Journal of Applied Analysis and Computation
Volume 10, Number 3, June 2020, 892–903
Website:http://jaac-online.com/
DOI:10.11948/20190096

FRACTIONAL ORDER NONLINEAR MIXED COUPLED SYSTEMS WITH COUPLED INTEGRO-DIFFERENTIAL BOUNDARY CONDITIONS

B. Ahmad[1,†], A. Alsaedi[1] and S. K. Ntouyas[2,1]

Abstract We study the existence and uniqueness of solutions for a class of coupled fractional differential equations involving both Riemann-Liouville and Caputo fractional derivatives, and coupled integro-differential boundary conditions. We derive the desired results with the aid of modern methods of functional analysis. An example illustrating the abstract results is also presented.

Keywords Riemann-Liouville fractional derivative, Caputo fractional derivative, Riemann-Liouville fractional integral, coupled equations, nonlocal boundary conditions.

MSC(2010) 26A33, 34B15.

1. Introduction

The study of systems of fractional differential equations is an important area of investigation as such systems appear in various problems of applied nature; for instance, see [5, 6, 9]. For theoretical development (existence theory) of coupled fractional differential equations, we refer the reader to the articles [1–3, 7, 10, 11] and the references cited therein.

In this paper, we initiate the study of fractional order mixed nonlinear coupled systems involving both Riemann-Liouville and Caputo fractional derivatives, supplemented with coupled integro-differential boundary conditions. Precisely, we are concerned with the existence and uniqueness of solutions for the following problem:

$$\begin{cases} {}^{RL}D^q({}^{C}D^r x(t)) = f(t, x(t), y(t)), \quad 0 < t < T, \\ {}^{C}D^r({}^{RL}D^q y(t)) = g(t, x(t), y(t)), \quad 0 < t < T, \\ x'(\xi) = \lambda \, {}^{C}D^{\nu} y(\eta), \quad x(T) = \mu \, I^p y(\zeta), \quad \xi, \eta, \zeta \in (0, T), \\ y(0) = 0, \quad y(T) = \mu_1 I^{p_1} x(\zeta_1), \quad \zeta_1 \in (0, T), \end{cases} \tag{1.1}$$

[†]the corresponding author. Email address:bashirahmad_gau@yahoo.com. (B. Ahmad)
[1]Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
[2]Department of Mathematics, University of Ioannina,451 10 Ioannina, Greece

where ${}^{RL}D^q$ is the standard Riemann-Liouville fractional derivative of order $q \in (0,1)$, ${}^{C}D^r, {}^{C}D^\nu$ are the Caputo fractional derivatives of order $r \in (0,1)$ and $\nu \in (0,1)$ respectively with $q + r > 1$, I^p, I^{p_1} are the Riemann-Liouville fractional integrals of order $p > 0, p_1 > 0$, $f, g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions and $\lambda, \mu, \mu_1 \in \mathbb{R}$.

Here the left-hand side of the first equation in (1.1) can be interpreted as follows: the quantity $x(t)$, expressed as ${}^{C}D^r x(t) = x_m(t)$ (say), means that the input values of $x(t)$ appear in form of the power-weighted sum in terms of Caputo-fractional differential operator. Thus ${}^{RL}D^q x_m(t)$ represents the Riemann-Liouville fractional derivative of the input values $x_m(t)$. So the system (1.1) consists of fractional variational functional equations. For details, we refer the reader to the text [8].

We apply Leray-Schauder alternative and Banach fixed point theorem to obtain the existence and uniqueness results for the problem at hand. Our results are new and significantly enhance the literature on the topic.

The rest of the paper is organized as follows. In Section 2, we recall some basic definitions of fractional calculus and present an auxiliary lemma, which plays a key role in obtaining the main results presented in Section 3. We also discuss an example illustrating the existence and uniqueness result.

2. Preliminaries

We recall some basic definitions of fractional calculus.

Definition 2.1. The Riemann–Liouville fractional integral of order $\alpha \in \mathbb{R}$ $(\alpha > 0)$ for a locally integrable real-valued function g on $-\infty \le a < t < b \le +\infty$ is defined as

$$I_a^\alpha g(t) = (g * K_\alpha)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} g(s) ds,$$

where $K_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ and $\Gamma(r) = \int_0^\infty t^{r-1} e^{-t} dt$.

Definition 2.2. Let $g \in L^1[a,b]$, $-\infty \le a < t < b \le +\infty$ and $g * K_{m-\alpha} \in W^{m,1}[a,b], m = [\alpha]+1, \alpha > 0$, where $W^{m,1}[a,b]$ is the Sobolev space defined as

$$W^{m,1}[a,b] = \left\{ g \in L^1[a,b] : \frac{d^m}{dt^m} g \in L^1[a,b] \right\}.$$

The Riemann–Liouville fractional derivative $D_a^\alpha g$ of order $\alpha > 0$ $(m-1 < \alpha < m,\ m \in \mathbb{N})$ is defined as

$$D_a^\alpha g(t) = \frac{d^m}{dt^m} I_a^{m-\alpha} g(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t (t-s)^{m-1-\alpha} g(s) ds.$$

In terms of Riemann–Liouville fractional differential operator D_a^α, the Caputo fractional derivative ${}^{c}D_a^\alpha g(t)$ is defined by

$${}^{c}D_a^\alpha g(t) = D_a^\alpha \left[g(t) - g(a) - g'(a) \frac{(t-a)}{1!} - \ldots - g^{(m-1)}(a) \frac{(t-a)^{m-1}}{(m-1)!} \right].$$

Remark 2.1. If $g \in C^m[a,b]$, then the Caputo fractional derivative ${}^cD_a^\alpha$ of order $\alpha \in \mathbb{R}$ $(m-1<\alpha<m,\, m \in \mathbb{N})$ is defined as

$$ {}^cD_a^\alpha [g](t) = I_a^{m-\alpha} g^{(m)}(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-1-\alpha} g^{(m)}(s)ds. $$

In the sequel, the Riemann–Liouville fractional integral I_a^α and the Caputo fractional derivative ${}^cD_a^\alpha$ with $a=0$ are respectively denoted by I^α and ${}^cD^\alpha$.

Lemma 2.1 (see [6]). *For $y \in C(0,T) \cap L(0,T)$ and $q>0$, the following relation holds:*

$$ {}^{RL}I^q\left({}^CD^q y\right)(t) = y(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, $$

where $c_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n-1$ and $n = [q]+1$.

In the following lemma, we solve the linear variant of the system (1.1), which plays a fundamental role in the forthcoming analysis.

Lemma 2.2. *Let $\Lambda := -A\Gamma_2 + B(A_1\Gamma_2 - A_2\Gamma_1) + A_2\Gamma_0 \neq 0$ and $f_1, g_1 \in C([0,T],\mathbb{R})$. Then the linear system*

$$ \begin{cases} {}^{RL}D^q({}^CD^r x(t)) = f_1(t), \quad 0<t<T, \\ {}^CD^r({}^{RL}D^q y(t)) = g_1(t), \quad 0<t<T, \\ x'(\xi) = \lambda\, {}^CD^\nu y(\eta), \quad x(T) = \mu\, I^p y(\zeta), \quad \xi,\eta,\zeta \in (0,T), \\ y(0)=0, \quad y(T) = \mu_1 I^{p_1} x(\zeta_1), \quad \zeta_1 \in (0,T), \end{cases} \tag{2.1} $$

is equivalent to a system of integral equations:

$$ \begin{aligned} x(t) = {} & I^{q+r} f_1(t) + \frac{\Gamma(q)}{\Gamma(q+r)} \frac{t^{q+r-1}}{\Lambda}\Big[-\Gamma_2\Big(I^{q+r}g_1(T) - \mu_1 I^{p_1+q+r} f_1(\zeta_1)\Big) \\ & + B\Gamma_2\Big(\mu I^{q+r+p} g_1(\xi) - I^{q+r} f_1(T)\Big) \\ & + (\Gamma_0 - B\Gamma_1)\Big(\lambda I^{q+r-\nu} g_1(\eta) - I^{q+r-1} f_1(\xi)\Big)\Big] \\ & + \frac{1}{\Lambda}\Big[(A_1\Gamma_2 - A_2\Gamma_1)\Big(I^{q+r}g_1(T) - \mu_1 I^{p_1+q+r} f_1(\zeta_1)\Big) \\ & + (A_2\Gamma_0 - A\Gamma_2)\Big(\mu I^{q+r+p} g_1(\zeta) - I^{q+r} f_1(T)\Big) \\ & + (A\Gamma_1 - A_1\Gamma_0)\Big(\lambda I^{q+r-\nu} g_1(\eta) - I^{q+r-1} f_1(\xi)\Big)\Big], \end{aligned} \tag{2.2} $$

$$ \begin{aligned} y(t) = {} & I^{q+r} g_1(t) + \frac{\Gamma(q)}{\Gamma(q+r)} \frac{t^q}{\Lambda}\Big[-A_2\Big(I^{q+r}g_1(T) - \mu_1 I^{p_1+q+r} f_1(\zeta_1)\Big) \\ & + A_2 B\Big(\mu I^{q+r+p} g_1(\zeta) - I^{q+r} f_1(T)\Big) \\ & + (A - A_1 B)\Big(\lambda I^{q+r-\nu} g_1(\eta) - I^{q+r-1} f_1(\xi)\Big)\Big], \end{aligned} \tag{2.3} $$

where

$$ A = \mu_1 \frac{\Gamma(q)}{\Gamma(q+r)} \zeta_1^{p_1+q+r-1}, \quad A_1 = \frac{\Gamma(q)}{\Gamma(q+r)} T^{q+r-1}, \quad A_2 = \frac{\Gamma(q)}{\Gamma(q+r-1)} \xi^{q+r-2}, $$

$$ B = \frac{\mu_1}{\Gamma(1+p_1)} \zeta_1^{p_1}, \ \Gamma_0 = \frac{1}{\Gamma(1+q)} T^q, \ \Gamma_1 = \frac{\mu}{\Gamma(q+r+1)}, \ \Gamma_2 = \lambda \frac{1}{\Gamma(q+1-\nu)} \eta^{q-\nu}. $$

Proof. Applying the Riemann-Liouville fractional integral of order q to both sides of the first equation in (2.1) and using Lemma 2.1, we get

$$ {}^{C}D^{r}x(t) = I^{q}f_1(t) + c_1 t^{q-1}, \tag{2.4} $$

where $c_1 \in \mathbb{R}$ is an unknown arbitrary constant. Operating the Riemann-Liouville fractional integral of order r on both sides (2.4) yields

$$ x(t) = I^{q+r}f_1(t) + c_1\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1} + c_2, \tag{2.5} $$

where $c_2 \in \mathbb{R}$ is an unknown arbitrary constant.

Now applying firstly the Riemann-Liouville fractional integral of order r to both sides of the second equation in (2.1), and then Riemann-Liouville fractional integral of order q to the resulting equation, we obtain

$$ y(t) = I^{q+r}g_1(t) + d_1\frac{1}{\Gamma(1+q)}t^{q} + d_2 t^{q-1}, \tag{2.6} $$

where $d_1, d_2 \in \mathbb{R}$ are unknown arbitrary constants. Using $y(0) = 0$ in (2.6) implies that $d_2 = 0$. Thus, from (2.5) and (2.6), we have

$$ \begin{aligned} x'(t) &= I^{q+r-1}f_1(t) + c_1\frac{\Gamma(q)}{\Gamma(q+r-1)}t^{q+r-2}, \\ {}^{C}D^{\nu}y(t) &= I^{q+r-\nu}g_1(t) + d_1\frac{\Gamma(q)}{\Gamma(q+1-\nu)}t^{q-\nu}, \\ I^{p}y(t) &= I^{q+r+p}g_1(t) + d_1\frac{\Gamma(q)}{\Gamma(q+p+1)}t^{q+p}, \\ I^{p_1}x(t) &= I^{q+r+p_1}f_1(t) + c_1\frac{\Gamma(q)}{\Gamma(q+r+p_1)}t^{q+r+p_1-1} + c_2\frac{1}{\Gamma(1+p_1)}t^{p_1}. \end{aligned} $$

Making use of the remaining boundary conditions given by (2.1) into (2.5) and (2.6) together with above expressions, and setting

$$ \begin{aligned} P &= I^{q+r}g_1(T) - \mu_1 I^{p_1+q+r}f_1(\zeta_1), \\ Q &= \mu I^{q+r+p}g_1(\xi) - I^{q+r}f_1(T), \\ R &= \lambda I^{q+r-\nu}g_1(\eta) - I^{q+r-1}f_1(\xi), \end{aligned} $$

we obtain the following system:

$$ \begin{aligned} &Ac_1 + Bc_2 - \Gamma_0 d_1 = P, \\ &A_1c_1 + c_2 - \Gamma_1 d_1 = Q, \\ &A_2c_1 - \Gamma_2 d_1 = R. \end{aligned} $$

Solving the above system for the unknown constants c_1, c_2, and d_1 yields

$$ \begin{aligned} c_1 &= \frac{1}{\Lambda}\Big[-\Gamma_2P + B\Gamma_2Q + (\Gamma_0 - B\Gamma_1)R\Big], \\ c_2 &= \frac{1}{\Lambda}\Big[(A_1\Gamma_2 - A_2\Gamma_1)P + (A_2\Gamma_0 - A\Gamma_2)Q + (A\Gamma_1 - A_1\Gamma_0)R\Big], \\ d_1 &= \frac{1}{\Lambda}\Big[-A_2P + A_2BQ + (A - A_1B)R\Big]. \end{aligned} $$

Inserting the values of c_1, c_2 and d_1 in (2.5) and (2.6), we find the required solution given by (2.2) and (2.3). The converse follows by direct computation. The proof is completed. □

3. Main Results

Let us introduce the space $X = C([0,T])$ endowed with the norm $\|x\| = \sup\{|x(t)|, t \in [0,T]\}$. Obviously $(X, \|\cdot\|)$ is a Banach space. Then the product space $(X \times X, \|(x,y)\|)$ is also a Banach space equipped with norm $\|(x,y)\| = \|x\| + \|y\|$.

In view of Lemma 2.2, we define an operator $T : X \times X \to X \times X$ by

$$T(x,y)(t) = \begin{pmatrix} T_1(x,y)(t) \\ T_2(x,y)(t) \end{pmatrix}, \tag{3.1}$$

where

$$\begin{aligned} T_1(x,y)(t) = {} & I^{q+r}\bar{f}(t) + \frac{\Gamma(q)}{\Gamma(q+r)}\frac{t^{q+r-1}}{\Lambda}\Big[-\Gamma_2\Big(I^{q+r}\bar{g}(T) - \mu_1 I^{p_1+q+r}\bar{f}(\zeta_1)\Big) \\ & +B\Gamma_2\Big(\mu I^{q+r+p}\bar{g}(\xi) - I^{q+r}\bar{f}(T)\Big) \\ & +(\Gamma_0 - B\Gamma_1)\Big(\lambda I^{q+r-\nu}\bar{g}(\eta) - I^{q+r-1}\bar{f}(\xi)\Big)\Big] \\ & +\frac{1}{\Lambda}\Big[(A_1\Gamma_2 - A_2\Gamma_1)\Big(I^{q+r}\bar{g}(T) - \mu_1 I^{p_1+q+r}\bar{f}(\zeta_1)\Big) \\ & +(A_2\Gamma_0 - A\Gamma_2)\Big(\mu I^{q+r+p}\bar{g}(\zeta) - I^{q+r}\bar{f}(T)\Big) \\ & +(A\Gamma_1 - A_1\Gamma_0)\Big(\lambda I^{q+r-\nu}\bar{g}(\eta) - I^{q+r-1}\bar{f}(\xi)\Big)\Big] \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} T_2(x,y)(t) = {} & I^{q+r}g_1(t) + \frac{\Gamma(q)}{\Gamma(q+r)}\frac{t^{q}}{\Lambda}\Big[-A_2\Big(I^{q+r}\bar{g}(T) - \mu_1 I^{p_1+q+r}\bar{f}(\zeta_1)\Big) \\ & +A_2B\Big(\mu I^{q+r+p}\bar{g}(\zeta) - I^{q+r}\bar{f}(T)\Big) \\ & +(A - A_1B)\Big(\lambda I^{q+r-\nu}\bar{g}(\eta) - I^{q+r-1}\bar{f}(\xi)\Big)\Big], \end{aligned} \tag{3.3}$$

where

$$\bar{f}(t) = f(t, x(t), y(t)), \quad \bar{g}(t) = g(t, x(t), y(t)).$$

For computational convenience, we introduce the notations:

$$Q_0 = \frac{\Gamma(q)}{\Gamma(q+r)}\frac{T^{q+r-1}}{|\Lambda|}, \quad \bar{Q}_0 = \frac{\Gamma(q)}{\Gamma(q+r)}\frac{T^{q}}{|\Lambda|}, \tag{3.4}$$

$$\begin{aligned} Q_1 = {} & \frac{T^{q+r}}{\Gamma(q+r+1)} + Q_0\Big[|\mu_1|\frac{\zeta_1^{p_1+q+r}}{\Gamma(p_1+q+r+1)} + |B\Gamma_2|\frac{T^{q+r}}{\Gamma(q+r+1)} \\ & +|\Gamma_0 - B\Gamma_1|\frac{\xi^{q+r-1}}{\Gamma(q+r)}\Big] + \frac{1}{|\Lambda|}\Big[|A_1\Gamma_2 - A_2\Gamma_1||\mu_1|\frac{\zeta_1^{p_1+q+r}}{\Gamma(p_1+q+r+1)} \\ & +|A_2\Gamma_0 - A\Gamma_2|\frac{T^{q+r}}{\Gamma(q+r+1)} + |A\Gamma_1 - A_1\Gamma|\frac{\xi^{q+r-1}}{\Gamma(q+r)}\Big], \end{aligned} \tag{3.5}$$

$$Q_2 = Q_0\Big[|\Gamma_2|\frac{T^{q+r}}{\Gamma(q+r+1)} + |B\Gamma_2||\mu|\frac{\xi^{q+r+p}}{\Gamma(q+r+p+1)}$$

$$+|\Gamma_0 - B\Gamma_1||\lambda|\frac{\eta^{q+r-\nu}}{\Gamma(q+r-\nu+1)}\Big] + \frac{1}{|\Lambda|}\Big[|A_1\Gamma_2 - A_2\Gamma_1|\frac{T^{q+r}}{\Gamma(q+r+1)} \quad (3.6)$$
$$+|A_2\Gamma_0 - A\Gamma_2||\mu|\frac{\zeta^{q+r+p}}{\Gamma(q+r+p+1)} + |A\Gamma_1 - A_1\Gamma||\lambda|\frac{\eta^{q+r-\nu}}{\Gamma(q+r-\nu+1)}\Big],$$

$$Q_3 = \bar{Q}_0\Big[|A_2||\mu_1|\frac{\zeta_1^{p_1+q+r}}{\Gamma(p_1+q+r+1)} + |A_2B|\frac{T^{q+r}}{\Gamma(q+r+1)}$$
$$+|A - A_1B|\frac{\xi^{q+r-1}}{\Gamma(q+r)}\Big], \tag{3.7}$$

$$Q_4 = \frac{T^{q+r}}{\Gamma(q+r+1)} + \bar{Q}_0\Big[|A_2|\frac{T^{q+r}}{\Gamma(q+r+1)} + |A_2B||\mu|\frac{\zeta^{q+r+p}}{\Gamma(q+r+p+1)}$$
$$+|A_2B||\lambda|\frac{\eta^{q+r-\nu}}{\Gamma(q+r-\nu)}\Big]. \tag{3.8}$$

In the following theorem, we prove the existence and uniqueness of solutions to the system (1.1) via Banach contraction mapping principle.

Theorem 3.1. *Assume that:*

(H_1) *$f, g : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions and that there exist positive constants ℓ_1 and ℓ_2 such that for all $t \in [0,T]$ and $x_i, y_i \in \mathbb{R}$, $i = 1,2$, we have*

$$|f(t,x_1,x_2) - f(t,y_1,y_2)| \le \ell_1(|x_1 - y_1| + |x_2 - y_2|),$$
$$|g(t,x_1,x_2) - g(t,y_1,y_2)| \le \ell_2(|x_1 - y_1| + |x_2 - y_2|).$$

Then there exists a unique solution for the the system (1.1) on $[0,T]$, provided that

$$(Q_1 + Q_3)\ell_1 + (Q_2 + Q_4)\ell_2 < 1,$$

where $Q_i, i = 0,1,2,3,4$ are given by (3.4)-(3.8).

Proof. Define $\sup_{t\in[0,T]} f(t,0,0) = N_1 < \infty$, $\sup_{t\in[0,T]} g(t,0,0) = N_2 < \infty$ and a positive number r such that

$$r > \frac{(Q_1 + Q_3)N_1 + (Q_2 + Q_4)N_2}{1 - (Q_1 + Q_3)\ell_1 - (Q_2 + Q_4)\ell_2}.$$

Then we show that $TB_r \subset B_r$, where $B_r = \{(x,y) \in X \times X : \|(x,y)\| \le r\}$.

By the assumption (H_1), for $(u,v) \in B_r$, $t \in [0,T]$, we have

$$\begin{aligned} |f(t,x(t),y(t))| &\le |f(t,x(t),y(t)) - f(t,0,0)| + |f(t,0,0)| \\ &\le \ell_1(|x(t)| + |y(t)|) + N_1 \\ &\le \ell_1(\|x\| + \|y\|) + N_1 \le \ell_1 r + N_1 \end{aligned}$$

and

$$|g(t,x(t),y(t))| \le \ell_2(\|x\| + \|y\|) + N_2 \le \ell_2 r + N_2.$$

In consequence, we obtain

$$
\begin{aligned}
&|T_1(x,y)(t)| \\
\leq\ & I^{q+r}|\bar{f}(t)| + \frac{\Gamma(q)}{\Gamma(q+r)}\frac{T^{q+r-1}}{|\Lambda|}\Big[|\Gamma_2|\Big(I^{q+r}|\bar{g}(T)| + |\mu_1|I^{p_1+q+r}|\bar{f}(\zeta_1)|\Big) \\
&+|B\Gamma_2|\Big(|\mu|I^{q+r+p}|\bar{g}(\xi)| + I^{q+r}|\bar{f}(T)|\Big) \\
&+|\Gamma_0 - B\Gamma_1|\Big(|\lambda|I^{q+r-\nu}|\bar{g}(\eta)| + I^{q+r-1}|\bar{f}(\xi)|\Big)\Big] \\
&+\frac{1}{|\Lambda|}\Big[|A_1\Gamma_2 - A_2\Gamma_1|\Big(I^{q+r}|\bar{g}(T)| + |\mu_1|I^{p_1+q+r}|\bar{f}(\zeta_1)|\Big) \\
&+|A_2\Gamma_0 - A\Gamma_2|\Big(|\mu|I^{q+r+p}|\bar{g}(\zeta)| + I^{q+r}|\bar{f}(T)|\Big) \\
&+|A\Gamma_1 - A_1\Gamma_0|\Big(|\lambda|I^{q+r-\nu}|\bar{g}(\eta)| + I^{q+r-1}|\bar{f}(\xi)|\Big)\Big] \\
\leq\ & \frac{T^{q+r}}{\Gamma(q+r+1)}(\ell_1 r + N_1) + \frac{\Gamma(q)}{\Gamma(q+r)}\frac{T^{q+r-1}}{|\Lambda|}\Big[|\Gamma_2|\frac{T^{q+r}}{\Gamma(q+r+1)}(\ell_2 r + N_2) \\
&+|\mu_1|\frac{\zeta_1^{p_1+q+r}}{\Gamma(p_1+q+r+1)}(\ell_1 r + N_1)\Big) \\
&+|B\Gamma_2|\Big(|\mu|\frac{\xi^{q+r+p}}{\Gamma(q+r+p+1)}(\ell_2 r + N_2) + \frac{T^{q+r}}{\Gamma(q+r+1)}(\ell_1 r + N_1)\Big) \\
&+|\Gamma_0 - B\Gamma_1|\Big(|\lambda|\frac{\eta^{q+r-\nu}}{\Gamma(q+r-\nu+1)}(\ell_2 r + N_2) + \frac{\xi^{q+r-1}}{\Gamma(q+r)}(\ell_1 r + N_1)\Big)\Big] \\
&+\frac{1}{|\Lambda|}\Big[|A_1\Gamma_2 - A_2\Gamma_1|\Big(\frac{T^{q+r}}{\Gamma(q+r+1)}(\ell_2 r + N_2) + |\mu_1|\frac{\zeta_1^{p_1+q+r}}{\Gamma(p_1+q+r+1)}(\ell_1 r + N_1)\Big) \\
&+|A_2\Gamma_0 - A\Gamma_2|\Big(|\mu|\frac{\zeta^{q+r+p}}{\Gamma(q+r+p+1)}(\ell_2 r + N_2) + \frac{T^{q+r}}{\Gamma(q+r+1)}(\ell_1 r + N_1)\Big) \\
&+|A\Gamma_1 - A_1\Gamma_0|\Big(|\lambda|\frac{\eta^{q+r-\nu}}{\Gamma(q_r-\nu+1)}(\ell_2 r + N_2) + \frac{\xi^{q+r-1}}{\Gamma(q+r)}(\ell_1 r + N_1)\Big)\Big] \\
=\ & (Q_1\ell_1 + Q_2\ell_2)r + Q_1N_1 + Q_2N_2,
\end{aligned}
$$

which implies that

$$\|T_1(x,y)\| \leq (Q_1\ell_1 + Q_2\ell_2)r + Q_1N_1 + Q_2N_2.$$

In the same way, we can find that

$$\|T_2(x,y)\| \leq (Q_3\ell_1 + Q_4\ell_2)r + Q_3N_1 + Q_4N_2.$$

From the above inequalities, it follows that

$$\|T(x,y)\| \leq [(Q_1+Q_3)\ell_1 + (Q_2+Q_4)\ell_2]r + (Q_1+Q_3)N_1 + (Q_2+Q_4)N_2 \leq r.$$

Next, for $(x_2,y_2),(x_1,y_1) \in X \times X$, and for any $t \in [0,T]$, we get

$$
\begin{aligned}
&|T_1(x_2,y_2)(t) - T_1(x_1,y_1)(t)| \\
\leq\ & \frac{T^{q+r}}{\Gamma(q+r+1)}\ell_1(\|x_2-x_1\| + \|y_2-y_1\|)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\Gamma(q)}{\Gamma(q+r)}\frac{T^{q+r-1}}{|\Lambda|}\Big[|\Gamma_2|\Big(\frac{T^{q+r}}{\Gamma(q+r+1)}\ell_2(\|x_2-x_1\|+\|y_2-y_1\|)\\
&+|\mu_1|\frac{\zeta_1^{p_1+q+r}}{\Gamma(p_1+q+r+1)}\ell_1(\|x_2-x_1\|+\|y_2-y_1\|)\Big)\\
&+|B\Gamma_2|\Big(|\mu|\frac{\xi^{q+r+p}}{\Gamma(q+r+p+1)}\ell_2(\|x_2-x_1\|+\|y_2-y_1\|)\\
&+\frac{T^{q+r}}{\Gamma(q+r+1)}\ell_1(\|x_2-x_1\|+\|y_2-y_1\|)\Big)\\
&+|\Gamma_0-B\Gamma_1|\Big(|\lambda|\frac{\eta^{q+r-\nu}}{\Gamma(q+r-\nu+1)}\ell_2(\|x_2-x_1\|+\|y_2-y_1\|)\\
&+\frac{\xi^{q+r-1}}{\Gamma(q+r)}\ell_1(\|x_2-x_1\|+\|y_2-y_1\|)\Big)\Big]\\
&+\frac{1}{|\Lambda|}\Big[|A_1\Gamma_2-A_2\Gamma_1|\Big(\frac{T^{q+r}}{\Gamma(q+r+1)}\ell_2(\|x_2-x_1\|+\|y_2-y_1\|)\\
&+|\mu_1|\frac{\zeta_1^{p_1+q+r}}{\Gamma(p_1+q+r+1)}\ell_1(\|x_2-x_1\|+\|y_2-y_1\|)\Big)\\
&+|A_2\Gamma_0-A\Gamma_2|\Big(|\mu|\frac{\zeta^{q+r+p}}{\Gamma(q+r+p+1)}\ell_2(\|x_2-x_1\|+\|y_2-y_1\|)\\
&+\frac{T^{q+r}}{\Gamma(q+r+1)}\ell_1(\|x_2-x_1\|+\|y_2-y_1\|)\Big)\\
&+|A\Gamma_1-A_1\Gamma_0|\Big(|\lambda|\frac{\eta^{q+r-\nu}}{\Gamma(q_r-\nu+1)}\ell_2(\|x_2-x_1\|+\|y_2-y_1\|)\\
&+\frac{\xi^{q+r-1}}{\Gamma(q+r)}\ell_1(\|x_2-x_1\|+\|y_2-y_1\|)\Big)\Big]\\
&\le (Q_1\ell_1+Q_2\ell_2)(\|x_2-x_1\|+\|y_2-y_1\|),
\end{aligned}
$$

which leads to

$$\|T_1(x_2,y_2)-T_1(x_1,y_1)\| \le (Q_1\ell_1+Q_2\ell_2)(\|x_2-x_1\|+\|y_2-y_1\|). \tag{3.9}$$

Similarly, one can obtain

$$\|T_2(x_2,y_2)(t)-T_2(x_1,y_1)\| \le (Q_3\ell_1+Q_4\ell_2)(\|x_2-x_1\|+\|y_2-y_1\|). \tag{3.10}$$

From (3.9) and (3.10), we deduce that

$$\|T(x_2,y_2)-T(x_1,y_1)\| \le [(Q_1+Q_3)\ell_1+(Q_2+Q_4)\ell_2](\|x_2-x_1\|+\|y_2-y_1\|).$$

Since $(Q_1+Q_3)\ell_1+(Q_2+Q_4)\ell_2<1$, therefore, T is a contraction. So, by Banach fixed point theorem, the operator T has a unique fixed point, which corresponds to a unique solution of problem (1.1). This completes the proof. □

The second result is based on Leray-Schauder alternative.

Lemma 3.1 (Leray-Schauder alternative, [4] p. 4.)**.** *Let $F : E \to E$ be a completely continuous operator (i.e., a map that restricted to any bounded set in E is compact). Let*

$$\mathcal{E}(F)=\{x\in E : x=\lambda F(x) \text{ for some } 0<\lambda<1\}.$$

Then either the set $\mathcal{E}(F)$ is unbounded, or F has at least one fixed point.

Theorem 3.2. *Assume that:*

(H_3) $f, g : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *are continuous functions and that there exist real constants* $k_i, \gamma_i \geq 0$, $(i = 0,1,2)$ *with* $k_0 > 0, \gamma_0 > 0$ *such that,* $\forall x_i \in \mathbb{R}$, $(i = 1,2)$,

$$|f(t,x_1,x_2)| \leq k_0 + k_1|x_1| + k_2|x_2|,$$
$$|g(t,x_1,x_2)| \leq \gamma_0 + \gamma_1|x_1| + \gamma_2|x_2|.$$

If

$$(Q_1 + Q_3)k_1 + (Q_2 + Q_4)\gamma_1 < 1 \quad \text{and} \quad (Q_1 + Q_3)k_2 + (Q_2 + Q_4)\gamma_2 < 1, \tag{3.11}$$

where $Q_i, i = 0,1,2,3,4$, *are given by (3.4)-(3.8), then the system (1.1) has at least one solution on* $[0,T]$.

Proof. Firstly we show that the operator $T : X \times X \to X \times X$ defined by (3.1) is completely continuous. Notice that continuity of the operator T follows from that of the functions f and g.

Let $\Omega \subset X \times X$ be bounded. Then there exist positive constants L_1 and L_2 such that $|f(t,x(t),y(t))| \leq L_1, |g(t,x(t),y(t))| \leq L_2, \quad \forall (x,y) \in \Omega$. Then, for any $(x,y) \in \Omega$, we have

$$\begin{aligned}
|T_1(x,y)(t)| \leq{}& \frac{T^{q+r}}{\Gamma(q+r+1)}L_1 + \frac{\Gamma(q)}{\Gamma(q+r)}\frac{T^{q+r-1}}{|\Lambda|}\Big[|\Gamma_2|\Big(\frac{T^{q+r}}{\Gamma(q+r+1)}L_2 \\
&+|\mu_1|\frac{\zeta_1^{p_1+q+r}}{\Gamma(p_1+q+r+1)}L_1\Big) \\
&+|B\Gamma_2|\Big(|\mu|\frac{\xi^{q+r+p}}{\Gamma(q+r+p+1)}L_2 + \frac{T^{q+r}}{\Gamma(q+r+1)}L_1\Big) \\
&+|\Gamma_0 - B\Gamma_1|\Big(|\lambda|\frac{\eta^{q+r-\nu}}{\Gamma(q+r-\nu+1)}L_2 + \frac{\xi^{q+r-1}}{\Gamma(q+r)}L_1\Big)\Big] \\
&+\frac{1}{|\Lambda|}\Big[|A_1\Gamma_2 - A_2\Gamma_1|\Big(\frac{T^{q+r}}{\Gamma(q+r+1)}L_2 + |\mu_1|\frac{\zeta_1^{p_1+q+r}}{\Gamma(p_1+q+r+1)}L_1\Big) \\
&+|A_2\Gamma_0 - A\Gamma_2|\Big(|\mu|\frac{\zeta^{q+r+p}}{\Gamma(q+r+p+1)}L_2 + \frac{T^{q+r}}{\Gamma(q+r+1)}L_1\Big) \\
&+|A\Gamma_1 - A_1\Gamma_0|\Big(|\lambda|\frac{\eta^{q+r-\nu}}{\Gamma(q_r-\nu+1)}L_2 + \frac{\xi^{q+r-1}}{\Gamma(q+r)}L_1\Big)\Big] \\
={}& Q_1L_1 + Q_2L_2,
\end{aligned}$$

which implies that

$$\|T_1(x,y)\| \leq Q_1L_1 + Q_2L_2.$$

In a similar manner, one can find that

$$\|T_2(x,y)\| \leq Q_3L_1 + Q_4L_2.$$

Thus, it follows from the above inequalities that the operator T is uniformly bounded, since $\|T(x,y)\| \leq (Q_1 + Q_3)L_1 + (Q_2 + Q_4)L_2$.

Next, we show that T is equicontinuous. Let $t_1, t_2 \in [0, T]$ with $t_1 < t_2$. Then we have

$$\begin{aligned}
&|T_1(x(t_2), y(t_2)) - T_1(x(t_1), y(t_1))| \\
\le\ & L_1\left\{\frac{1}{\Gamma(q+r)}\int_0^{t_1}[(t_2-s)^{q+r-1}-(t_1-s)^{q+r-1}]ds+\frac{1}{\Gamma(q+r)}\int_{t_1}^{t_2}(t_2-s)^{q+r-1}ds\right\} \\
&+\frac{\Gamma(q)}{\Gamma(q+r)}\frac{|t_2^{q+r-1}-t_1^{q+r-1}|}{|\Lambda|}\Big[|\Gamma_2|\Big(\frac{T^{q+r}}{\Gamma(q+r+1)}L_2+|\mu_1|\frac{\zeta_1^{p_1+q+r}}{\Gamma(p_1+q+r+1)}L_1\Big) \\
&+|B\Gamma_2|\Big(|\mu|\frac{\xi^{q+r+p}}{\Gamma(q+r+p+1)}L_2+\frac{T^{q+r}}{\Gamma(q+r+1)}L_1\Big) \\
&+|\Gamma_0-B\Gamma_1|\Big(|\lambda|\frac{\eta^{q+r-\nu}}{\Gamma(q+r-\nu+1)}L_2+\frac{\xi^{q+r-1}}{\Gamma(q+r)}L_1\Big)\Big] \\
\le\ & \frac{L_1}{\Gamma(q+r+1)}[2(t_2-t_1)^{q+r}+|t_2^{q+r}-t_1^{q+r}|] \\
&+\frac{\Gamma(q)}{\Gamma(q+r)}\frac{|t_2^{q+r-1}-t_1^{q+r-1}|}{|\Lambda|}\Big[|\Gamma_2|\Big(\frac{T^{q+r}}{\Gamma(q+r+1)}L_2+|\mu_1|\frac{\zeta_1^{p_1+q+r}}{\Gamma(p_1+q+r+1)}L_1\Big) \\
&+|B\Gamma_2|\Big(|\mu|\frac{\xi^{q+r+p}}{\Gamma(q+r+p+1)}L_2+\frac{T^{q+r}}{\Gamma(q+r+1)}L_1\Big) \\
&+|\Gamma_0-B\Gamma_1|\Big(|\lambda|\frac{\eta^{q+r-\nu}}{\Gamma(q+r-\nu+1)}L_2+\frac{\xi^{q+r-1}}{\Gamma(q+r)}L_1\Big)\Big].
\end{aligned}$$

Analogously, we can obtain

$$\begin{aligned}
&|T_2(x(t_2), y(t_2)) - T_2(x(t_1), y(t_1))| \\
\le\ & \frac{L_2}{\Gamma(q+r+1)}[2(t_2-t_1)^{q+r}+|t_2^{q+r}-t_1^{q+r}|] \\
&+\frac{\Gamma(q)}{\Gamma(q+r)}\frac{|t_2^{q}-t_1^{q}|}{|\Lambda|}\Big[|A_2|\Big(\frac{T^{q+r}}{\Gamma(q+r+1)}L_2+|\mu_1|\frac{\zeta_1^{p_1+q+r}}{\Gamma(p_1+q+r+1)}L_1\Big) \\
&+|A_2B|\Big(|\mu|\frac{\zeta^{q+r+p}}{\Gamma(q+r+p+1)}L_2+\frac{T^{q+r}}{\Gamma(q+r+1)}L_1\Big) \\
&+|A-A_1B|\Big(|\lambda|\frac{\eta^{q+r-\nu}}{\Gamma(q+r-\nu)}L_2+\frac{\xi^{q+r-1}}{\Gamma(q+r)}L_1\Big)\Big].
\end{aligned}$$

Thus the operator $T(x, y)$ is equicontinuous. In view of the foregoing arguments, we deduce that the operator $T(x, y)$ is completely continuous.

Finally, it will be verified that the set $\mathcal{E} = \{(x, y) \in X \times X | (x, y) = \lambda T(x, y), 0 \le \lambda \le 1\}$ is bounded. Let $(x, y) \in \mathcal{E}$, with $(x, y) = \lambda T(x, y)$. For any $t \in [0, T]$, we have

$$x(t) = \lambda T_1(x, y)(t), \quad y(t) = \lambda T_2(x, y)(t).$$

Then

$$\begin{aligned}
|x(t)| &\le Q_1(k_0 + k_1|x| + k_2|y|) + Q_2(\gamma_0 + \gamma_1|x| + \gamma_2|y|) \\
&= Q_1k_0 + Q_2\gamma_0 + (Q_1k_1 + Q_2\gamma_1)|x| + (Q_1k_2 + Q_2\gamma_2)|y|
\end{aligned}$$

and

$$\begin{aligned}
|y(t)| &\le Q_3(k_0 + k_1|x| + k_2|y|) + Q_4(\gamma_0 + \gamma_1|x| + \gamma_2|y|) \\
&= Q_3k_0 + Q_4\gamma_0 + (Q_3k_1 + Q_4\gamma_1)|x| + (Q_3k_2 + Q_4\gamma_2)|y|.
\end{aligned}$$

In consequence, we have

$$\|x\| \le Q_1k_0 + Q_2\gamma_0 + (Q_1k_1 + Q_2\gamma_1)\|x\| + (Q_1k_2 + Q_2\gamma_2)\|y\|$$

and

$$\|y\| \le Q_3k_0 + Q_4\gamma_0 + (Q_3k_1 + Q_4\gamma_1)\|x\| + (Q_3k_2 + Q_4\gamma_2)\|y\|,$$

which imply that

$$\|x\| + \|y\| \le (Q_1 + Q_3)k_0 + (Q_2 + Q_4)\gamma_0 + [(Q_1 + Q_3)k_1 + (Q_2 + Q_4)\gamma_1]\|x\| \\ +[(Q_1 + Q_3)k_2 + (Q_2 + Q_4)\gamma_2)]\|y\|.$$

Thus we have

$$\|(x, y)\| \le \frac{(Q_1 + Q_3)k_0 + (Q_2 + Q_4)\gamma_0}{M_0},$$

where $M_0 = \min\{1-[(Q_1+Q_3)k_1+(Q_2+Q_4)\gamma_1], 1-[(Q_1+Q_3)k_2+(Q_2+Q_4)\gamma_2)]\}$, which establishes that the set $\mathcal{E}$ is bounded. Thus, by Lemma 3.1, the operator T has at least one fixed point. Hence the system (1.1) has at least one solution. The proof is complete. □

Example 3.1. Consider the following boundary value problem of coupled nonlinear fractional differential equations

$$\begin{cases} {}^{RL}D^{4/5}\left({}^{C}D^{1/2}x(t)\right)(t) = \dfrac{1}{8(t+2)^2}\dfrac{|x|}{1+|x|} + 1 + \dfrac{1}{64}\sin^2 y, \quad 0 < t < 3, \\ {}^{C}D^{1/2}({}^{RL}D^{4/5}y(t)) = \dfrac{1}{64\pi}\sin(2\pi x) + \dfrac{|y|}{32(1+|y|)} + \dfrac{1}{2}, \quad 0 < t < 3, \\ x'(1/2) = (1/12)\ {}^{C}D^{1/2}y(1/3), \quad x(3) = 2\ I^{2/3}y(1/2), \\ y(0) = 0, \quad y(3) = (1/2)I^{1/2}x(1/4). \end{cases} \tag{3.12}$$

Here $q = 4/5$, $r = 1/2$, $\xi = 1/2$, $\lambda = 1/12$, $\nu = 1/2$, $\eta = 1/3$, $T = 3$, $\mu = 2$, $p = 2/3$, $\zeta = 1/2$, $\mu_1 = 1/2$, $p_1 = 1/2$, $\zeta_1 = 1/4$. With these data we find $A \approx 0.069498$, $A_1 \approx 1.546083$, $A_2 \approx 0.684503$, $B \approx 0.282040$, $\Gamma_0 \approx 2.585643$, $\Gamma_1 \approx 1.714220$, $\Gamma_2 \approx 0.092930$, $\Lambda \approx 1.473003$, $Q_0 \approx 1.049991$, $\bar{Q}_0 \approx 1.817985$, $Q_1 \approx 6.24681$, $Q_2 \approx 4.728334$, $Q_3 \approx 1.889017$, $Q_4 \approx 8.732422$. From $f(t, x, y) = \dfrac{1}{8(t+2)^2}\dfrac{|x|}{1+|x|} + 1 + \dfrac{1}{64}\sin^2 y$, and $g(t, x, y) = \dfrac{1}{64\pi}\sin(2\pi x) + \dfrac{|y|}{32(1+|y|)} + \dfrac{1}{2}$, we have $\ell_1 = \ell_2 = \frac{1}{32}$as $|f(t, x_1, x_2) - f(t, y_1, y_2)| \le \dfrac{1}{32}(|x_1 - x_2| + |y_1 - y_2|)$, $|g(t, x_1, x_2)-g(t, y_1, y_2)| \le \dfrac{1}{32}(|x_1-x_2|+|y_1-y_2|)$. Then $(Q_1+Q_3)\ell_1+(Q_2+Q_4)\ell_2 \approx 0.674892 < 1$. Thus all the conditions of Theorem 3.1 are satisfied and consequently, its conclusion applies to the problem (3.12).

4. Conclusions

We proved the existence of solutions for a nonlinear mixed coupled system of fractional differential equations involving Riemann-Liouville as well as Caputo fractional derivatives, equipped with coupled integro-differential boundary conditions.

By fixing the parameters involved in the given problem, we can obtain some new results as special cases of the present work. For instance, if we take $\lambda = 0, \mu = 0, \mu_1 = 0$, our results correspond to the uncoupled three-point nonlocal boundary conditions of the form: $x'(\xi) = 0, x(T) = 0; y(0) = 0, y(T) = 0, \xi \in (0,T)$. Letting $\mu_1 = 0$ in the results of this paper, we obtain the ones for a coupled system of differential equations involving both Riemann-Liouville and Caputo fractional derivatives, subject to the boundary conditions: $x'(\xi) = \lambda\ {}^{C}D^{\nu} y(\eta), \quad x(T) = \mu\ I^{p} y(\zeta); y(0) = 0, y(T) = 0, \xi, \eta, \zeta \in (0,T)$.

Acknowledgment. This project was funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, Saudi Arabia, under grant no. (KEP-MSc-23-130-40). The author, therefore, acknowledge with thanks DSR technical and financial support. We also thank the reviewers for their useful comments on our work.

References

[1] B. Ahmad, S. Ntouyas, A. Alsaedi, *On a coupled system of fractional differential equations with coupled nonlocal and integral boundary conditions*, Chaos Solitons Fractals, 2016, 83, 234–241.

[2] B. Ahmad, R. Luca, *Existence of solutions for a system of fractional differential equations with coupled nonlocal boundary conditions,* Fract. Calc. Appl. Anal., 2018, 21, 423–441.

[3] A. Alsaedi, S.K. Ntouyas, D. Garout, B. Ahmad, *Coupled fractional-order systems with nonlocal coupled integral and discrete boundary conditions*, Bull. Malays. Math. Sci. Soc., 2019, 42, 241–266.

[4] A. Granas, J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2005.

[5] M. Javidi, B. Ahmad, *Dynamic analysis of time fractional order phytoplankton-toxic phytoplankton–zooplankton system,* Ecological Modelling, 2015, 318, 8–18.

[6] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204, Elsevier Science B.V., Amsterdam, 2006.

[7] S. Liang, J. Zhang, *Existence of multiple positive solutions for m-point fractional boundary value problems on an infinite interval,* Math. Comput. Modelling, 2011, 54, 1334–1346.

[8] A.B. Malinowska, D.F.M. Torres, *Introduction to the Fractional Calculus of Variations*, Imperial College Press, London, 2012.

[9] J. Sabatier, O.P. Agrawal, J.A.T. Machado (Eds.), *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*, Springer, Dordrecht, 2007.

[10] J. R. Wang, Y. Zhang, *Analysis of fractional order differential coupled systems,* Math. Methods Appl. Sci., 2015, 38, 3322–3338.

[11] C. Zhai, L. Xu, *Properties of positive solutions to a class of four-point boundary value problem of Caputo fractional differential equations with a parameter*, Commun. Nonlinear Sci. Numer. Simul., 2014, 19, 2820–2827.