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PERIODICITY AND SOLUTIONS OF SOME RATIONAL DIFFERENCE EQUATIONS SYSTEMS

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Abstract In this paper we are interested in a technique for solving some nonlinear rational systems of difference equations of third order, in three-dimensional case. Moreover, we study the periodicity of solutions for such systems. Finally, some numerical examples are presented.

Keywords Difference equations, recursive sequences, periodic solutions, system of difference equations, stability.

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1. Introduction

The nature of many biological systems naturally leads to their study by means of a discrete variable. Particular examples include population dynamics and genetics. Some elementary models of biological phenomena, including a single species population model, harvesting of fish, the production of red blood cells, ventilation volume and blood CO_2 levels, a simple epidemics model and a model of waves of disease that can be analyzed by difference equations are shown in [28]. Recently, there has been interest in so-called dynamical diseases, which correspond to physiological disorders for which a generally stable control system becomes unstable. One of the first papers on this subject was that of Mackey and Glass [27]. In that paper they investigated a simple first order difference-delay equation that models the concentration of blood-level CO_2. They also discussed models of a second class of diseases associated with the production of red cells, white cells, and platelets in the bone marrow.

The dynamical characteristics of population system have been modelled, among others by differential equations in the case of species with overlapping generations and by difference equations in the case of species with non-overlapping generations.

In practice, one can formulate a discrete model directly from experiments and observations. Sometimes, for numerical purposes one wants to propose a finite-difference scheme to numerically solved a given differential equation model, especially when the differential equation cannot be solved explicitly. For a given differential equation, a difference equation approximation would be most acceptable

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if the solution of the difference equation is the same as the differential equation at the discrete points [21]. But unless we can explicitly solve both equations, it is impossible to satisfy this requirements. Most of the time, it is desirable that a differential equation, when derived from a difference equation, preserves the dynamical features of the corresponding continuous-time model such as equilibria, their local and global stability characteristics and bifurcation behaviors. If such discrete models can be derived from continuous-time models and it will preserve the considered realities; such discrete-time models can be called 'dynamically consistent' with the continuous-time models.

The study of asymptotic stability and oscillatory properties of solutions of difference equations is extremely useful in the behavior of mathematical models of various biological systems and other applications. This is due to the fact that difference equations are appropriate models for describing situations where the variable is assumed to take only a discrete set of values and they arise frequently in the study of biological models, in the formulation and analysis of discrete time systems, the numerical integration of differential equations by finite-difference schemes, the study of deterministic chaos, etc. For example, [26] the study of oscillation of positive solutions about the positive steady state N in the delay logistic difference equation

$$N_{n+1} = N_n \ \exp\left[r\left(1-\sum_{j=0}^{m} p_j N_{n-j}\right)\right],$$

where $r,\ p_m \in (0,\infty), p_0, p_1, ..., p_{m-1} \in [0,\infty)$ and $m+r \neq 1$, which describes situations where population growth is not continuous but seasonal with non-overlapping generations, leads to the study of oscillations about zero of a linear difference equation of the form

$$x_{n+1} - x_n + \sum_{i=0}^{m} p_i x_{n-k_i} = 0, \quad n = 0, 1, ... \ .$$

Also, difference equations are appropriate models for describing situations where population growth is not continuous but seasonal with overlapping generations.

For example, the difference equation,

$$y_{n+1} = y_n \exp\left[r\left(1-\frac{y_n}{K}\right)\right],$$

has been used to model various animal populations. This equation is considered by some to be the discrete analogue of the logistic differential equation

$$y'(t) = ry(t)\left(1-\frac{y(t)}{k}\right),$$

where r and k are the growth rate and the carrying capacity of population, respectively.

El-Metwally et al. [13] investigated the asymptotic behavior of the population model:

$$x_{n+1} = \alpha + \beta x_{n-1} e^{-x_n},$$

where α is the immigration rate and β is the population growth rate.

Ding et al. [10] studied the following discrete delay Mosquito population equation

$$x_{n+1} = (\alpha x_n + \beta x_{n-1})\, e^{-x_n}.$$

The generalized Beverton-Holt stock recruitment model has investigated in [4,8]:

$$x_{n+1} = a x_n + \frac{b x_{n-1}}{1 + c x_{n-1} + d x_n}.$$

See also [1–22]. The long term behavior of the solutions of nonlinear difference equations systems of order greater than one has been extensively studied during the last decade. For example, various results about boundedness, stability and periodic character of the solutions of the second-order nonlinear difference equations and systems of difference equations see [23, 27, 30–42].

Many researchers have investigated the behavior of the solution of difference equations systems for example:

The periodicity of the positive solutions of the rational difference equations system

$$x_{n+1} = \frac{1}{z_n},\ y_{n+1} = \frac{y_n}{x_{n-1} y_{n-1}},\ z_{n+1} = \frac{1}{x_{n-1}},$$

has been obtained by Cinar in [6].

In [15] Elsayed et al. dealed with the solutions of the systems of the difference equations

$$x_{n+1} = \frac{1}{x_{n-p} y_{n-p}},\quad y_{n+1} = \frac{x_{n-p} y_{n-p}}{x_{n-q} y_{n-q}},$$

and

$$x_{n+1} = \frac{1}{x_{n-p} y_{n-p} z_{n-p}},\quad y_{n+1} = \frac{x_{n-p} y_{n-p} z_{n-p}}{x_{n-q} y_{n-q} z_{n-q}},\ z_{n+1} = \frac{x_{n-q} y_{n-q} z_{n-q}}{x_{n-r} y_{n-r} z_{n-r}}.$$

In [39] Yalcinkaya and Cinar, showed that every solution of the following system of the difference equations

$$\begin{aligned} x_{n+1}^{(1)} &= \frac{x_n^{(2)} x_{n-1}^{(3)}}{x_n^{(2)} x_{n-1}^{(3)} - x_n^{(2)} - x_{n-1}^{(3)}}, \\ x_{n+1}^{(2)} &= \frac{x_n^{(3)} x_{n-1}^{(4)}}{x_n^{(3)} x_{n-1}^{(4)} - x_n^{(3)} - x_{n-1}^{(4)}}, \\ &\ldots, \\ x_{n+1}^{(z_0)} &= \frac{x_n^{(1)} x_{n-1}^{(2)}}{x_n^{(1)} x_{n-1}^{(2)} - x_n^{(1)} - x_{n-1}^{(2)}}, \end{aligned}$$

is periodic with known period.

Kurbanli [24, 25] dealed with the behavior of the solutions of the following systems of difference equations

$$x_{n+1} = \frac{x_{n-1}}{x_{n-1} y_n - 1},\ y_{n+1} = \frac{y_{n-1}}{y_{n-1} x_n - 1},\quad z_{n+1} = \frac{z_{n-1}}{z_{n-1} y_n - 1},$$
$$x_{n+1} = \frac{x_{n-1}}{x_{n-1} y_n - 1},\ y_{n+1} = \frac{y_{n-1}}{y_{n-1} x_n - 1},\ z_{n+1} = \frac{x_n}{z_{n-1} y_n}.$$

Zkan and Kurbanli [41] have investigated the periodical solutions of the following system of third order rational difference equations

$$x_{n+1} = \frac{y_{n-2}}{-1 \pm y_{n-2}x_{n-1}y_n},\ y_{n+1} = \frac{x_{n-2}}{-1 \pm x_{n-2}y_{n-1}x_n},\ z_{n+1} = \frac{x_{n-2}+y_{n-2}}{-1 \pm x_{n-2}y_{n-1}x_n}.$$

Similar to difference equations and nonlinear systems of rational difference equations were investigated see [23–30].

Definition 1.1 (Periodicity)**.** A sequence $\{x_n\}_{n=-z_0}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -z_0$.

Definition 1.2 (Fibonacci Sequence)**.** The sequence $\{f_m\}_{m=1}^{\infty} = \{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...\}$ i.e. $f_{m+1} = f_m + f_{m-1},\ m \geq 0,\ f_{-1} = 1,\ f_0 = 0$ is called Fibonacci Sequence.

The main goal of this paper, is to study a class of nonlinear rational systems of difference equations of order three, in three-dimensional case,given by

$$x_{n+1} = \frac{y_{n-1}x_{n-2}}{x_{n-2} \pm z_n},\ y_{n+1} = \frac{z_{n-1}y_{n-2}}{y_{n-2} \pm x_n},\ z_{n+1} = \frac{x_{n-1}z_{n-2}}{z_{n-2} \pm y_n},$$

with the initial conditions are nonzero real numbers.

2. Some Systems and Their Solutions

Here we interest to investigate the following systems of difference equations

$$x_{n+1} = \frac{y_{n-1}x_{n-2}}{x_{n-2} + z_n},\ y_{n+1} = \frac{z_{n-1}y_{n-2}}{y_{n-2} + x_n},\ z_{n+1} = \frac{x_{n-1}z_{n-2}}{z_{n-2} + y_n}. \tag{2.1}$$

$$x_{n+1} = \frac{y_{n-1}x_{n-2}}{x_{n-2} + z_n},\ y_{n+1} = \frac{z_{n-1}y_{n-2}}{y_{n-2} + x_n},\ z_{n+1} = \frac{x_{n-1}z_{n-2}}{z_{n-2} - y_n}. \tag{2.2}$$

$$x_{n+1} = \frac{y_{n-1}x_{n-2}}{x_{n-2} + z_n},\ y_{n+1} = \frac{z_{n-1}y_{n-2}}{y_{n-2} - x_n},\ z_{n+1} = \frac{x_{n-1}z_{n-2}}{z_{n-2} - y_n}. \tag{2.3}$$

where $n \in \mathbb{N}_0$ and the initial conditions are arbitrary nonzero real numbers.

The following theorems are devoted to the form of the solutions of previous systems.

Theorem 2.1. *Assume that* $\{x_n, y_n, z_n\}$ *are solutions of system* (2.1)*. Then for* $n = 0, 1, 2, ...,$ *we see that*

$$x_{6n-2} = x_{-2} \prod_{i=0}^{n-1} \frac{(f_{6n-1}z_0 + f_{6n}x_{-2})(f_{6n+1}z_0 + f_{6n+2}x_{-2})(f_{6n+3}z_0 + f_{6n+4}x_{-2})}{(f_{6n}z_0 + f_{6n+1}x_{-2})(f_{6n+2}z_0 + f_{6n+3}x_{-2})(f_{6n+4}z_0 + f_{6n+5}x_{-2})},$$

$$x_{6n-1} = x_{-1} \prod_{i=0}^{n-1} \frac{(f_{6n}y_0 + f_{6n+1}z_{-2})(f_{6n+2}y_0 + f_{6n+3}z_{-2})(f_{6n+4}y_0 + f_{6n+5}z_{-2})}{(f_{6n+1}y_0 + f_{6n+2}z_{-2})(f_{6n+3}y_0 + f_{6n+4}z_{-2})(f_{6n+5}y_0 + f_{6n+6}z_{-2})},$$

$$x_{6n} = x_0 \prod_{i=0}^{n-1} \frac{(f_{6n+1}x_0 + f_{6n+2}y_{-2})(f_{6n+3}x_0 + f_{6n+4}y_{-2})(f_{6n+5}x_0 + f_{6n+6}y_{-2})}{(f_{6n+2}x_0 + f_{6n+3}y_{-2})(f_{6n+4}x_0 + f_{6n+5}y_{-2})(f_{6n+6}x_0 + f_{6n+7}y_{-2})},$$

$$x_{6n+1} = \frac{x_{-2}y_{-1}}{(z_0+x_{-2})} \prod_{i=0}^{n-1} \frac{(f_{6n+2}z_0+f_{6n+3}x_{-2})(f_{6n+4}z_0+f_{6n+5}x_{-2})(f_{6n+6}z_0+f_{6n+7}x_{-2})}{(f_{6n+3}z_0+f_{6n+4}x_{-2})(f_{6n+5}z_0+f_{6n+6}x_{-2})(f_{6n+7}z_0+f_{6n+8}x_{-2})},$$

$$x_{6n+2} = \frac{y_0(y_0+z_{-2})}{(y_0+2z_{-2})} \prod_{i=0}^{n-1} \frac{(f_{6n+3}y_0 + f_{6n+4}z_{-2})(f_{6n+5}y_0+f_{6n+6}z_{-2})(f_{6n+7}y_0+f_{6n+8}z_{-2})}{(f_{6n+4}y_0+f_{6n+5}z_{-2})(f_{6n+6}y_0+f_{6n+7}z_{-2})(f_{6n+8}y_0+f_{6n+9}z_{-2})},$$

$$x_{6n+3}=\frac{y_{-2}z_{-1}(x_0+2y_{-2})}{(x_0+y_{-2})(2x_0+3y_{-2})}\prod_{i=0}^{n-1}\frac{(f_{6n+4}x_0+f_{6n+5}y_{-2})(f_{6n+6}x_0+f_{6n+7}y_{-2})(f_{6n+8}x_0+f_{6n+9}y_{-2})}{(f_{6n+5}x_0+f_{6n+6}y_{-2})(f_{6n+7}x_0+f_{6n+8}y_{-2})(f_{6n+9}x_0+f_{6n+10}y_{-2})},$$

$$y_{6n-2}=y_{-2}\prod_{i=0}^{n-1}\frac{(f_{6n-1}x_0+f_{6n}y_{-2})(f_{6n+1}x_0+f_{6n+2}y_{-2})(f_{6n+3}x_0+f_{6n+4}y_{-2})}{(f_{6n}x_0+f_{6n+1}y_{-2})(f_{6n+2}x_0+f_{6n+3}y_{-2})(f_{6n+4}x_0+f_{6n+5}y_{-2})},$$

$$y_{6n-1}=y_{-1}\prod_{i=0}^{n-1}\frac{(f_{6n}z_0+f_{6n+1}x_{-2})(f_{6n+2}z_0+f_{6n+3}x_{-2})(f_{6n+4}z_0+f_{6n+5}x_{-2})}{(f_{6n+1}z_0+f_{6n+2}x_{-2})(f_{6n+3}z_0+f_{6n+4}x_{-2})(f_{6n+5}z_0+f_{6n+6}x_{-2})},$$

$$y_{6n}=y_0\prod_{i=0}^{n-1}\frac{(f_{6n+1}y_0+f_{6n+2}z_{-2})(f_{6n+3}y_0+f_{6n+4}z_{-2})(f_{6n+5}y_0+f_{6n+6}z_{-2})}{(f_{6n+2}y_0+f_{6n+3}z_{-2})(f_{6n+4}y_0+f_{6n+5}z_{-2})(f_{6n+6}y_0+f_{6n+7}z_{-2})},$$

$$y_{6n+1}=\frac{y_{-2}z_{-1}}{(x_0+y_{-2})}\prod_{i=0}^{n-1}\frac{(f_{6n+2}x_0+f_{6n+3}y_{-2})(f_{6n+4}x_0+f_{6n+5}y_{-2})(f_{6n+6}x_0+f_{6n+7}y_{-2})}{(f_{6n+3}x_0+f_{6n+4}y_{-2})(f_{6n+5}x_0+f_{6n+6}y_{-2})(f_{6n+7}x_0+f_{6n+8}y_{-2})},$$

$$y_{6n+2}=\frac{z_0(z_0+x_{-2})}{(z_0+2x_{-2})}\prod_{i=0}^{n-1}\frac{(f_{6n+3}z_0+f_{6n+4}x_{-2})(f_{6n+5}z_0+f_{6n+6}x_{-2})(f_{6n+7}z_0+f_{6n+8}x_{-2})}{(f_{6n+4}z_0+f_{6n+5}x_{-2})(f_{6n+6}z_0+f_{6n+7}x_{-2})(f_{6n+8}z_0+f_{6n+9}x_{-2})},$$

$$y_{6n+3}=\frac{x_{-1}z_{-2}(y_0+2z_{-2})}{(y_0+z_{-2})(2y_0+3z_{-2})}\prod_{i=0}^{n-1}\frac{(f_{6n+4}y_0+f_{6n+5}z_{-2})(f_{6n+6}y_0+f_{6n+7}z_{-2})(f_{6n+8}y_0+f_{6n+9}z_{-2})}{(f_{6n+5}y_0+f_{6n+6}z_{-2})(f_{6n+7}y_0+f_{6n+8}z_{-2})(f_{6n+9}y_0+f_{6n+10}z_{-2})},$$

and

$$z_{6n-2}=z_{-2}\prod_{i=0}^{n-1}\frac{(f_{6n-1}y_0+f_{6n}z_{-2})(f_{6n+1}y_0+f_{6n+2}z_{-2})(f_{6n+3}y_0+f_{6n+4}z_{-2})}{(f_{6n}y_0+f_{6n+1}z_{-2})(f_{6n+2}y_0+f_{6n+3}z_{-2})(f_{6n+4}y_0+f_{6n+5}z_{-2})},$$

$$z_{6n-1}=z_{-1}\prod_{i=0}^{n-1}\frac{(f_{6n}x_0+f_{6n+1}y_{-2})(f_{6n+2}x_0+f_{6n+3}y_{-2})(f_{6n+4}x_0+f_{6n+5}y_{-2})}{(f_{6n+1}x_0+f_{6n+2}y_{-2})(f_{6n+3}x_0+f_{6n+4}y_{-2})(f_{6n+5}x_0+f_{6n+6}y_{-2})},$$

$$z_{6n}=z_0\prod_{i=0}^{n-1}\frac{(f_{6n+1}z_0+f_{6n+2}x_{-2})(f_{6n+3}z_0+f_{6n+4}x_{-2})(f_{6n+5}z_0+f_{6n+6}x_{-2})}{(f_{6n+2}z_0+f_{6n+3}x_{-2})(f_{6n+4}z_0+f_{6n+5}x_{-2})(f_{6n+6}z_0+f_{6n+7}x_{-2})},$$

$$z_{6n+1}=\frac{x_{-1}z_{-2}}{(y_0+z_{-2})}\prod_{i=0}^{n-1}\frac{(f_{6n+2}y_0+f_{6n+3}z_{-2})(f_{6n+4}y_0+f_{6n+5}z_{-2})(f_{6n+6}y_0+f_{6n+7}z_{-2})}{(f_{6n+3}y_0+f_{6n+4}z_{-2})(f_{6n+5}y_0+f_{6n+6}z_{-2})(f_{6n+7}y_0+f_{6n+8}z_{-2})},$$

$$z_{6n+2}=\frac{x_0(x_0+y_{-2})}{(x_0+2y_{-2})}\prod_{i=0}^{n-1}\frac{(f_{6n+3}x_0+f_{6n+4}y_{-2})(f_{6n+5}x_0+f_{6n+6}y_{-2})(f_{6n+7}x_0+f_{6n+8}y_{-2})}{(f_{6n+4}x_0+f_{6n+5}y_{-2})(f_{6n+6}x_0+f_{6n+7}y_{-2})(f_{6n+8}x_0+f_{6n+9}y_{-2})},$$

$$z_{6n+3}=\frac{x_{-2}y_{-1}(z_0+2x_{-2})}{(z_0+x_{-2})(2z_0+3x_{-2})}\prod_{i=0}^{n-1}\frac{(f_{6n+4}z_0+f_{6n+5}x_{-2})(f_{6n+6}z_0+f_{6n+7}x_{-2})(f_{6n+8}z_0+f_{6n+9}x_{-2})}{(f_{6n+5}z_0+f_{6n+6}x_{-2})(f_{6n+7}z_0+f_{6n+8}x_{-2})(f_{6n+9}z_0+f_{6n+10}x_{-2})},$$

where $\{f_n\}_{n=-1}^{\infty}=\{1,0,1,1,2,3,5,8,13,...\}$.

Proof. For $n=0$ the result holds. Now suppose that $n>1$ and that our assumption holds for $n-1$. that is,

$$x_{6n-8}=x_{-2}\prod_{i=0}^{n-2}\frac{(f_{6n-1}z_0+f_{6n}x_{-2})(f_{6n+1}z_0+f_{6n+2}x_{-2})(f_{6n+3}z_0+f_{6n+4}x_{-2})}{(f_{6n}z_0+f_{6n+1}x_{-2})(f_{6n+2}z_0+f_{6n+3}x_{-2})(f_{6n+4}z_0+f_{6n+5}x_{-2})},$$

$$x_{6n-7}=x_{-1}\prod_{i=0}^{n-2}\frac{(f_{6n}y_0+f_{6n+1}z_{-2})(f_{6n+2}y_0+f_{6n+3}z_{-2})(f_{6n+4}y_0+f_{6n+5}z_{-2})}{(f_{6n+1}y_0+f_{6n+2}z_{-2})(f_{6n+3}y_0+f_{6n+4}z_{-2})(f_{6n+5}y_0+f_{6n+6}z_{-2})},$$

$$x_{6n-6}=x_0\prod_{i=0}^{n-2}\frac{(f_{6n+1}x_0+f_{6n+2}y_{-2})(f_{6n+3}x_0+f_{6n+4}y_{-2})(f_{6n+5}x_0+f_{6n+6}y_{-2})}{(f_{6n+2}x_0+f_{6n+3}y_{-2})(f_{6n+4}x_0+f_{6n+5}y_{-2})(f_{6n+6}x_0+f_{6n+7}y_{-2})},$$

$$x_{6n-5}=\frac{x_{-2}y_{-1}}{(z_0+x_{-2})}\prod_{i=0}^{n-2}\frac{(f_{6n+2}z_0+f_{6n+3}x_{-2})(f_{6n+4}z_0+f_{6n+5}x_{-2})(f_{6n+6}z_0+f_{6n+7}x_{-2})}{(f_{6n+3}z_0+f_{6n+4}x_{-2})(f_{6n+5}z_0+f_{6n+6}x_{-2})(f_{6n+7}z_0+f_{6n+8}x_{-2})},$$

$$x_{6n-4}=\frac{y_0(y_0+z_{-2})}{(y_0+2z_{-2})}\prod_{i=0}^{n-2}\frac{(f_{6n+3}y_0+f_{6n+4}z_{-2})(f_{6n+5}y_0+f_{6n+6}z_{-2})(f_{6n+7}y_0+f_{6n+8}z_{-2})}{(f_{6n+4}y_0+f_{6n+5}z_{-2})(f_{6n+6}y_0+f_{6n+7}z_{-2})(f_{6n+8}y_0+f_{6n+9}z_{-2})},$$

$$x_{6n-3}=\frac{y_{-2}z_{-1}(x_0+2y_{-2})}{(x_0+y_{-2})(2x_0+3y_{-2})}\prod_{i=0}^{n-2}\frac{(f_{6n+4}x_0+f_{6n+5}y_{-2})(f_{6n+6}x_0+f_{6n+7}y_{-2})(f_{6n+8}x_0+f_{6n+9}y_{-2})}{(f_{6n+5}x_0+f_{6n+6}y_{-2})(f_{6n+7}x_0+f_{6n+8}y_{-2})(f_{6n+9}x_0+f_{6n+10}y_{-2})},$$

$$y_{6n-8}=y_{-2}\prod_{i=0}^{n-2}\frac{(f_{6n-1}x_0+f_{6n}y_{-2})(f_{6n+1}x_0+f_{6n+2}y_{-2})(f_{6n+3}x_0+f_{6n+4}y_{-2})}{(f_{6n}x_0+f_{6n+1}y_{-2})(f_{6n+2}x_0+f_{6n+3}y_{-2})(f_{6n+4}x_0+f_{6n+5}y_{-2})},$$

$$y_{6n-7}=y_{-1}\prod_{i=0}^{n-2}\frac{(f_{6n}z_0+f_{6n+1}x_{-2})(f_{6n+2}z_0+f_{6n+3}x_{-2})(f_{6n+4}z_0+f_{6n+5}x_{-2})}{(f_{6n+1}z_0+f_{6n+2}x_{-2})(f_{6n+3}z_0+f_{6n+4}x_{-2})(f_{6n+5}z_0+f_{6n+6}x_{-2})},$$

$$y_{6n-6}=y_0\prod_{i=0}^{n-2}\frac{(f_{6n+1}y_0+f_{6n+2}z_{-2})(f_{6n+3}y_0+f_{6n+4}z_{-2})(f_{6n+5}y_0+f_{6n+6}z_{-2})}{(f_{6n+2}y_0+f_{6n+3}z_{-2})(f_{6n+4}y_0+f_{6n+5}z_{-2})(f_{6n+6}y_0+f_{6n+7}z_{-2})},$$

$$y_{6n-5}=\frac{y_{-2}z_{-1}}{(x_0+y_{-2})}\prod_{i=0}^{n-2}\frac{(f_{6n+2}x_0+f_{6n+3}y_{-2})(f_{6n+4}x_0+f_{6n+5}y_{-2})(f_{6n+6}x_0+f_{6n+7}y_{-2})}{(f_{6n+3}x_0+f_{6n+4}y_{-2})(f_{6n+5}x_0+f_{6n+6}y_{-2})(f_{6n+7}x_0+f_{6n+8}y_{-2})},$$

$$y_{6n-4}=\frac{z_0(z_0+x_{-2})}{(z_0+2x_{-2})}\prod_{i=0}^{n-2}\frac{(f_{6n+3}z_0+f_{6n+4}x_{-2})(f_{6n+5}z_0+f_{6n+6}x_{-2})(f_{6n+7}z_0+f_{6n+8}x_{-2})}{(f_{6n+4}z_0+f_{6n+5}x_{-2})(f_{6n+6}z_0+f_{6n+7}x_{-2})(f_{6n+8}z_0+f_{6n+9}x_{-2})},$$

$$y_{6n-3}=\frac{x_{-1}z_{-2}(y_0+2z_{-2})}{(y_0+z_{-2})(2y_0+3z_{-2})}\prod_{i=0}^{n-2}\frac{(f_{6n+4}y_0+f_{6n+5}z_{-2})(f_{6n+6}y_0+f_{6n+7}z_{-2})(f_{6n+8}y_0+f_{6n+9}z_{-2})}{(f_{6n+5}y_0+f_{6n+6}z_{-2})(f_{6n+7}y_0+f_{6n+8}z_{-2})(f_{6n+9}y_0+f_{6n+10}z_{-2})},$$

and

$$z_{6n-8}=z_{-2}\prod_{i=0}^{n-2}\frac{(f_{6n-1}y_0+f_{6n}z_{-2})(f_{6n+1}y_0+f_{6n+2}z_{-2})(f_{6n+3}y_0+f_{6n+4}z_{-2})}{(f_{6n}y_0+f_{6n+1}z_{-2})(f_{6n+2}y_0+f_{6n+3}z_{-2})(f_{6n+4}y_0+f_{6n+5}z_{-2})},$$

$$z_{6n-7}=z_{-1}\prod_{i=0}^{n-2}\frac{(f_{6n}x_0+f_{6n+1}y_{-2})(f_{6n+2}x_0+f_{6n+3}y_{-2})(f_{6n+4}x_0+f_{6n+5}y_{-2})}{(f_{6n+1}x_0+f_{6n+2}y_{-2})(f_{6n+3}x_0+f_{6n+4}y_{-2})(f_{6n+5}x_0+f_{6n+6}y_{-2})},$$

$$z_{6n-6}=z_0\prod_{i=0}^{n-2}\frac{(f_{6n+1}z_0+f_{6n+2}x_{-2})(f_{6n+3}z_0+f_{6n+4}x_{-2})(f_{6n+5}z_0+f_{6n+6}x_{-2})}{(f_{6n+2}z_0+f_{6n+3}x_{-2})(f_{6n+4}z_0+f_{6n+5}x_{-2})(f_{6n+6}z_0+f_{6n+7}x_{-2})},$$

$$z_{6n-5}=\frac{x_{-1}z_{-2}}{(y_0+z_{-2})}\prod_{i=0}^{n-2}\frac{(f_{6n+2}y_0+f_{6n+3}z_{-2})(f_{6n+4}y_0+f_{6n+5}z_{-2})(f_{6n+6}y_0+f_{6n+7}z_{-2})}{(f_{6n+3}y_0+f_{6n+4}z_{-2})(f_{6n+5}y_0+f_{6n+6}z_{-2})(f_{6n+7}y_0+f_{6n+8}z_{-2})},$$

$$z_{6n-4}=\frac{x_0(x_0+y_{-2})}{(x_0+2y_{-2})}\prod_{i=0}^{n-2}\frac{(f_{6n+3}x_0+f_{6n+4}y_{-2})(f_{6n+5}x_0+f_{6n+6}y_{-2})(f_{6n+7}x_0+f_{6n+8}y_{-2})}{(f_{6n+4}x_0+f_{6n+5}y_{-2})(f_{6n+6}x_0+f_{6n+7}y_{-2})(f_{6n+8}x_0+f_{6n+9}y_{-2})},$$

$$z_{6n-3}=\frac{x_{-2}y_{-1}(z_0+2x_{-2})}{(z_0+x_{-2})(2z_0+3x_{-2})}\prod_{i=0}^{n-2}\frac{(f_{6n+4}z_0+f_{6n+5}x_{-2})(f_{6n+6}z_0+f_{6n+7}x_{-2})(f_{6n+8}z_0+f_{6n+9}x_{-2})}{(f_{6n+5}z_0+f_{6n+6}x_{-2})(f_{6n+7}z_0+f_{6n+8}x_{-2})(f_{6n+9}z_0+f_{6n+10}x_{-2})}.$$

It follows from Eq. (2.1) that

$$x_{6n-2}=\frac{y_{6n-4}x_{6n-5}}{x_{6n-5}+z_{6n-3}}$$

$$=\frac{\left(\frac{z_0(z_0+x_{-2})}{(z_0+2x_{-2})}\prod_{i=0}^{n-2}\frac{(f_{6n+3}z_0+f_{6n+4}x_{-2})(f_{6n+5}z_0+f_{6n+6}x_{-2})(f_{6n+7}z_0+f_{6n+8}x_{-2})}{(f_{6n+4}z_0+f_{6n+5}x_{-2})(f_{6n+6}z_0+f_{6n+7}x_{-2})(f_{6n+8}z_0+f_{6n+9}x_{-2})}\right)\left(\frac{x_{-2}y_{-1}}{(z_0+x_{-2})}\prod_{i=0}^{n-2}\frac{(f_{6n+2}z_0+f_{6n+3}x_{-2})(f_{6n+4}z_0+f_{6n+5}x_{-2})(f_{6n+6}z_0+f_{6n+7}x_{-2})}{(f_{6n+3}z_0+f_{6n+4}x_{-2})(f_{6n+5}z_0+f_{6n+6}x_{-2})(f_{6n+7}z_0+f_{6n+8}x_{-2})}\right)}{\left(\frac{x_{-2}y_{-1}}{(z_0+x_{-2})}\prod_{i=0}^{n-2}\frac{(f_{6n+2}z_0+f_{6n+3}x_{-2})(f_{6n+4}z_0+f_{6n+5}x_{-2})(f_{6n+6}z_0+f_{6n+7}x_{-2})}{(f_{6n+3}z_0+f_{6n+4}x_{-2})(f_{6n+5}z_0+f_{6n+6}x_{-2})(f_{6n+7}z_0+f_{6n+8}x_{-2})}\right)+\left(\frac{x_{-2}y_{-1}(z_0+2x_{-2})}{(z_0+x_{-2})(2z_0+3x_{-2})}\prod_{i=0}^{n-2}\frac{(f_{6n+4}z_0+f_{6n+5}x_{-2})(f_{6n+6}z_0+f_{6n+7}x_{-2})(f_{6n+8}z_0+f_{6n+9}x_{-2})}{(f_{6n+5}z_0+f_{6n+6}x_{-2})(f_{6n+7}z_0+f_{6n+8}x_{-2})(f_{6n+9}z_0+f_{6n+10}x_{-2})}\right)}$$

$$=\frac{\left(\frac{z_0(z_0+x_{-2})}{(z_0+2x_{-2})}\prod_{i=0}^{n-2}\frac{(f_{6n+3}z_0+f_{6n+4}x_{-2})(f_{6n+5}z_0+f_{6n+6}x_{-2})(f_{6n+7}z_0+f_{6n+8}x_{-2})}{(f_{6n+4}z_0+f_{6n+5}x_{-2})(f_{6n+6}z_0+f_{6n+7}x_{-2})(f_{6n+8}z_0+f_{6n+9}x_{-2})}\right)}{1+\left(\frac{(z_0+2x_{-2})}{(2z_0+3x_{-2})}\prod_{i=0}^{n-2}\frac{(f_{6n+3}z_0+f_{6n+4}x_{-2})(f_{6n+8}z_0+f_{6n+9}x_{-2})}{(f_{6n+2}z_0+f_{6n+3}x_{-2})(f_{6n+9}z_0+f_{6n+10}x_{-2})}\right)}$$

$$=\frac{\left(\frac{z_0(z_0+x_{-2})}{(z_0+2x_{-2})}\prod_{i=0}^{n-2}\frac{(f_{6n+3}z_0+f_{6n+4}x_{-2})(f_{6n+5}z_0+f_{6n+6}x_{-2})(f_{6n+7}z_0+f_{6n+8}x_{-2})}{(f_{6n+4}z_0+f_{6n+5}x_{-2})(f_{6n+6}z_0+f_{6n+7}x_{-2})(f_{6n+8}z_0+f_{6n+9}x_{-2})}\right)}{1+\left(\frac{(f_{6n-4}z_0+f_{6n-3}x_{-2})}{(f_{6n-3}z_0+f_{6n-2}x_{-2})}\right)}$$

$$=\frac{z_0(z_0+x_{-2})}{(z_0+2x_{-2})}\prod_{i=0}^{n-2}\frac{(f_{6n+3}z_0+f_{6n+4}x_{-2})(f_{6n+5}z_0+f_{6n+6}x_{-2})(f_{6n+7}z_0+f_{6n+8}x_{-2})}{(f_{6n+4}z_0+f_{6n+5}x_{-2})(f_{6n+6}z_0+f_{6n+7}x_{-2})(f_{6n+8}z_0+f_{6n+9}x_{-2})}\left(\frac{f_{6n-3}z_0+f_{6n-2}x_{-2}}{f_{6n-2}z_0+f_{6n-1}x_{-2}}\right).$$

Then we see that

$$x_{6n-2}=x_{-2}\prod_{i=0}^{n-1}\frac{(f_{6n-1}z_0+f_{6n}x_{-2})(f_{6n+1}z_0+f_{6n+2}x_{-2})(f_{6n+3}z_0+f_{6n+4}x_{-2})}{(f_{6n}z_0+f_{6n+1}x_{-2})(f_{6n+2}z_0+f_{6n+3}x_{-2})(f_{6n+4}z_0+f_{6n+5}x_{-2})}.$$

Also, we see from Eq. (2.1) that

$$y_{6n-2}=\frac{z_{6n-4}y_{6n-5}}{y_{6n-5}+x_{6n-3}}$$

$$=\frac{\left(\frac{x_0(x_0+y_{-2})}{(x_0+2y_{-2})}\prod_{i=0}^{n-2}\frac{(f_{6n+3}x_0+f_{6n+4}y_{-2})(f_{6n+5}x_0+f_{6n+6}y_{-2})(f_{6n+7}x_0+f_{6n+8}y_{-2})}{(f_{6n+4}x_0+f_{6n+5}y_{-2})(f_{6n+6}x_0+f_{6n+7}y_{-2})(f_{6n+8}x_0+f_{6n+9}y_{-2})}\right)\left(\frac{y_{-2}z_{-1}}{(x_0+y_{-2})}\prod_{i=0}^{n-2}\frac{(f_{6n+2}x_0+f_{6n+3}y_{-2})(f_{6n+4}x_0+f_{6n+5}y_{-2})(f_{6n+6}x_0+f_{6n+7}y_{-2})}{(f_{6n+3}x_0+f_{6n+4}y_{-2})(f_{6n+5}x_0+f_{6n+6}y_{-2})(f_{6n+7}x_0+f_{6n+8}y_{-2})}\right)}{\left(\frac{y_{-2}z_{-1}}{(x_0+y_{-2})}\prod_{i=0}^{n-2}\frac{(f_{6n+2}x_0+f_{6n+3}y_{-2})(f_{6n+4}x_0+f_{6n+5}y_{-2})(f_{6n+6}x_0+f_{6n+7}y_{-2})}{(f_{6n+3}x_0+f_{6n+4}y_{-2})(f_{6n+5}x_0+f_{6n+6}y_{-2})(f_{6n+7}x_0+f_{6n+8}y_{-2})}\right)+\left(\frac{y_{-2}z_{-1}(x_0+2y_{-2})}{(x_0+y_{-2})(2x_0+3y_{-2})}\prod_{i=0}^{n-2}\frac{(f_{6n+4}x_0+f_{6n+5}y_{-2})(f_{6n+6}x_0+f_{6n+7}y_{-2})(f_{6n+8}x_0+f_{6n+9}y_{-2})}{(f_{6n+5}x_0+f_{6n+6}y_{-2})(f_{6n+7}x_0+f_{6n+8}y_{-2})(f_{6n+9}x_0+f_{6n+10}y_{-2})}\right)}$$

$$=\frac{\left(\frac{x_0(x_0+y_{-2})}{(x_0+2y_{-2})}\prod_{i=0}^{n-2}\frac{(f_{6n+3}x_0+f_{6n+4}y_{-2})(f_{6n+5}x_0+f_{6n+6}y_{-2})(f_{6n+7}x_0+f_{6n+8}y_{-2})}{(f_{6n+4}x_0+f_{6n+5}y_{-2})(f_{6n+6}x_0+f_{6n+7}y_{-2})(f_{6n+8}x_0+f_{6n+9}y_{-2})}\right)}{1+\left(\frac{(x_0+2y_{-2})}{(2x_0+3y_{-2})}\prod_{i=0}^{n-2}\frac{(f_{6n+3}x_0+f_{6n+4}y_{-2})(f_{6n+8}x_0+f_{6n+9}y_{-2})}{(f_{6n+2}x_0+f_{6n+3}y_{-2})(f_{6n+9}x_0+f_{6n+10}y_{-2})}\right)}$$

$$=\frac{\left(\frac{x_0(x_0+y_{-2})}{(x_0+2y_{-2})}\prod_{i=0}^{n-2}\frac{(f_{6n+3}x_0+f_{6n+4}y_{-2})(f_{6n+5}x_0+f_{6n+6}y_{-2})(f_{6n+7}x_0+f_{6n+8}y_{-2})}{(f_{6n+4}x_0+f_{6n+5}y_{-2})(f_{6n+6}x_0+f_{6n+7}y_{-2})(f_{6n+8}x_0+f_{6n+9}y_{-2})}\right)}{1+\left(\frac{(f_{6n-4}x_0+f_{6n-3}y_{-2})}{(f_{6n-3}x_0+f_{6n-2}y_{-2})}\right)}$$

$$=\frac{x_0(x_0+y_{-2})}{(x_0+2y_{-2})}\prod_{i=0}^{n-2}\frac{(f_{6n+3}x_0+f_{6n+4}y_{-2})(f_{6n+5}x_0+f_{6n+6}y_{-2})(f_{6n+7}x_0+f_{6n+8}y_{-2})}{(f_{6n+4}x_0+f_{6n+5}y_{-2})(f_{6n+6}x_0+f_{6n+7}y_{-2})(f_{6n+8}x_0+f_{6n+9}y_{-2})}\left(\frac{f_{6n-3}x_0+f_{6n-2}y_{-2}}{f_{6n-2}x_0+f_{6n-1}y_{-2}}\right).$$

Then

$$y_{6n-2}=y_{-2}\prod_{i=0}^{n-1}\frac{(f_{6n-1}x_0+f_{6n}y_{-2})(f_{6n+1}x_0+f_{6n+2}y_{-2})(f_{6n+3}x_0+f_{6n+4}y_{-2})}{(f_{6n}x_0+f_{6n+1}y_{-2})(f_{6n+2}x_0+f_{6n+3}y_{-2})(f_{6n+4}x_0+f_{6n+5}y_{-2})}.$$

Finally from Eq. (2.1), we see that

$$z_{6n-2} = \frac{x_{6n-4}z_{6n-5}}{z_{6n-5}+y_{6n-3}}$$

$$= \frac{\left(\frac{y_0(y_0+z_{-2})}{(y_0+2z_{-2})}\prod_{i=0}^{n-2}\frac{(f_{6n+3}y_0+f_{6n+4}z_{-2})(f_{6n+5}y_0+f_{6n+6}z_{-2})(f_{6n+7}y_0+f_{6n+8}z_{-2})}{(f_{6n+4}y_0+f_{6n+5}z_{-2})(f_{6n+6}y_0+f_{6n+7}z_{-2})(f_{6n+8}y_0+f_{6n+9}z_{-2})}\right)\left(\frac{x_{-1}z_{-2}}{(y_0+z_{-2})}\prod_{i=0}^{n-2}\frac{(f_{6n+2}y_0+f_{6n+3}z_{-2})(f_{6n+4}y_0+f_{6n+5}z_{-2})(f_{6n+6}y_0+f_{6n+7}z_{-2})}{(f_{6n+3}y_0+f_{6n+4}z_{-2})(f_{6n+5}y_0+f_{6n+6}z_{-2})(f_{6n+7}y_0+f_{6n+8}z_{-2})}\right)}{\left(\frac{x_{-1}z_{-2}}{(y_0+z_{-2})}\prod_{i=0}^{n-2}\frac{(f_{6n+2}y_0+f_{6n+3}z_{-2})(f_{6n+4}y_0+f_{6n+5}z_{-2})(f_{6n+6}y_0+f_{6n+7}z_{-2})}{(f_{6n+3}y_0+f_{6n+4}z_{-2})(f_{6n+5}y_0+f_{6n+6}z_{-2})(f_{6n+7}y_0+f_{6n+8}z_{-2})}\right)+\left(\frac{x_{-1}z_{-2}(y_0+2z_{-2})}{(y_0+z_{-2})(2y_0+3z_{-2})}\prod_{i=0}^{n-2}\frac{(f_{6n+4}y_0+f_{6n+5}z_{-2})(f_{6n+6}y_0+f_{6n+7}z_{-2})(f_{6n+8}y_0+f_{6n+9}z_{-2})}{(f_{6n+5}y_0+f_{6n+6}z_{-2})(f_{6n+7}y_0+f_{6n+8}z_{-2})(f_{6n+9}y_0+f_{6n+10}z_{-2})}\right)}$$

$$= \frac{\left(\frac{y_0(y_0+z_{-2})}{(y_0+2z_{-2})}\prod_{i=0}^{n-2}\frac{(f_{6n+3}y_0+f_{6n+4}z_{-2})(f_{6n+5}y_0+f_{6n+6}z_{-2})(f_{6n+7}y_0+f_{6n+8}z_{-2})}{(f_{6n+4}y_0+f_{6n+5}z_{-2})(f_{6n+6}y_0+f_{6n+7}z_{-2})(f_{6n+8}y_0+f_{6n+9}z_{-2})}\right)}{1+\left(\frac{(y_0+2z_{-2})}{(2y_0+3z_{-2})}\prod_{i=0}^{n-2}\frac{(f_{6n+3}y_0+f_{6n+4}z_{-2})(f_{6n+8}y_0+f_{6n+9}z_{-2})}{(f_{6n+2}y_0+f_{6n+3}z_{-2})(f_{6n+9}y_0+f_{6n+10}z_{-2})}\right)}$$

$$= \frac{\left(\frac{y_0(y_0+z_{-2})}{(y_0+2z_{-2})}\prod_{i=0}^{n-2}\frac{(f_{6n+3}y_0+f_{6n+4}z_{-2})(f_{6n+5}y_0+f_{6n+6}z_{-2})(f_{6n+7}y_0+f_{6n+8}z_{-2})}{(f_{6n+4}y_0+f_{6n+5}z_{-2})(f_{6n+6}y_0+f_{6n+7}z_{-2})(f_{6n+8}y_0+f_{6n+9}z_{-2})}\right)}{1+\left(\frac{(f_{6n-4}y_0+f_{6n-3}z_{-2})}{(f_{6n-3}y_0+f_{6n-2}z_{-2})}\right)}.$$

Thus

$$z_{6n-2} = z_{-2}\prod_{i=0}^{n-1}\frac{(f_{6n-1}y_0+f_{6n}z_{-2})(f_{6n+1}y_0+f_{6n+2}z_{-2})(f_{6n+3}y_0+f_{6n+4}z_{-2})}{(f_{6n}y_0+f_{6n+1}z_{-2})(f_{6n+2}y_0+f_{6n+3}z_{-2})(f_{6n+4}y_0+f_{6n+5}z_{-2})}.$$

Similarly we can prove the other relations. This completes the proof. □

Lemma 2.1. *Let* $\{x_n, y_n, z_n\}$ *be a positive solution of system* (2.1), *then every solution of system* (2.1) *is bounded and converges to zero.*

Proof. It follows from Eq. (2.1) that

$$x_{n+1} = \frac{y_{n-1}x_{n-2}}{x_{n-2}+z_n} \leq y_{n-1}, \qquad y_{n+1} = \frac{z_{n-1}y_{n-2}}{y_{n-2}+x_n} \leq z_{n-1},$$
$$z_{n+1} = \frac{x_{n-1}z_{n-2}}{z_{n-2}+y_n} \leq x_{n-1},$$

we see that

$$x_{n+4} \leq y_{n+2}, \quad y_{n+2} \leq z_n, \quad z_n \leq x_{n-2}, \quad \Rightarrow x_{n+4} < x_{n-2},$$
$$y_{n+4} \leq z_{n+2}, \quad z_{n+2} \leq x_n, \quad x_n \leq y_{n-2}, \quad \Rightarrow y_{n+4} < y_{n-2},$$
$$z_{n+4} \leq x_{n+2}, \quad x_{n+2} \leq y_n, \quad y_n \leq z_{n-2}, \quad \Rightarrow z_{n+4} < z_{n-2},$$

Then the subsequences $\{x_{6n-2}\}_{n=0}^{\infty}$, $\{x_{6n-1}\}_{n=0}^{\infty}$, $\{x_{6n}\}_{n=0}^{\infty}$, $\{x_{6n+1}\}_{n=0}^{\infty}$, $\{x_{6n+2}\}_{n=0}^{\infty}$, $\{x_{6n+3}\}_{n=0}^{\infty}$ are decreasing and so are bounded from above by $M = \max\{x_{-2}, x_{-1}, x_0, x_1, x_2, x_3\}$. Also, the subsequences $\{y_{6n-2}\}_{n=0}^{\infty}$, $\{y_{6n-1}\}_{n=0}^{\infty}$, $\{y_{6n}\}_{n=0}^{\infty}$, $\{y_{6n+1}\}_{n=0}^{\infty}$, $\{y_{6n+2}\}_{n=0}^{\infty}$, $\{y_{6n+3}\}_{n=0}^{\infty}$ are decreasing and so are bounded from above by $M = \max\{y_{-2}, y_{-1}, y_0, y_1, y_2, y_3\}$ and $\{z_{6n-2}\}_{n=0}^{\infty}$, $\{z_{6n-1}\}_{n=0}^{\infty}$, $\{z_{6n}\}_{n=0}^{\infty}$, $\{z_{6n+1}\}_{n=0}^{\infty}$, $\{z_{6n+2}\}_{n=0}^{\infty}$, $\{z_{6n+3}\}_{n=0}^{\infty}$, are decreasing and also, bounded from above by $M = \max\{z_{-2}, z_{-1}, z_0, z_1, z_2, z_3\}$. □

Theorem 2.2. *Suppose that $\{x_n, y_n, z_n\}$ are solutions of system (2.2). Then the solution of system (2.2) are given by the following formula for $n = 0, 1, 2, ...,$*

$$x_{6n-2} = x_{-2}\prod_{i=0}^{n-1}\frac{(z_0+(4i)x_{-2})\,(z_0+(4i+1)x_{-2})}{(z_0+(4i+2)x_{-2})\,(z_0+(4i+3)x_{-2})},$$

$$x_{6n-1} = x_{-1}\prod_{i=0}^{n-1}\frac{((4i)y_0-(4i+1)z_{-2})\,((4i+1)y_0-(4i+2)z_{-2})}{((4i+2)y_0-(4i+3)z_{-2})\,((4i+3)y_0-(4i+4)z_{-2})},$$

$$x_{6n} = x_0\prod_{i=0}^{n-1}\frac{(y_{-2}+(4i+1)x_0)\,(y_{-2}+(4i+2)x_0)}{(y_{-2}+(4i+3)x_0)\,(y_{-2}+(4i+4)x_0)},$$

$$x_{6n+1} = \frac{x_{-2}y_{-1}}{(z_0+x_{-2})}\prod_{i=0}^{n-1}\frac{(z_0+(4i+2)x_{-2})\,(z_0+(4i+3)x_{-2})}{(z_0+(4i+4)x_{-2})\,(z_0+(4i+5)x_{-2})},$$

$$x_{6n+2} = \frac{y_0(y_0-z_{-2})}{(y_0-2z_{-2})}\prod_{i=0}^{n-1}\frac{((4i+2)y_0-(4i+3)z_{-2})\,((4i+3)y_0-(4i+4)z_{-2})}{((4i+4)y_0-(4i+5)z_{-2})\,((4i+5)y_0-(4i+6)z_{-2})},$$

$$x_{6n+3} = \frac{x_0y_{-2}z_{-1}}{(y_{-2}+x_0)(y_{-2}+2x_0)}\prod_{i=0}^{n-1}\frac{(y_{-2}+(4i+3)x_0)\,(y_{-2}+(4i+4)x_0)}{(y_{-2}+(4i+5)x_0)\,(y_{-2}+(4i+6)x_0)},$$

$$y_{6n-2} = y_{-2}\prod_{i=0}^{n-1}\frac{(y_{-2}+(4i+1)x_0)\,(y_{-2}+(4i+2)x_0)}{(y_{-2}+(4i)x_0)\,(y_{-2}+(4i+3)x_0)},$$

$$y_{6n-1} = y_{-1}\prod_{i=0}^{n-1}\frac{(z_0+(4i+2)x_{-2})\,(z_0+(4i+3)x_{-2})}{(z_0+(4i+1)x_{-2})\,(z_0+(4i+4)x_{-2})},$$

$$y_{6n} = y_0\prod_{i=0}^{n-1}\frac{((4i+2)y_0-(4i+3)z_{-2})\,((4i+3)y_0-(4i+4)z_{-2})}{((4i+1)y_0-(4i+2)z_{-2})\,((4i+4)y_0-(4i+5)z_{-2})},$$

$$y_{6n+1} = \frac{y_{-2}z_{-1}}{(y_{-2}+x_0)}\prod_{i=0}^{n-1}\frac{(y_{-2}+(4i+3)x_0)\,(y_{-2}+(4i+4)x_0)}{(y_{-2}+(4i+2)x_0)\,(y_{-2}+(4i+5)x_0)},$$

$$y_{6n+2} = \frac{z_0(z_0+x_{-2})}{(z_0+2x_{-2})}\prod_{i=0}^{n-1}\frac{(z_0+(4i+4)x_{-2})\,(z_0+(4i+5)x_{-2})}{(z_0+(4i+3)x_{-2})\,(z_0+(4i+6)x_{-2})},$$

$$y_{6n+3} = \frac{-x_{-1}z_{-2}(y_0-2z_{-2})}{(y_0-z_{-2})(2y_0-3z_{-2})}\prod_{i=0}^{n-1}\frac{((4i+4)y_0-(4i+5)z_{-2})((4i+5)y_0-(4i+6)z_{-2})}{((4i+3)y_0-(4i+4)z_{-2})\,((4i+6)y_0-(4i+7)z_{-2})},$$

and

$$z_{6n-2} = z_{-2}\prod_{i=0}^{n-1}\frac{((4i-1)y_0-(4i)z_{-2})\,((4i+2)y_0-(4i+3)z_{-2})}{((4i)y_0-(4i+1)z_{-2})\,((4i+1)y_0-(4i+2)z_{-2})},$$

$$z_{6n-1} = z_{-1}\prod_{i=0}^{n-1}\frac{(y_{-2}+(4i)x_0)\,(y_{-2}+(4i+3)x_0)}{(y_{-2}+(4i+1)x_0)\,(y_{-2}+(4i+2)x_0)},$$

$$z_{6n} = z_0\prod_{i=0}^{n-1}\frac{(z_0+(4i+1)x_{-2})\,(z_0+(4i+4)x_{-2})}{(z_0+(4i+2)x_{-2})\,(z_0+(4i+3)x_{-2})},$$

$$z_{6n+1} = \frac{-x_{-1}z_{-2}}{(y_0 - z_{-2})}\prod_{i=0}^{n-1}\frac{((4i+1)y_0 - (4i+2)z_{-2})\,((4i+4)y_0 - (4i+5)z_{-2})}{((4i+2)y_0 - (4i+3)z_{-2})\,((4i+3)y_0 - (4i+4)z_{-2})},$$
$$z_{6n+2} = (x_0 + y_{-2})\prod_{i=0}^{n-1}\frac{(y_{-2} + (4i+2)x_0)\,(y_{-2} + (4i+5)x_0)}{(y_{-2} + (4i+3)x_0)\,(y_{-2} + (4i+4)x_0)},$$
$$z_{6n+3} = \frac{y_{-1}(z_0 + 2x_{-2})}{(z_0 + x_{-2})}\prod_{i=0}^{n-1}\frac{(z_0 + (4i+3)x_{-2})\,(z_0 + (4i+6)x_{-2})}{(z_0 + (4i+4)x_{-2})\,(z_0 + (4i+5)x_{-2})}.$$

Proof. For $n = 0$ the result holds. Now suppose that $n > 1$ and that our assumption holds for $n-1$. that is,

$$x_{6n-5} = \frac{x_{-2}y_{-1}}{(z_0 + x_{-2})}\prod_{i=0}^{n-2}\frac{(z_0 + (4i+2)x_{-2})\,(z_0 + (4i+3)x_{-2})}{(z_0 + (4i+4)x_{-2})\,(z_0 + (4i+5)x_{-2})},$$
$$x_{6n-4} = \frac{y_0(y_0 - z_{-2})}{(y_0 - 2z_{-2})}\prod_{i=0}^{n-2}\frac{((4i+2)y_0 - (4i+3)z_{-2})\,((4i+3)y_0 - (4i+4)z_{-2})}{((4i+4)y_0 - (4i+5)z_{-2})\,((4i+5)y_0 - (4i+6)z_{-2})},$$
$$x_{6n-3} = \frac{x_0y_{-2}z_{-1}}{(y_{-2} + x_0)(y_{-2} + 2x_0)}\prod_{i=0}^{n-2}\frac{(y_{-2} + (4i+3)x_0)\,(y_{-2} + (4i+4)x_0)}{(y_{-2} + (4i+5)x_0)\,(y_{-2} + (4i+6)x_0)},$$
$$y_{6n-5} = \frac{y_{-2}z_{-1}}{(y_{-2} + x_0)}\prod_{i=0}^{n-2}\frac{(y_{-2} + (4i+3)x_0)\,(y_{-2} + (4i+4)x_0)}{(y_{-2} + (4i+2)x_0)\,(y_{-2} + (4i+5)x_0)},$$
$$y_{6n-4} = \frac{z_0(z_0 + x_{-2})}{(z_0 + 2x_{-2})}\prod_{i=0}^{n-2}\frac{(z_0 + (4i+4)x_{-2})\,(z_0 + (4i+5)x_{-2})}{(z_0 + (4i+3)x_{-2})\,(z_0 + (4i+6)x_{-2})},$$
$$y_{6n-3} = \frac{-x_{-1}z_{-2}(y_0 - 2z_{-2})}{(y_0 - z_{-2})(2y_0 - 3z_{-2})}\prod_{i=0}^{n-2}\frac{((4i+4)y_0 - (4i+5)z_{-2})\,((4i+5)y_0 - (4i+6)z_{-2})}{((4i+3)y_0 - (4i+4)z_{-2})((4i+6)y_0 - (4i+7)z_{-2})},$$

and

$$z_{6n-5} = \frac{-x_{-1}z_{-2}}{(y_0 - z_{-2})}\prod_{i=0}^{n-2}\frac{((4i+1)y_0 - (4i+2)z_{-2})\,((4i+4)y_0 - (4i+5)z_{-2})}{((4i+2)y_0 - (4i+3)z_{-2})\,((4i+3)y_0 - (4i+4)z_{-2})},$$
$$z_{6n-4} = (x_0 + y_{-2})\prod_{i=0}^{n-2}\frac{(y_{-2} + (4i+2)x_0)\,(y_{-2} + (4i+5)x_0)}{(y_{-2} + (4i+3)x_0)\,(y_{-2} + (4i+4)x_0)},$$
$$z_{6n-3} = \frac{y_{-1}(z_0 + 2x_{-2})}{(z_0 + x_{-2})}\prod_{i=0}^{n-2}\frac{(z_0 + (4i+3)x_{-2})\,(z_0 + (4i+6)x_{-2})}{(z_0 + (4i+4)x_{-2})\,(z_0 + (4i+5)x_{-2})}.$$

We see from Eq. (2.2) that

$$x_{6n-2} = \frac{y_{6n-4}x_{6n-5}}{x_{6n-5} + z_{6n-3}}$$
$$= \frac{\left(\frac{z_0(z_0+x_{-2})}{(z_0+2x_{-2})}\prod_{i=0}^{n-2}\frac{(z_0+(4i+4)x_{-2})(z_0+(4i+5)x_{-2})}{(z_0+(4i+3)x_{-2})(z_0+(4i+6)x_{-2})}\right)\left(\frac{x_{-2}y_{-1}}{(z_0+x_{-2})}\prod_{i=0}^{n-2}\frac{(z_0+(4i+2)x_{-2})(z_0+(4i+3)x_{-2})}{(z_0+(4i+4)x_{-2})(z_0+(4i+5)x_{-2})}\right)}{\left(\frac{x_{-2}y_{-1}}{(z_0+x_{-2})}\prod_{i=0}^{n-2}\frac{(z_0+(4i+2)x_{-2})(z_0+(4i+3)x_{-2})}{(z_0+(4i+4)x_{-2})(z_0+(4i+5)x_{-2})}\right)+\left(\frac{y_{-1}(z_0+2x_{-2})}{(z_0+x_{-2})}\prod_{i=0}^{n-2}\frac{(z_0+(4i+3)x_{-2})(z_0+(4i+6)x_{-2})}{(z_0+(4i+4)x_{-2})(z_0+(4i+5)x_{-2})}\right)}$$

$$= \frac{\left(\frac{x_{-2}z_0(z_0+x_{-2})}{(z_0+2x_{-2})}\prod_{i=0}^{n-2}\frac{(z_0+(4i+4)x_{-2})(z_0+(4i+5)x_{-2})}{(z_0+(4i+3)x_{-2})(z_0+(4i+6)x_{-2})}\right)}{x_{-2}+\left((z_0+2x_{-2})\prod_{i=0}^{n-2}\frac{(z_0+(4i+6)x_{-2})}{(z_0+(4i+2)x_{-2})}\right)}$$

$$= \frac{\left(\frac{x_{-2}z_0(z_0+x_{-2})}{(z_0+2x_{-2})}\prod_{i=0}^{n-2}\frac{(z_0+(4i+4)x_{-2})(z_0+(4i+5)x_{-2})}{(z_0+(4i+3)x_{-2})(z_0+(4i+6)x_{-2})}\right)}{x_{-2}+z_0+(4n-2)x_{-2}}$$

$$= \frac{x_{-2}z_0(z_0+x_{-2})}{(z_0+2x_{-2})}\prod_{i=0}^{n-2}\frac{(z_0+(4i+4)x_{-2})\,(z_0+(4i+5)x_{-2})}{(z_0+(4i+3)x_{-2})\,(z_0+(4i+6)x_{-2})}\left(\frac{1}{z_0+(4n-1)x_{-2}}\right).$$

Thus

$$x_{6n-2} = x_{-2}\prod_{i=0}^{n-1}\frac{(z_0+(4i)x_{-2})\,(z_0+(4i+1)x_{-2})}{(z_0+(4i+2)x_{-2})\,(z_0+(4i+3)x_{-2})}.$$

Also, we see from System (2.2) that

$$y_{6n} = \frac{z_{6n-2}y_{6n-3}}{y_{6n-3}+x_{6n-1}}$$

$$= \frac{\left(z_{-2}\prod_{i=0}^{n-1}\frac{((4i-1)y_0-(4i)z_{-2})((4i+2)y_0-(4i+3)z_{-2})}{((4i)y_0-(4i+1)z_{-2})((4i+1)y_0-(4i+2)z_{-2})}\right)\left(\frac{-x_{-1}z_{-2}(y_0-2z_{-2})}{(y_0-z_{-2})(2y_0-3z_{-2})}\prod_{i=0}^{n-2}\frac{((4i+4)y_0-(4i+5)z_{-2})((4i+5)y_0-(4i+6)z_{-2})}{((4i+3)y_0-(4i+4)z_{-2})((4i+6)y_0-(4i+7)z_{-2})}\right)}{\left(\frac{-x_{-1}z_{-2}(y_0-2z_{-2})}{(y_0-z_{-2})(2y_0-3z_{-2})}\prod_{i=0}^{n-2}\frac{((4i+4)y_0-(4i+5)z_{-2})((4i+5)y_0-(4i+6)z_{-2})}{((4i+3)y_0-(4i+4)z_{-2})((4i+6)y_0-(4i+7)z_{-2})}\right)+\left(x_{-1}\prod_{i=0}^{n-1}\frac{((4i)y_0-(4i+1)z_{-2})((4i+1)y_0-(4i+2)z_{-2})}{((4i+2)y_0-(4i+3)z_{-2})((4i+3)y_0-(4i+4)z_{-2})}\right)}$$

$$= \frac{\left(z_{-2}\prod_{i=0}^{n-1}\frac{((4i-1)y_0-(4i)z_{-2})((4i+2)y_0-(4i+3)z_{-2})}{((4i)y_0-(4i+1)z_{-2})((4i+1)y_0-(4i+2)z_{-2})}\right)}{1+\left(\left(\prod_{i=0}^{n-1}\frac{((4i)y_0-(4i+1)z_{-2})((4i+1)y_0-(4i+2)z_{-2})}{((4i+2)y_0-(4i+3)z_{-2})((4i+3)y_0-(4i+4)z_{-2})}\right)\left(\frac{(y_0-z_{-2})(2y_0-3z_{-2})}{-z_{-2}(y_0-2z_{-2})}\prod_{i=0}^{n-2}\frac{((4i+3)y_0-(4i+4)z_{-2})((4i+6)y_0-(4i+7)z_{-2})}{((4i+4)y_0-(4i+5)z_{-2})((4i+5)y_0-(4i+6)z_{-2})}\right)\right)}$$

$$= \frac{\left(z_{-2}\prod_{i=0}^{n-1}\frac{((4i-1)y_0-(4i)z_{-2})((4i+2)y_0-(4i+3)z_{-2})}{((4i)y_0-(4i+1)z_{-2})((4i+1)y_0-(4i+2)z_{-2})}\right)}{1+\left(\prod_{i=0}^{n-1}\frac{1}{((4i+3)y_0-(4i+4)z_{-2})}\right)\left((y_0-z_{-2})\prod_{i=0}^{n-2}((4i+3)y_0-(4i+4)z_{-2})\right)}$$

$$= \frac{\left(z_{-2}\prod_{i=0}^{n-1}\frac{((4i-1)y_0-(4i)z_{-2})((4i+2)y_0-(4i+3)z_{-2})}{((4i)y_0-(4i+1)z_{-2})((4i+1)y_0-(4i+2)z_{-2})}\right)}{1+\left(\frac{(y_0-z_{-2})}{((4n-1)y_0-(4n)z_{-2})}\right)}$$

$$= z_{-2}\prod_{i=0}^{n-1}\frac{((4i-1)y_0-(4i)z_{-2})((4i+2)y_0-(4i+3)z_{-2})}{((4i)y_0-(4i+1)z_{-2})((4i+1)y_0-(4i+2)z_{-2})}\left(\frac{(4n-1)y_0-(4n)z_{-2}}{(4n)y_0-(4n+1)z_{-2}}\right).$$

Then

$$y_{6n} = y_0\prod_{i=0}^{n-1}\frac{((4i+2)y_0-(4i+3)z_{-2})\,((4i+3)y_0-(4i+4)z_{-2})}{((4i+1)y_0-(4i+2)z_{-2})\,((4i+4)y_0-(4i+5)z_{-2})}.$$

Although, from Eq. (2.2), we see that

$$z_{6n+2} = \frac{x_{6n}z_{6n-1}}{z_{6n-1} - y_{6n+1}}$$

$$= \frac{\left(x_0 \prod_{i=0}^{n-1} \frac{(y_{-2}+(4i+1)x_0)(y_{-2}+(4i+2)x_0)}{(y_{-2}+(4i+3)x_0)(y_{-2}+(4i+4)x_0)}\right)\left(z_{-1} \prod_{i=0}^{n-1} \frac{(y_{-2}+(4i)x_0)(y_{-2}+(4i+3)x_0)}{(y_{-2}+(4i+1)x_0)(y_{-2}+(4i+2)x_0)}\right)}{\left(z_{-1} \prod_{i=0}^{n-1} \frac{(y_{-2}+(4i)x_0)(y_{-2}+(4i+3)x_0)}{(y_{-2}+(4i+1)x_0)(y_{-2}+(4i+2)x_0)}\right) - \left(\frac{y_{-2}z_{-1}}{(y_{-2}+x_0)} \prod_{i=0}^{n-1} \frac{(y_{-2}+(4i+3)x_0)(y_{-2}+(4i+4)x_0)}{(y_{-2}+(4i+2)x_0)(y_{-2}+(4i+5)x_0)}\right)}$$

$$= \frac{\left(x_0 \prod_{i=0}^{n-1} \frac{(y_{-2}+(4i+1)x_0)(y_{-2}+(4i+2)x_0)}{(y_{-2}+(4i+3)x_0)(y_{-2}+(4i+4)x_0)}\right)}{1 - \left(\frac{y_{-2}}{(y_{-2}+x_0)} \prod_{i=0}^{n-1} \frac{(y_{-2}+(4i+1)x_0)(y_{-2}+(4i+4)x_0)}{(y_{-2}+(4i)x_0)(y_{-2}+(4i+5)x_0)}\right)}$$

$$= \frac{\left(x_0 \prod_{i=0}^{n-1} \frac{(y_{-2}+(4i+1)x_0)(y_{-2}+(4i+2)x_0)}{(y_{-2}+(4i+3)x_0)(y_{-2}+(4i+4)x_0)}\right)}{1 - \left(\frac{(y_{-2}+(4n)x_0)}{(y_{-2}+(4n+1)x_0)}\right)}$$

$$= x_0 \prod_{i=0}^{n-1} \frac{(y_{-2}+(4i+1)x_0)(y_{-2}+(4i+2)x_0)}{(y_{-2}+(4i+3)x_0)(y_{-2}+(4i+4)x_0)} \left(\frac{(y_{-2}+(4n+1)x_0)}{x_0}\right).$$

Thus

$$z_{6n+2} = (x_0 + y_{-2}) \prod_{i=0}^{n-1} \frac{(y_{-2} + (4i+2)x_0)(y_{-2} + (4i+5)x_0)}{(y_{-2} + (4i+3)x_0)(y_{-2} + (4i+4)x_0)}.$$

Similarly one can prove the other formulas. This completes the proof. □

Theorem 2.3. *If $\{x_n, y_n, z_n\}$ are solutions of difference equation system (2.3). Then the solution of system (2.3) are takes the following form for $n = 0, 1, 2, ...$,*

$$x_{6n-2} = x_{-2}\left(\frac{x_{-2}+z_0}{x_{-2}-z_0}\right)^n, \quad x_{6n-1} = x_{-1}\left(\frac{y_0-2z_{-2}}{y_0}\right)^n, \qquad x_{6n} = x_0\left(\frac{2x_0-y_{-2}}{y_{-2}}\right)^n,$$

$$x_{6n+1} = \frac{x_{-2}y_{-1}}{x_{-2}+z_0}\left(\frac{x_{-2}-z_0}{x_{-2}+z_0}\right)^n, \quad x_{6n+2} = (y_0 - z_{-2})\left(\frac{y_0}{y_0-2z_{-2}}\right)^{n+1},$$

$$x_{6n+3} = \frac{-x_0z_{-1}}{(x_0-y_{-2})}\left(\frac{y_{-2}}{2x_0-y_{-2}}\right)^{n+1},$$

$$y_{6n-2} = y_{-2}\left(\frac{2x_0-y_{-2}}{y_{-2}}\right)^n, \quad y_{6n-1} = y_{-1}\left(\frac{x_{-2}-z_0}{x_{-2}+z_0}\right)^n, \quad y_{6n} = y_0\left(\frac{y_0}{y_0-2z_{-2}}\right)^n,$$

$$y_{6n+1} = \frac{z_{-1}y_{-2}}{y_{-2}-x_0}\left(\frac{y_{-2}}{2x_0-y_{-2}}\right)^n, \quad y_{6n+2} = (x_{-2} + z_0)\left(\frac{x_{-2}+z_0}{x_{-2}-z_0}\right)^n,$$

$$y_{6n+3} = \frac{x_{-1}y_0}{y_0-z_{-2}}\left(\frac{y_0-2z_{-2}}{y_0}\right)^{n+1},$$

and

$$z_{6n-2} = z_{-2}\left(\frac{y_0}{y_0-2z_{-2}}\right)^n, \quad z_{6n-1} = z_{-1}\left(\frac{y_{-2}}{2x_0-y_{-2}}\right)^n, \quad z_{6n} = z_0\left(\frac{x_{-2}+z_0}{x_{-2}-z_0}\right)^n,$$

$$z_{6n+1} = \frac{x_{-1}z_{-2}}{z_{-2}-y_0}\left(\frac{y_0-2z_{-2}}{y_0}\right)^n, \quad z_{6n+2} = (x_0 - y_{-2})\left(\frac{2x_0-y_{-2}}{y_{-2}}\right)^n,$$

$$z_{6n+3} = \frac{-y_{-1}z_0}{(x_{-2}+z_0)}\left(\frac{x_{-2}-z_0}{x_{-2}+z_0}\right)^n,$$

where $y_0 \neq 2z_{-2}$, $y_0 \neq z_{-2}$, $y_{-2} \neq 2x_0$, $y_{-2} \neq x_0$, and $x_{-2} \neq \pm z_0$.

Proof. The proof as in the proof of Theorems 2.1, 2.2 and so will be left to the reader. □

Here for confirming the results of this section, we consider an interesting numerical examples of the systems (2.1)–(2.3).

Example 2.1. We consider this example for the difference system (2.1) with the initial conditions $x_{-2} = 12$, $x_{-1} = 6$, $x_0 = -8$, $y_{-2} = 3$, $y_{-1} = -7$, $y_0 = -4$, $z_{-2} = 5$, $z_{-1} = -16$ and $z_0 = 9$. (See Fig. 1).

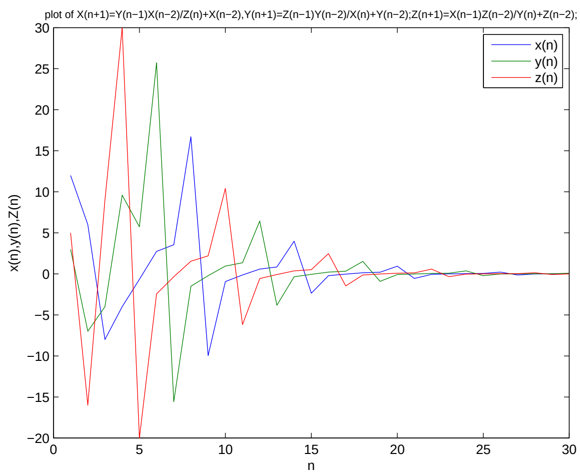

Figure 1.

Example 2.2. See Figure 2 for an example for the system (2.2) with the initial values $x_{-2} = -.2$, $x_{-1} = -.6$, $x_0 = .8$, $y_{-2} = .3$, $y_{-1} = .7$, $y_0 = .4$, $z_{-2} = -.5$, $z_{-1} = .6$ and $z_0 = .9$.

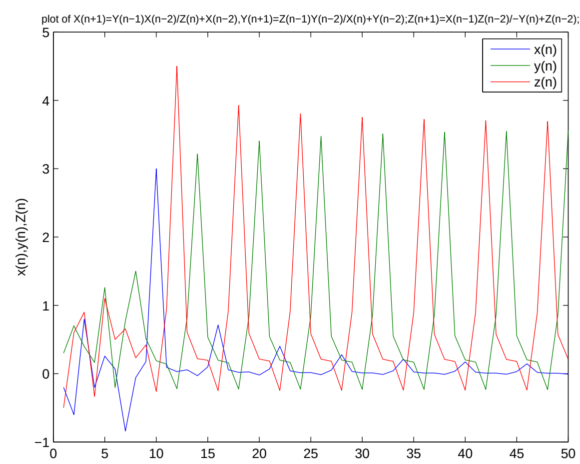

Figure 2.

Example 2.3. We assume the initial conditions $x_{-2} = .5$, $x_{-1} = .56$, $x_0 = 2$, $y_{-2} = -3$, $y_{-1} = .7$, $y_0 = 2.5$, $z_{-2} = 1.5$, $z_{-1} = .6$ and $z_0 = -1.5$, for the difference system (2.3), see Fig. 3.

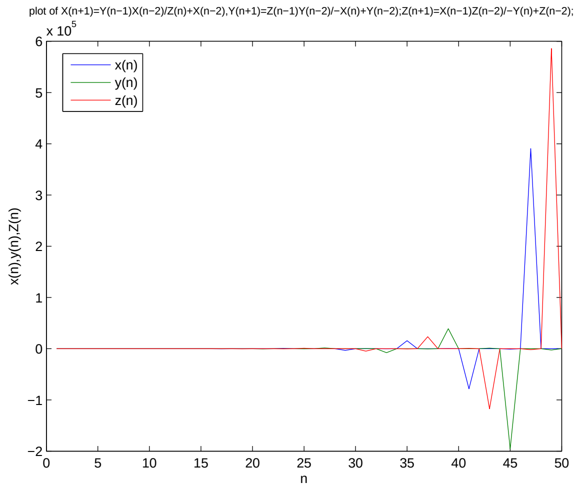

Figure 3.

3. Periodicity of Some Systems:

In this section, we investigate the periodic nature of the solutions of the following systems of three difference equations

$$x_{n+1} = \frac{y_{n-1}x_{n-2}}{x_{n-2} - z_n},\ y_{n+1} = \frac{z_{n-1}y_{n-2}}{y_{n-2} - x_n},\ z_{n+1} = \frac{x_{n-1}z_{n-2}}{z_{n-2} - y_n}. \tag{3.1}$$

$$x_{n+1} = \frac{y_{n-1}x_{n-2}}{-x_{n-2} - z_n},\ y_{n+1} = \frac{z_{n-1}y_{n-2}}{-y_{n-2} - x_n},\ z_{n+1} = \frac{x_{n-1}z_{n-2}}{-z_{n-2} - y_n}. \tag{3.2}$$

where $n \in \mathbb{N}_0$ and the initial conditions are arbitrary non zero real numbers.

The following theorem is devoted to the expressions and the periodicity of the solutions of systems (3.1), (3.2).

Theorem 3.1. *Suppose that* $\{x_n, y_n, z_n\}$ *are solutions of system* (3.1) *such that* $y_0 \neq z_{-2},\ y_{-2} \neq x_0,\ x_{-2} \neq z_0$. *Then every solutions of system* (3.1) *are periodic with period twelve and given by the following formula for* $n = 0, 1, 2, ...,$

$$x_{12n-2} = x_{-2},\ x_{12n-1} = x_{-1},\ x_{12n} = x_0,\ x_{12n+1} = \frac{x_{-2}y_{-1}}{x_{-2} - z_0},\ x_{12n+2} = y_0 - z_{-2},$$

$$x_{12n+3} = \frac{x_0 z_{-1}}{y_{-2} - x_0},\quad x_{12n+4} = -x_{-2},\quad x_{12n+5} = -x_{-1},\quad x_{12n+6} = -x_0,$$

$$x_{12n+7} = -\frac{x_{-2}y_{-1}}{x_{-2} - z_0},\quad x_{12n+8} = -(y_0 - z_{-2}),\ x_{12n+9} = -\frac{x_0 z_{-1}}{y_{-2} - x_0},$$

$$y_{12n-2} = y_{-2},\ y_{12n-1} = y_{-1},\ y_{12n} = y_0,\ y_{12n+1} = \frac{y_{-2}z_{-1}}{y_{-2} - x_0},\ y_{12n+2} = z_0 - x_{-2},$$

$$y_{12n+3}=\frac{x_{-1}y_0}{z_{-2}-y_0},\quad y_{12n+4}=-y_{-2},\quad y_{12n+5}=-y_{-1},\quad y_{12n+6}=-y_0,$$
$$y_{12n+7}=-\frac{y_{-2}z_{-1}}{y_{-2}-x_0},\quad y_{12n+8}=-(z_0-x_{-2}),\ y_{12n+9}=-\frac{x_{-1}y_0}{z_{-2}-y_0},$$

and

$$z_{12n-2}=z_{-2},\ z_{12n-1}=z_{-1},\ z_{12n}=z_0,\ z_{12n+1}=\frac{x_{-1}z_{-2}}{z_{-2}-y_0},\ z_{12n+2}=x_0-y_{-2},$$
$$z_{12n+3}=\frac{y_{-1}z_0}{x_{-2}-z_0},\quad z_{12n+4}=-z_{-2},\quad z_{12n+5}=-z_{-1},\quad z_{12n+6}=-z_0,$$
$$z_{12n+7}=-\frac{x_{-1}z_{-2}}{z_{-2}-y_0},\quad z_{12n+8}=-(x_0-y_{-2}),\ z_{12n+9}=-\frac{y_{-1}z_0}{x_{-2}-z_0}.$$

Or equivalently,

$$\{x_n\}_{n=-2}^{+\infty}=\left\{\begin{array}{c} x_{-2},x_{-1},x_0,\frac{x_{-2}y_{-1}}{x_{-2}-z_0},y_0-z_{-2},\frac{x_0z_{-1}}{y_{-2}-x_0},-x_{-2},-x_{-1},\\ -x_0,-\frac{x_{-2}y_{-1}}{x_{-2}-z_0},-(y_0-z_{-2}),-\frac{x_0z_{-1}}{y_{-2}-x_0},x_{-2},x_{-1},... \end{array}\right\},$$
$$\{y_n\}_{n=-2}^{+\infty}=\left\{\begin{array}{c} y_{-2},y_{-1},y_0,\frac{y_{-2}z_{-1}}{y_{-2}-x_0},z_0-x_{-2},\frac{x_{-1}y_0}{z_{-2}-y_0},-y_{-2},-y_{-1},\\ -y_0,-\frac{y_{-2}z_{-1}}{y_{-2}-x_0},-(z_0-x_{-2}),-\frac{x_{-1}y_0}{z_{-2}-y_0},y_{-2},y_{-1},... \end{array}\right\},$$
$$\{y_n\}_{n=-2}^{+\infty}=\left\{\begin{array}{c} z_{-2},z_{-1},z_0,\frac{x_{-1}z_{-2}}{z_{-2}-y_0},x_0-y_{-2},\frac{y_{-1}z_0}{x_{-2}-z_0},-z_{-2},-z_{-1},\\ -z_0,-\frac{x_{-1}z_{-2}}{z_{-2}-y_0},-(x_0-y_{-2}),-\frac{y_{-1}z_0}{x_{-2}-z_0},z_{-2},z_{-1},... \end{array}\right\}.$$

Proof. For $n=0$ the result holds. Now suppose that $n>0$ and that our assumption holds for $n-1$. that is,

$$x_{12n-5}=-\frac{x_{-2}y_{-1}}{x_{-2}-z_0},\quad x_{12n-4}=-(y_0-z_{-2}),\ x_{12n-3}=-\frac{x_0z_{-1}}{y_{-2}-x_0},$$
$$y_{12n-5}=-\frac{y_{-2}z_{-1}}{y_{-2}-x_0},\quad y_{12n-4}=-(z_0-x_{-2}),\ y_{12n-3}=-\frac{x_{-1}y_0}{z_{-2}-y_0},$$
$$z_{12n-5}=-\frac{x_{-1}z_{-2}}{z_{-2}-y_0},\quad z_{12n-4}=-(x_0-y_{-2}),\ z_{12n-3}=-\frac{y_{-1}z_0}{x_{-2}-z_0}.$$

Now from Eq. (3.1) it follows that

$$x_{6n-2}=\frac{y_{6n-4}x_{6n-5}}{x_{6n-5}-z_{6n-3}}=\frac{-(z_0-x_{-2})\left(-\frac{x_{-2}y_{-1}}{x_{-2}-z_0}\right)}{\left(-\frac{x_{-2}y_{-1}}{x_{-2}-z_0}\right)-\left(-\frac{y_{-1}z_0}{x_{-2}-z_0}\right)}=x_{-2},$$
$$y_{6n-1}=\frac{z_{6n-3}y_{6n-4}}{y_{6n-4}-x_{6n-2}}=\frac{\frac{y_{-1}z_0}{x_{-2}-z_0}(z_0-x_{-2})}{-(z_0-x_{-2})-x_{-2}}=y_{-1},$$
$$z_{6n}=\frac{x_{6n-2}z_{6n-3}}{z_{6n-3}-y_{6n-1}}=\frac{x_{-2}\left(-\frac{y_{-1}z_0}{x_{-2}-z_0}\right)}{\left(-\frac{y_{-1}z_0}{x_{-2}-z_0}\right)-y_{-1}}=z_0.$$

The other relations can be proved similarly. The proof is complete. □

Theorem 3.2. *Assume that* $\{x_n,y_n,z_n\}$ *are solutions of system* (3.2), *with* $y_0\neq -z_{-2},\ y_{-2}\neq -x_0,\ x_{-2}\neq -z_0$, *then the following statements are true:-*

1. $\{x_n\}_{n=-2}^{+\infty}$, $\{y_n\}_{n=-2}^{+\infty}$ *and* $\{z_n\}_{n=-2}^{+\infty}$ *are periodic with period six i.e.,* $x_{n+6} = x_n,\ y_{n+6} = y_n,\ z_{n+6} = z_n$ *for* $n \geq -2$.

2. We have

$$x_{6n-2} = x_{-2},\ x_{6n-1} = x_{-1},\ x_{6n} = x_0,\ x_{6n+1} = \tfrac{-x_{-2}y_{-1}}{x_{-2}+z_0},$$
$$x_{6n+2} = -y_0 - z_{-2},\ x_{6n+3} = \tfrac{-x_0 z_{-1}}{x_0+y_{-2}},$$
$$y_{6n-2} = y_{-2},\ y_{6n-1} = y_{-1},\ y_{6n} = y_0,\ y_{6n+1} = \tfrac{-y_{-2}z_{-1}}{x_0+y_{-2}},$$
$$y_{6n+2} = -z_0 - x_{-2},\ y_{6n+3} = \tfrac{-x_{-1}y_0}{z_{-2}+y_0},$$
$$z_{6n-2} = z_{-2},\ z_{6n-1} = z_{-1},\ z_{6n} = z_0,\ z_{6n+1} = \tfrac{-x_{-1}z_{-2}}{z_{-2}+y_0},$$
$$z_{6n+2} = -x_0 - y_{-2},\ z_{6n+3} = \tfrac{-y_{-1}z_0}{x_{-2}+z_0}.$$

Or equivalently,

$$\{x_n\}_{n=-2}^{+\infty} = \left\{x_{-2}, x_{-1}, x_0, \frac{-x_{-2}y_{-1}}{x_{-2}+z_0}, -y_0 - z_{-2}, \frac{-x_0 z_{-1}}{y_{-2}+x_0}, x_{-2}, x_{-1}, x_0, ...\right\},$$
$$\{y_n\}_{n=-2}^{+\infty} = \left\{y_{-2}, y_{-1}, y_0, \frac{-y_{-2}z_{-1}}{y_{-2}+x_0}, -z_0 - x_{-2}, \frac{-x_{-1}y_0}{z_{-2}+y_0}, y_{-2}, y_{-1}, y_0, ...\right\},$$
$$\{y_n\}_{n=-2}^{+\infty} = \left\{z_{-2}, z_{-1}, z_0, \frac{-x_{-1}z_{-2}}{z_{-2}+y_0}, -x_0 - y_{-2}, \frac{-y_{-1}z_0}{x_{-2}+z_0}, z_{-2}, z_{-1}, z_0, ...\right\}.$$

Example 3.1. Figure 4 shows the behavior of the solution of the difference system (3.1) with the initial conditions $x_{-2} = .9,\ x_{-1} = 6,\ x_0 = -2,\ y_{-2} = -3,\ y_{-1} = 7,\ y_0 = 2,\ z_{-2} = 1.5,\ z_{-1} = .8$ and $z_0 = 11$.

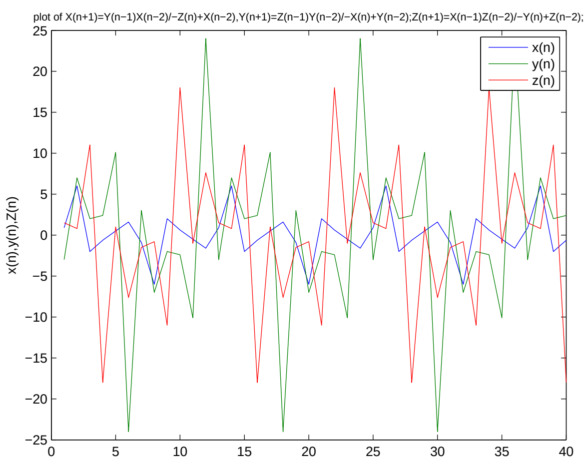

Figure 4.

Example 3.2. See Figure 5 to know the periodicity of the solution of the difference system (3.2) with the initial conditions $x_{-2} = 7,\ x_{-1} = -2,\ x_0 = 5,\ y_{-2} = -3,\ y_{-1} = 9,\ y_0 = .7,\ z_{-2} = -11,\ z_{-1} = 8$ and $z_0 = .14$.

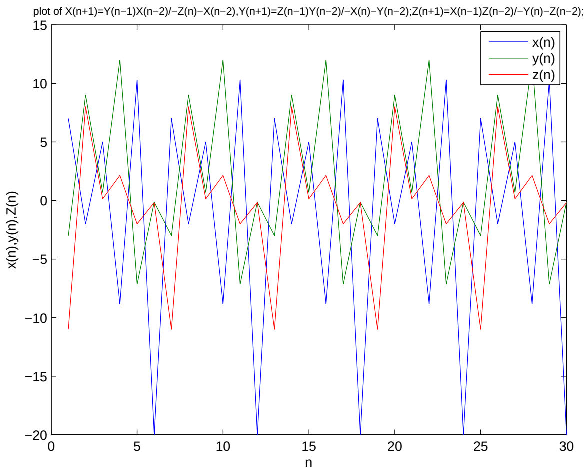

Figure 5.

4. Other Systems:

In this section, we get the solutions of the following systems of the difference equations

$$x_{n+1} = \frac{y_{n-1}x_{n-2}}{x_{n-2}+z_n},\ y_{n+1} = \frac{z_{n-1}y_{n-2}}{y_{n-2}-x_n},\ z_{n+1} = \frac{x_{n-1}z_{n-2}}{z_{n-2}+y_n}. \tag{4.1}$$

$$x_{n+1} = \frac{y_{n-1}x_{n-2}}{x_{n-2}-z_n},\ y_{n+1} = \frac{z_{n-1}y_{n-2}}{y_{n-2}+x_n},\ z_{n+1} = \frac{x_{n-1}z_{n-2}}{z_{n-2}+y_n}. \tag{4.2}$$

$$x_{n+1} = \frac{y_{n-1}x_{n-2}}{x_{n-2}-z_n},\ y_{n+1} = \frac{z_{n-1}y_{n-2}}{y_{n-2}+x_n},\ z_{n+1} = \frac{x_{n-1}z_{n-2}}{z_{n-2}-y_n}. \tag{4.3}$$

$$x_{n+1} = \frac{y_{n-1}x_{n-2}}{x_{n-2}-z_n},\ y_{n+1} = \frac{z_{n-1}y_{n-2}}{y_{n-2}-x_n},\ z_{n+1} = \frac{x_{n-1}z_{n-2}}{z_{n-2}+y_n}. \tag{4.4}$$

where $n \in \mathbb{N}_0$ and the initial conditions are arbitrary nonzero real numbers.

Theorem 4.1. *If* $\{x_n, y_n, z_n\}$ *are solutions of difference equation system* (4.1). *Then the solution of system* (4.1) *are takes the following form for* $n = 0, 1, 2, \ldots$,

$$x_{6n-2} = x_{-2}\prod_{i=0}^{n-1}\frac{(x_{-2}+(4i+1)z_0)\,(x_{-2}+(4i+2)z_0)}{(x_{-2}+(4i)z_0)\,(x_{-2}+(4i+3)z_0)},$$

$$x_{6n-1} = x_{-1}\prod_{i=0}^{n-1}\frac{(y_0+(4i+2)z_{-2})\,(y_0+(4i+3)z_{-2})}{(y_0+(4i+1)z_{-2})\,(y_0+(4i+4)z_{-2})},$$

$$x_{6n} = x_0\prod_{i=0}^{n-1}\frac{((4i+2)x_0-(4i+3)y_{-2})\,((4i+3)x_0-(4i+4)y_{-2})}{((4i+1)x_0-(4i+2)y_{-2})\,((4i+4)x_0-(4i+5)y_{-2})},$$

$$x_{6n+1} = \frac{x_{-2}y_{-1}}{(z_0+x_{-2})}\prod_{i=0}^{n-1}\frac{(x_{-2}+(4i+3)z_0)\,(x_{-2}+(4i+4)z_0)}{(x_{-2}+(4i+2)z_0)\,(x_{-2}+(4i+5)z_0)},$$

$$x_{6n+2} = \frac{y_0(y_0+z_{-2})}{(y_0+2z_{-2})}\prod_{i=0}^{n-1}\frac{(y_0+(4i+4)z_{-2})\,(y_0+(4i+5)z_{-2})}{(y_0+(4i+3)z_{-2})\,(y_0+(4i+6)z_{-2})},$$

$$x_{6n+3} = \frac{-y_{-2}z_{-1}(x_0-2y_{-2})}{(x_0-y_{-2})(2x_0-3y_{-2})}\prod_{i=0}^{n-1}\frac{((4i+4)x_0-(4i+5)y_{-2})\,((4i+5)x_0-(4i+6)y_{-2})}{((4i+3)x_0-(4i+4)y_{-2})((4i+6)x_0-(4i+7)y_{-2})},$$

$$y_{6n-2} = y_{-2}\prod_{i=0}^{n-1}\frac{((4i-1)x_0-(4i)y_{-2})\,((4i+2)x_0-(4i+3)y_{-2})}{((4i)x_0-(4i+1)y_{-2})\,((4i+1)x_0-(4i+2)y_{-2})},$$

$$y_{6n-1} = y_{-1}\prod_{i=0}^{n-1}\frac{(x_{-2}+(4i)z_0)\,(x_{-2}+(4i+3)z_0)}{(x_{-2}+(4i+1)z_0)\,(x_{-2}+(4i+2)z_0)},$$

$$y_{6n} = y_0\prod_{i=0}^{n-1}\frac{(y_0+(4i+1)z_{-2})\,(y_0+(4i+4)z_{-2})}{(y_0+(4i+2)z_{-2})\,(y_0+(4i+3)z_{-2})},$$

$$y_{6n+1} = \frac{-y_{-2}z_{-1}}{(x_0-y_{-2})}\prod_{i=0}^{n-1}\frac{((4i+1)x_0-(4i+2)y_{-2})\,((4i+4)x_0-(4i+5)y_{-2})}{((4i+2)x_0-(4i+3)y_{-2})\,((4i+3)x_0-(4i+4)y_{-2})},$$

$$y_{6n+2} = (x_{-2}+z_0)\prod_{i=0}^{n-1}\frac{(x_{-2}+(4i+2)z_0)\,(x_{-2}+(4i+5)z_0)}{(x_{-2}+(4i+3)z_0)\,(x_{-2}+(4i+4)z_0)},$$

$$y_{6n+3} = \frac{x_{-1}(y_0+2z_{-2})}{(y_0+z_{-2})}\prod_{i=0}^{n-1}\frac{(y_0+(4i+3)z_{-2})\,(y_0+(4i+6)z_{-2})}{(y_0+(4i+4)z_{-2})\,(y_0+(4i+5)z_{-2})},$$

$$z_{6n-2} = z_{-2}\prod_{i=0}^{n-1}\frac{(y_0+(4i)z_{-2})\,(y_0+(4i+1)z_{-2})}{(y_0+(4i+2)z_{-2})\,(y_0+(4i+3)z_{-2})},$$

$$z_{6n-1} = z_{-1}\prod_{i=0}^{n-1}\frac{((4i)x_0-(4i+1)y_{-2})\,((4i+1)x_0-(4i+2)y_{-2})}{((4i+2)x_0-(4i+3)y_{-2})\,((4i+3)x_0-(4i+4)y_{-2})},$$

$$z_{6n} = z_0\prod_{i=0}^{n-1}\frac{(x_{-2}+(4i+1)z_0)\,(x_{-2}+(4i+2)z_0)}{(x_{-2}+(4i+3)z_0)\,(x_{-2}+(4i+4)z_0)},$$

$$z_{6n+1} = \frac{x_{-1}z_{-2}}{(y_0+z_{-2})}\prod_{i=0}^{n-1}\frac{(y_0+(4i+2)z_{-2})\,(y_0+(4i+3)z_{-2})}{(y_0+(4i+4)z_{-2})\,(y_0+(4i+5)z_{-2})},$$

$$z_{6n+2} = \frac{x_0(x_0-y_{-2})}{(x_0-2y_{-2})}\prod_{i=0}^{n-1}\frac{((4i+2)x_0-(4i+3)y_{-2})\,((4i+3)x_0-(4i+4)y_{-2})}{((4i+4)x_0-(4i+5)y_{-2})\,((4i+5)x_0-(4i+6)y_{-2})},$$

$$z_{6n+3} = \frac{x_{-2}y_{-1}z_0}{(x_{-2}+z_0)(x_{-2}+2z_0)}\prod_{i=0}^{n-1}\frac{(x_{-2}+(4i+3)z_0)\,(x_{-2}+(4i+4)z_0)}{(x_{-2}+(4i+5)z_0)\,(x_{-2}+(4i+6)z_0)}.$$

Theorem 4.2. *Assume that* $\{x_n, y_n, z_n\}$ *are solutions of system* (4.2)*. Then for* $n = 0, 1, 2, ...,$

$$x_{6n-2} = x_{-2}\prod_{i=0}^{n-1}\frac{((4i)x_{-2}-(4i-1)z_0)\,((4i+3)x_{-2}-(4i+2)z_0)}{((4i+1)x_{-2}-(4i)z_0)\,((4i+2)x_{-2}-(4i+1)z_0)},$$

$$x_{6n-1} = x_{-1}\prod_{i=0}^{n-1}\frac{(z_{-2}+(4i)y_0)\,(z_{-2}+(4i+3)y_0)}{(z_{-2}+(4i+1)y_0)\,(z_{-2}+(4i+2)y_0)},$$

$$x_{6n} = x_0\prod_{i=0}^{n-1}\frac{(x_0+(4i+1)y_{-2})\,(x_0+(4i+4)y_{-2})}{(x_0+(4i+2)y_{-2})\,(x_0+(4i+3)y_{-2})},$$

$$x_{6n+1} = \frac{x_{-2}y_{-1}}{(x_{-2}-z_0)} \prod_{i=0}^{n-1} \frac{((4i+2)x_{-2}-(4i+1)z_0)\,((4i+5)x_{-2}-(4i+4)z_0)}{((4i+3)x_{-2}-(4i+2)z_0)\,((4i+4)x_{-2}-(4i+3)z_0)},$$

$$x_{6n+2} = (y_0+z_{-2}) \prod_{i=0}^{n-1} \frac{(z_{-2}+(4i+2)y_0)\,(z_{-2}+(4i+5)y_0)}{(z_{-2}+(4i+3)y_0)\,(z_{-2}+(4i+4)y_0)},$$

$$x_{6n+3} = \frac{z_{-1}(x_0+2y_{-2})}{(x_0+y_{-2})} \prod_{i=0}^{n-1} \frac{(x_0+(4i+3)y_{-2})\,(x_0+(4i+6)y_{-2})}{(x_0+(4i+4)y_{-2})\,(x_0+(4i+5)y_{-2})},$$

$$y_{6n-2} = y_{-2} \prod_{i=0}^{n-1} \frac{(x_0+(4i)y_{-2})\,(x_0+(4i+1)y_{-2})}{(x_0+(4i+2)y_{-2})\,(x_0+(4i+3)y_{-2})},$$

$$y_{6n-1} = y_{-1} \prod_{i=0}^{n-1} \frac{((4i+1)x_{-2}-(4i)z_0)\,((4i+2)x_{-2}-(4i+1)z_0)}{((4i+3)x_{-2}-(4i+2)z_0)\,((4i+4)x_{-2}-(4i+3)z_0)},$$

$$y_{6n} = y_0 \prod_{i=0}^{n-1} \frac{(z_{-2}+(4i+1)y_0)\,(z_{-2}+(4i+2)y_0)}{(z_{-2}+(4i+3)y_0)\,(z_{-2}+(4i+4)y_0)},$$

$$y_{6n+1} = \frac{y_{-2}z_{-1}}{(x_0+y_{-2})} \prod_{i=0}^{n-1} \frac{(x_0+(4i+2)y_{-2})\,(x_0+(4i+3)y_{-2})}{(x_0+(4i+4)y_{-2})\,(x_0+(4i+5)y_{-2})},$$

$$y_{6n+2} = \frac{z_0(x_{-2}-z_0)}{(2x_{-2}-z_0)} \prod_{i=0}^{n-1} \frac{((4i+4)x_{-2}-(4i+3)z_0)\,((4i+3)x_{-2}-(4i+2)z_0)}{((4i+5)x_{-2}-(4i+4)z_0)\,((4i+6)x_{-2}-(4i+5)z_0)},$$

$$y_{6n+3} = \frac{x_{-1}y_0z_{-2}}{(z_{-2}+y_0)(z_{-2}+2y_0)} \prod_{i=0}^{n-1} \frac{(z_{-2}+(4i+3)y_0)\,(z_{-2}+(4i+4)y_0)}{(z_{-2}+(4i+5)y_0)\,(z_{-2}+(4i+6)y_0)},$$

$$z_{6n-2} = z_{-2} \prod_{i=0}^{n-1} \frac{(z_{-2}+(4i+1)y_0)\,(z_{-2}+(4i+2)y_0)}{(z_{-2}+(4i)y_0)\,(z_{-2}+(4i+3)y_0)},$$

$$z_{6n-1} = z_{-1} \prod_{i=0}^{n-1} \frac{(x_0+(4i+2)y_{-2})\,(x_0+(4i+3)y_{-2})}{(x_0+(4i+1)y_{-2})\,(x_0+(4i+4)y_{-2})},$$

$$z_{6n} = z_0 \prod_{i=0}^{n-1} \frac{((4i+3)x_{-2}-(4i+2)z_0)\,((4i+4)x_{-2}-(4i+3)z_0)}{((4i+2)x_{-2}-(4i+1)z_0)\,((4i+5)x_{-2}-(4i+4)z_0)},$$

$$z_{6n+1} = \frac{x_{-1}z_{-2}}{(y_0+z_{-2})} \prod_{i=0}^{n-1} \frac{(z_{-2}+(4i+3)y_0)\,(z_{-2}+(4i+4)y_0)}{(z_{-2}+(4i+2)y_0)\,(z_{-2}+(4i+5)y_0)},$$

$$z_{6n+2} = \frac{x_0(x_0+y_{-2})}{(x_0+2y_{-2})} \prod_{i=0}^{n-1} \frac{(x_0+(4i+4)y_{-2})\,(x_0+(4i+5)y_{-2})}{(x_0+(4i+3)y_{-2})\,(x_0+(4i+6)y_{-2})},$$

$$z_{6n+3} = \frac{x_{-2}y_{-1}(2x_{-2}-z_0)}{(x_{-2}-z_0)(3x_{-2}-2z_0)} \prod_{i=0}^{n-1} \frac{((4i+5)x_{-2}-(4i+4)z_0)\,((4i+6)x_{-2}-(4i+5)z_0)}{((4i+4)x_{-2}-(4i+3)z_0)\,((4i+7)x_{-2}-(4i+6)z_0)}.$$

Theorem 4.3. *The solutions of difference equation system* (4.3) *are given by the following relations for* $n = 0, 1, 2, ...,$

$$x_{6n-2} = (-1)^n x_{-2} \left(\frac{z_0}{2x_{-2}-z_0}\right)^n, x_{6n-1} = x_{-1}\left(\frac{z_{-2}}{2y_0-z_{-2}}\right)^n, x_{6n} = (-1)^n x_0 \left(\frac{x_0+y_{-2}}{x_0-y_{-2}}\right)^n,$$

$$x_{6n+1} = \frac{(-1)^n x_{-2}y_{-1}}{x_{-2}-z_0}\left(\frac{2x_{-2}-z_0}{z_0}\right)^n,\ x_{6n+2} = (y_0-z_{-2})\left(\frac{2y_0-z_{-2}}{z_{-2}}\right)^n,$$

$$x_{6n+3} = \frac{(-1)^{n+1}x_0z_{-1}}{(x_0+y_{-2})}\left(\frac{x_0-y_{-2}}{x_0+y_{-2}}\right)^n,$$

$$y_{6n-2} = (-1)^n y_{-2}\left(\frac{x_0+y_{-2}}{x_0-y_{-2}}\right)^n,\ y_{6n-1} = (-1)^n y_{-1}\left(\frac{2x_{-2}-z_0}{z_0}\right)^n,\ y_{6n} = y_0\left(\frac{2y_0-z_{-2}}{z_{-2}}\right)^n,$$

$$y_{6n+1} = \frac{(-1)^n z_{-1}y_{-2}}{x_0+y_{-2}}\left(\frac{x_0-y_{-2}}{x_0+y_{-2}}\right)^n,\ y_{6n+2} = (-1)^n(x_{-2}-z_0)\left(\frac{z_0}{2x_{-2}-z_0}\right)^{n+1},$$

$$y_{6n+3} = \frac{-x_{-1}y_0}{y_0-z_{-2}}\left(\frac{z_{-2}}{2y_0-z_{-2}}\right)^{n+1},$$

and

$$z_{6n-2} = z_{-2}\left(\frac{2y_0-z_{-2}}{z_{-2}}\right)^n,\ z_{6n-1} = (-1)^n z_{-1}\left(\frac{x_0-y_{-2}}{x_0+y_{-2}}\right)^n,\ z_{6n} = (-1)^n z_0\left(\frac{z_0}{2x_{-2}-z_0}\right)^n,$$

$$z_{6n+1} = \frac{-x_{-1}z_{-2}}{y_0-z_{-2}}\left(\frac{z_{-2}}{2y_0-z_{-2}}\right)^n,\ z_{6n+2} = (-1)^n(x_0+y_{-2})\left(\frac{x_0+y_{-2}}{x_0-y_{-2}}\right)^n,$$

$$z_{6n+3} = \frac{(-1)^n y_{-1}z_0}{(x_{-2}-z_0)}\left(\frac{2x_{-2}-z_0}{z_0}\right)^{n+1},$$

where $x_0 \neq \pm y_{-2},\ z_{-2} \neq 2y_0,\ z_{-2} \neq y_0,\ z_0 \neq 2x_{-2},\ z_0 \neq x_{-2}$.

Theorem 4.4. *Assume that* $\{x_n, y_n, z_n\}$ *are solutions of system* (4.4) *with* $y_0 \neq \pm z_{-2},\ x_{-2} \neq 2z_0,\ x_{-2} \neq z_0,\ x_0 \neq 2y_{-2},\ x_0 \neq y_{-2}$. *Then for* $n = 0, 1, 2, ...,$

$$x_{6n-2} = (-1)^n x_{-2}\left(\frac{x_{-2}-2z_0}{x_{-2}}\right)^n,\ x_{6n-1} = (-1)^n x_{-1}\left(\frac{y_0-z_{-2}}{y_0+z_{-2}}\right)^n,\ x_{6n} = x_0\left(\frac{x_0}{x_0-2y_{-2}}\right)^n,$$

$$x_{6n+1} = \frac{(-1)^n x_{-2}y_{-1}}{x_{-2}-z_0}\left(\frac{x_{-2}}{x_{-2}-2z_0}\right)^n,\ x_{6n+2} = (-1)^n(y_0+z_{-2})\left(\frac{y_0+z_{-2}}{y_0-z_{-2}}\right)^{n+1},$$

$$x_{6n+3} = \frac{x_0z_{-1}}{(x_0-y_{-2})}\left(\frac{x_0-2y_{-2}}{x_0}\right)^{n+1},$$

$$y_{6n-2} = y_{-2}\left(\frac{x_0}{x_0-2y_{-2}}\right)^n,\ y_{6n-1} = (-1)^n y_{-1}\left(\frac{x_{-2}}{x_{-2}-2z_0}\right)^n,\ y_{6n} = (-1)^n y_0\left(\frac{y_0+z_{-2}}{y_0-z_{-2}}\right)^n,$$

$$y_{6n+1} = \frac{-z_{-1}y_{-2}}{x_0-y_{-2}}\left(\frac{x_0-2y_{-2}}{x_0}\right)^n,\ y_{6n+2} = (-1)^{n+1}(x_{-2}-z_0)\left(\frac{x_{-2}-2z_0}{x_{-2}}\right)^n,$$

$$y_{6n+3} = \frac{(-1)^{n+1}x_{-1}y_0}{y_0+z_{-2}}\left(\frac{y_0-z_{-2}}{y_0+z_{-2}}\right)^n,$$

and

$$z_{6n-2} = (-1)^n z_{-2}\left(\frac{y_0+z_{-2}}{y_0-z_{-2}}\right)^n,\ z_{6n-1} = z_{-1}\left(\frac{x_0-2y_{-2}}{x_0}\right)^n,\ z_{6n} = (-1)^n z_0\left(\frac{x_{-2}-2z_0}{x_{-2}}\right)^n,$$

$$z_{6n+1} = \frac{(-1)^n x_{-1}z_{-2}}{y_0+z_{-2}}\left(\frac{y_0-z_{-2}}{y_0+z_{-2}}\right)^n,\ z_{6n+2} = (x_0-y_{-2})\left(\frac{x_0}{x_0-2y_{-2}}\right)^{n+1},$$

$$z_{6n+3} = \frac{(-1)^{n+1}y_{-1}z_0}{(x_{-2}-z_0)}\left(\frac{x_{-2}}{x_{-2}-2z_0}\right)^{n+1}.$$

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