = 0.18 + \int_0^{+\infty} \frac{0.7x}{1+0.04x}\mu_x(dx) > 0.$$

And

$$\begin{aligned}\lambda_1(\mu_y) &= r - \frac{\sigma_1^2}{2} - \int_0^{+\infty} \frac{\omega y}{1+\alpha_2 y}\mu_y(dy) \\ &= 0.775 - \int_0^{+\infty} \frac{0.9y}{1+0.1y}\frac{2.5^{0.56}y^{-0.44}e^{-2.5y}}{\Gamma(0.56)}dy.\end{aligned}$$

Mathematical software can compute that $\lambda_1(\mu_y) > 0$, hence $\tilde{\Pi}_t(\cdot) \to \pi(\cdot)$. Figure 9 shows that both x and y are persistent.

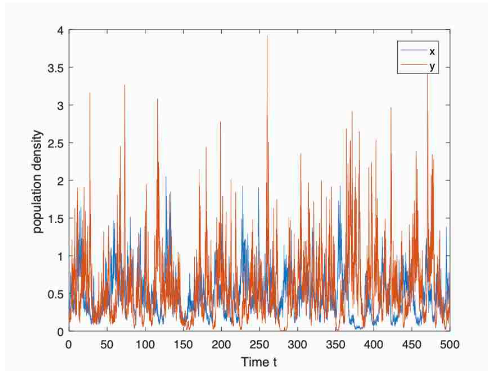

Figure 9. both x and y are persistent

3. Stochastic Hopf bifurcation

It follows from Section 2 that when t is sufficiently large, the statistical properties of sample paths can be used to replace spatial ones. Therefore, in this section, we will use numerical simulations of sample paths to study stochastic Hopf bifurcation of the system (1.2).

Now, we will use numerical simulations to illustrate that the stochastic predator-prey model can undergo a stochastic Hopf bifurcation phenomenon. Let $r = 1, a = 1, w = 10, \alpha_1 = 2.1, \alpha_2 = 1.1, \alpha_3 = 0.001, c = -0.4, f = 5, \sigma_1 = 0.05, \sigma_2 = 0.03, x(0) = 0.2, y(0) = 0.16$. Let $b = 0.5 \times 10^{-11}$ and $b = 2$ respectively, then the system (1.2) becomes the systems (3.1) and (3.2) respectively:

$$\begin{cases} dx = \left(1 - x - \frac{10y}{1+2.1x+1.1y+0.001xy}\right) xdt + 0.05xdB_1(t), \\ dy = \left(-0.4 - 0.5 \times 10^{-11}y + \frac{5x}{1+2.1x+1.1y+0.001xy}\right) ydt + 0.03ydB_2(t), \end{cases} \tag{3.1}$$

and

$$\begin{cases} dx = \left(1 - x - \frac{10y}{1+2.1x+1.1y+0.001xy}\right) xdt + 0.05xdB_1(t), \\ dy = \left(-0.4 - 2y + \frac{5x}{1+2.1x+1.1y+0.001xy}\right) ydt + 0.03ydB_2(t). \end{cases} \tag{3.2}$$

The deterministic system for the system (3.1) becomes

$$\begin{cases} dx = \left(1 - x - \frac{10y}{1+2.1x+1.1y+0.001xy}\right) xdt, \\ dy = \left(-0.4 - 0.5 \times 10^{-11}y + \frac{5x}{1+2.1x+1.1y+0.001xy}\right) ydt. \end{cases} \tag{3.3}$$

The deterministic system (3.3) exists a stable limit cycle according to the literature [26] (see Figures 10-12). Here, Figures 10-12 are given in comparison with the stochastic system (3.1).

The deterministic system for the system (3.2) becomes

$$\begin{cases} dx = \left(1 - x - \frac{10y}{1+2.1x+1.1y+0.001xy}\right) xdt, \\ dy = \left(-0.4 - 2y + \frac{5x}{1+2.1x+1.1y+0.001xy}\right) ydt. \end{cases} \tag{3.4}$$

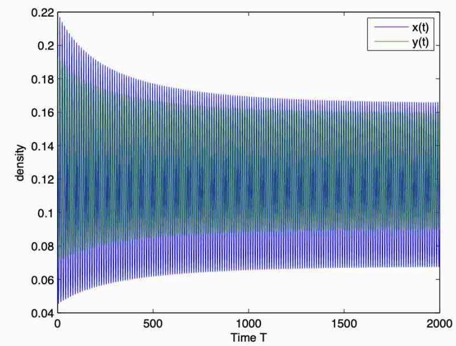

Figure 10. Periodic solution for the system (3.3).

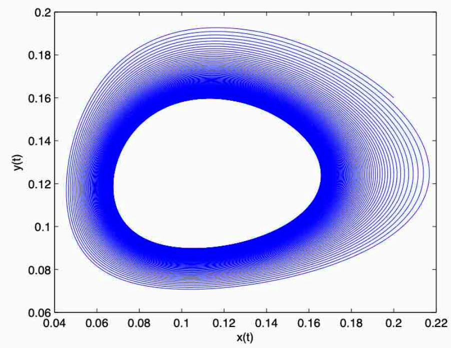

Figure 11. A stable limit cycle for the system (3.3) in phase space.

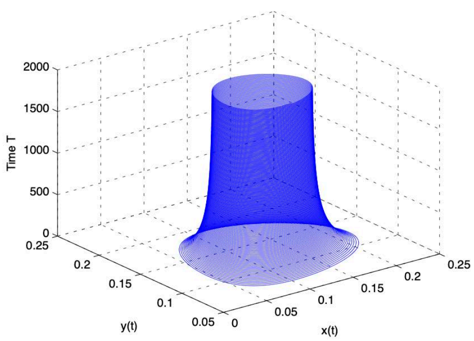

Figure 12. A stable limit cycle for the system (3.3) in three-dimensional space introducing time axis.

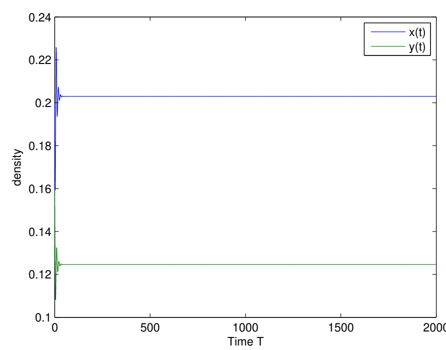

Figure 13. A stable positive equilibrium point for the system (3.4).

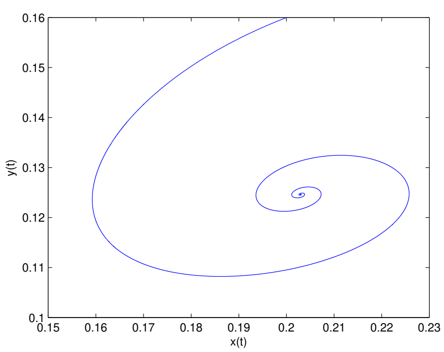

Figure 14. A stable positive equilibrium point for the system (3.4) in phase space.

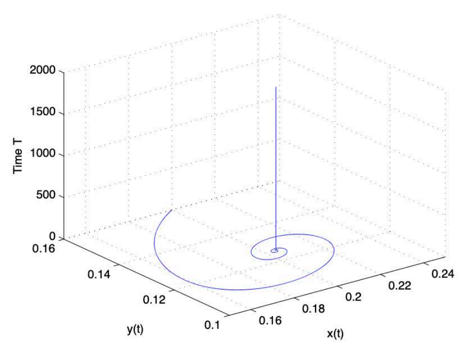

Figure 15. A stable positive equilibrium point for the system (3.4) in three-dimensional space introducing time axis.

The deterministic system (3.4) exists a stable positive equilibrium point according to the literature [26] (see Figures 13-15). Here, Figures 13-15 are given in comparison with stochastic system (3.2). It is observed that the deterministic system exists Hopf bifurcation phenomenon.

Figure 16 is the stationary distribution of the system (3.1) in the phase space. Figure 17 shows a stochastic limit cycle for the system (3.1) in three-dimensional space introducing time axis. Figure 18 implies that there is a crater-like stationary distribution for the stochastic system (3.1). Figure 19 is the stationary distribution of the system (3.2) in the phase space. Figure 20 shows the stochastic solution for the system (3.2) in three-dimensional space introducing time axis. Figure 21 implies that there is a peak-like stationary distribution for the stochastic system (3.2).

Now, from the viewpoint of numerical simulations, Figures 16-18 show the stochastic system (1.2) exists a crater-like stationary distribution, and Figures 19-21 show the stochastic system (1.2) exists a peak-like stationary distribution. Overall, the shapes of stationary distributions change from crater-like to peak-like. Therefore, the stochastic model (1.2) undergoes a stochastic Hopf-bifurcation phenomenon [4, 7, 14, 15, 17, 23, 34].

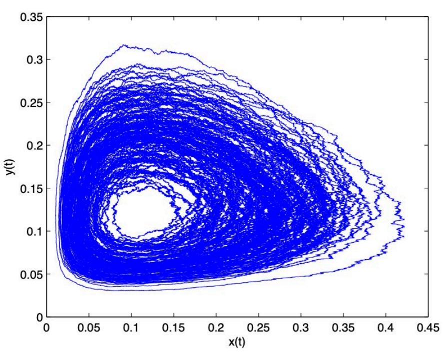

Figure 16. A stochastic limit cycle for the system (3.1) in phase space.

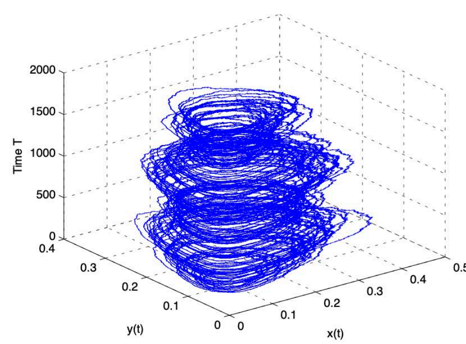

Figure 17. A stochastic limit cycle for the system (3.1) in three-dimensional space introducing time axis.

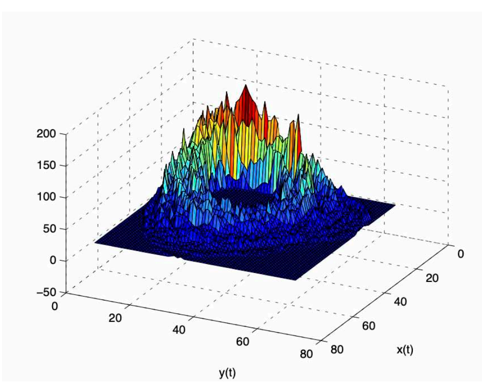

Figure 18. A crater-like stationary distribution for the system (3.1) in three-dimensional space.

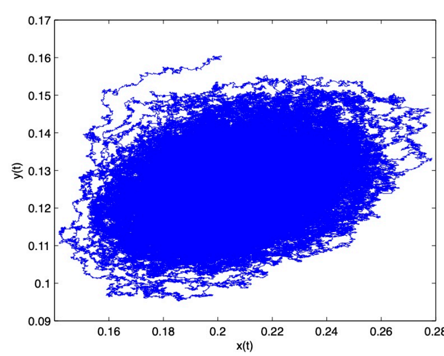

Figure 19. A stochastic solution process for the system (3.2) in phase space.

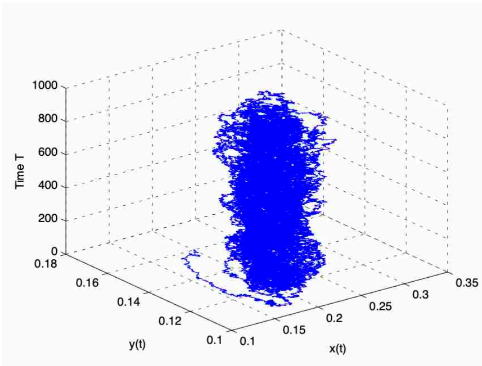

Figure 20. A stochastic solution process for the system (3.2) in three-dimensional space introducing time axis.

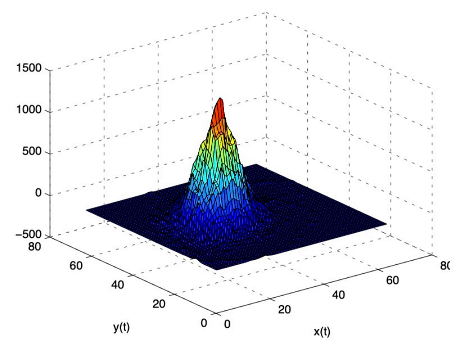

Figure 21. A peak-like stationary distribution for the system (3.2) in three-dimensional space.

4. Concluding remarks

Here, we consider a stochastic predator-prey model with Crowley-Martin functional response. The main results are as follows:

- We obtain the complete threshold analysis of coexistence and extinction of the stochastic system (1.2). Moreover, numerical simulations are introduced to support each conclusion in Table 1 and Table 2.
- From the perspective of numerical simulations, the stochastic model (1.2) exists peak-like stationary distribution and crater-like stationary distribution, that is, it undergoes a stochastic Hopf bifurcation.

Some interesting topics deserve further investigation. It will be interesting to study the stochastic high-order nonlinear systems. We will discuss these issues in the near future.

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