Journal of Applied Analysis and Computation
Volume 9, Number 3, June 2019, 1183–1192

Website:http://jaac.ijournal.cn
DOI:10.11948/2156-907X.20190095

THE EXISTENCE OF GLOBAL SOLUTIONS FOR THE FOURTH-ORDER NONLINEAR SCHRÖDINGER EQUATIONS*

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Abstract In this paper, the problem of a class of multidimensional fourth-order nonlinear Schrödinger equation including the derivatives of the unknown function in the nonlinear term is studied, and the existence of global weak solutions of nonlinear Schrödinger equation is proved by the Galerkin method according to the different values of λ.

Keywords Fourth-order nonlinear Schrödinger equation, Galerkin method, initial boundary value problem.

MSC(2010) 35Q40, 35Q55.

1. Introduction

In this paper, we consider the following multidimensional fourth-order nonlinear Schrödinger equation (1.1) with initial boundary values (1.2) and (1.3)

$$iu_t + \Delta^2 u + \lambda \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\big(|\frac{\partial u}{\partial x_j}|^{p-2}\frac{\partial u}{\partial x_j}\big) - \alpha\Delta|u|^2 u + \beta(x) f(|u|^2)u = 0, \tag{1.1}$$

$$u|_{t=0} = u_0(x), \qquad x \in \Omega, \tag{1.2}$$

$$u|_{\partial\Omega} = 0, \quad \Delta u|_{\partial\Omega} = 0, \quad t \geq 0, \tag{1.3}$$

where Ω is a bounded smooth domain, $u(x,t) = (u_1(x,t), u_2(x,t), \cdots, u_n(x,t))$ is a complex-valued vector function, $f(s)$ is a real-valued function satisfying $|f(s)| \leq As^q + B$, with $A, B > 0$, and $q \geq 0$, $\alpha > 0$ is a real number and $\beta(x)$ is a given real-valued bounded function. The initial data $u_0(x)$ is a given complex-valued function, and the boundary condition is taken to be the Dirichlet boundary condition for both u and Δu on the boundary $\partial\Omega$.

The nonlinear Schrödinger (NLS) equation arises from the study of nonlinear wave propagation in dispersive and inhomogeneous media, such as plasma phenomena and non-uniform dielectric media [8]. In recent years, many studies have

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*The authors were supported by National Natural Science Foundation of China (No.11771444), the Yue Qi Young Scholar project, China University of Mining and Technology(Beijing) and China Scholarship Council(CSC).

been devoted to the nonlinear Schrödinger equations with a variety of nonlinearities. Some methods, theoretical, numerical or analytical, have been used to deal with these problems. The fourth-order physical models occur in various areas of physics, including nonlinear optics, plasma physics, superconductivity and quantum mechanics [1, 4, 6]. Several exact solutions were obtained for the fourth-order nonlinear Schrodinger equation with cubic and power law nonlinearities in [11]. Karpman investigated the stability of the soliton solutions to the fourth-order nonlinear Schrödinger equations with arbitrary power nonlinearities in different space dimensions in [7]. Guo and Cui obtained the global existence of solutions for a fourth-order nonlinear Schrödinger equation in [3]. Huo and Jia in [5] considered the Cauchy problem for the fourth-order nonlinear Schrödinger equation related to the vortex filament. In this paper, we study the global existence of weak solutions for the initial boundary problem of the more general nonlinear Schrödinger equation. In particular, we will establish global existence of weak solutions for equations (1.1)-(1.3), which is different from that considered in [3] in several aspects. Also, the present results extend the results of [2] in the following aspects. Firstly, compared to the condition $n < 4$ in [2], we can prove that weak solutions exist for $n < 6$. Secondly, the emphasis of this paper will be the different values of λ. We define

$$E(t) \equiv \frac{1}{2}\|\Delta u\|^2 - \frac{\lambda}{p}\sum_{j=1}^{n}\int_\Omega \Big|\frac{\partial u}{\partial x_j}\Big|^p dx + \frac{\alpha}{4}\|\nabla|u|^2\|^2 + \frac{1}{2}\int_\Omega \beta(x)F(|u(t)|^2)dx,$$

where $F(s) = \int_0^s f(\tau)d\tau$. When $\lambda < 0$, the second term in energy $E(t)$ is positive, and we can obtain some useful estimates. But when $\lambda > 0$, we need some mathematical techniques to deal with this energy in order to obtain some estimates. One can refer to Lemma 2.2 and Lemma 2.3 for the details.

Let $L^p(\Omega)$ be the p-times integrable function space on Ω with norm denoted by $\|\cdot\|_{L^p}$. Of course, $\|\cdot\|_{L^2}$ and $\|\cdot\|$ coincide. Let H_0^s be the completion of $C_0^\infty(\Omega)$ in the H^s norm, and $W_0^{s,p}$ be the completion of $C_0^\infty(\Omega)$ in the $W^{s,p}$ norm. We use $\|\cdot\|_s$ to denote the H^s Sobolev-norm for $s \in R$, and use $\|\cdot\|_{W^{s,p}}$ to denote the $W^{s,p}$ norm. The letter C will denote some positive constant, which may change from one line to another.

Definition 1.1. A complex-valued vector function $u(x,t)$ is called a global weak solution of equations (1.1)-(1.3), if for any $T > 0$, $u(x,t) \in L^\infty(0,T; H_0^2(\Omega) \cap W_0^{1,p}(\Omega)) \cap W^{1,\infty}(0,T; H^{-2}(\Omega) \cap W^{-1,p'}(\Omega))$, $p \geq 2$, and it satisfies:

$$\begin{aligned} &-i\int_0^T (u, v_t)dt + \int_0^T (\Delta u, \Delta v)dt - \lambda \int_0^T \int_\Omega (\sum_{j=1}^n \Big|\frac{\partial u}{\partial x_j}\Big|^{p-2}\frac{\partial u}{\partial x_j}\frac{\partial \overline{v}}{\partial x_j})dxdt \\ &-\alpha\int_0^T (\Delta|u|^2 u, v)dt + \int_0^T (\beta(x)f(|u|^2)u, v)dt - i(u_0(x), v(0)) = 0, \end{aligned} \quad (1.4)$$

for all $v(x,t) \in C^1(0,T; L^2(\Omega)) \cap C(0,T; H_0^2(\Omega) \cap W_0^{1,p}(\Omega))$, and $v(x,T) \equiv 0$.

The main result of this paper is as follows:

Theorem 1.1. *Assume that* $n < 6$, $u_0 \in H_0^2(\Omega) \cap W_0^{1,p}(\Omega)$, *and* $\int_\Omega \beta(x)F(|u_0(x)|^2)dx < \infty$, $|\beta(x)| \leq M$. *If either* (P_1) *or* (P_2) *below is satisfied,*

(P_1) $\lambda < 0$, *and* $0 \leq q < \min\{\frac{4}{n}, \frac{p}{2} - 1 + \frac{p}{n}\}$, $p \geq 2$,

(P_2) $\lambda > 0,\ 0 \le q < \frac{4}{n}$, and $2 \le p < \frac{2n+8}{n+2}$,

then there exists a global weak solution for the equations (1.1)-(1.3).

This paper is organized as follows. In section 2, we give some *a prior* estimates according to the different value of λ; In section 3, we obtain the existence of weak solutions by the Galerkin method.

2. A priori estimates

Lemma 2.1. *Assume that $u_0(x) \in L^2(\Omega)$, $\beta(x)$ and $f(s)$ are real-valued functions, and $u(x,t)$ is the solution of the equations (1.1)-(1.3). Then $\|u(t)\|^2 = \|u_0\|^2$.*

Proof. Taking the inner product of the equation (1.1) with u, we have

$$\begin{aligned} i(u_t,u) + (\Delta^2 u, u) &+ \lambda\Big(\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\big(|\frac{\partial u}{\partial x_j}|^{p-2}\frac{\partial u}{\partial x_j}\big), u\Big) \\ &- \alpha(\Delta|u|^2 u, u) + (\beta(x) f(|u|^2)u, u) = 0. \end{aligned} \tag{2.1}$$

Notice that

$$(\Delta^2 u, u) = \int_\Omega \Delta^2 u \cdot \overline{u} dx = \int_\Omega \Delta u \cdot \Delta \overline{u} dx + \int_{\partial\Omega} \big(\frac{\partial \Delta u}{\partial n}\cdot \overline{u} - \frac{\partial \overline{u}}{\partial n}\cdot \Delta u\big) dS = \|\Delta u\|^2,$$

where we used the boundary condition (1.3).

For the third term on the left-hand side of equation (2.1), applying the integration by parts, we have

$$\begin{aligned} \lambda\Big(\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\big(|\frac{\partial u}{\partial x_j}|^{p-2}\frac{\partial u}{\partial x_j}\big), u\Big) &= \lambda \int_\Omega \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\big(|\frac{\partial u}{\partial x_j}|^{p-2}\frac{\partial u}{\partial x_j}\big)\overline{u} dx \\ &= -\lambda \sum_{j=1}^{n} \int_\Omega |\frac{\partial u}{\partial x_j}|^{p-2}\frac{\partial u}{\partial x_j}\frac{\partial \overline{u}}{\partial x_j} dx \\ &= -\lambda \sum_{j=1}^{n} \int_\Omega |\frac{\partial u}{\partial x_j}|^{p} dx. \end{aligned} \tag{2.2}$$

We can easily obtain the following estimates

$$-\alpha(\Delta|u|^2 u, u) = -\alpha \int_\Omega \Delta|u|^2 u\overline{u} dx = -\alpha \int_\Omega \Delta|u|^2 |u|^2 dx, \tag{2.3}$$

and

$$\big(\beta(x) f(|u|^2)u, u\big) = \int_\Omega \beta(x) f(|u|^2)|u|^2 dx, \tag{2.4}$$

where $\beta(x), f(s)$ are real-valued functions.

Hence, taking the imaginary part of the equation (2.1), we can obtain $\|u(t)\|^2 = \|u_0\|^2$. □

Lemma 2.2. *Let $u_0 \in H_0^2(\Omega) \cap W_0^{1,p}(\Omega)$, $\int_\Omega \beta(x) F(|u_0(x)|^2) dx < \infty$, and $u(x,t)$ be the solution of (1.1)-(1.3). Then we have*

$$E(t) \equiv \frac{1}{2}\|\Delta u\|^2 - \frac{\lambda}{p}\sum_{j=1}^{n}\int_\Omega |\frac{\partial u}{\partial x_j}|^p dx + \frac{\alpha}{4}\|\nabla |u|^2\|^2 + \frac{1}{2}\int_\Omega \beta(x) F(|u(t)|^2) dx = E(0),$$

where $F(s) = \int_0^s f(\tau)d\tau$.

Proof. Taking the inner product of (1.1) with u_t, we have

$$i(u_t,u_t) + (\Delta^2 u, u_t) + \lambda\Big(\sum_{j=1}^n \frac{\partial}{\partial x_j}(|\frac{\partial u}{\partial x_j}|^{p-2}\frac{\partial u}{\partial x_j}), u_t\Big) - \alpha(\Delta|u|^2u, u_t) + (\beta(x)f(|u|^2)u, u_t) = 0. \quad (2.5)$$

Next, we take the real part and estimate the terms one by one. First, for the second term on the left-hand side of equation (2.5), we get

$$Re(\Delta^2 u, u_t) = Re\int_\Omega \Delta u \cdot \Delta\overline{u_t}dx + Re\int_{\partial\Omega}(\overline{u_t}\frac{\partial \Delta u}{\partial n} - \Delta u\frac{\partial \overline{u_t}}{\partial n})dS = \frac{1}{2}\frac{d}{dt}\|\Delta u\|^2. \quad (2.6)$$

Applying the integration by parts, we obtain

$$\begin{aligned} &Re\lambda\Big(\sum_{j=1}^n \frac{\partial}{\partial x_j}(|\frac{\partial u}{\partial x_j}|^{p-2}\frac{\partial u}{\partial x_j}), u_t\Big) \\ &= -\lambda Re\sum_{j=1}^n\int_\Omega (|\frac{\partial u}{\partial x_j}|^{p-2}\frac{\partial u}{\partial x_j})\cdot\frac{\partial \overline{u_t}}{\partial x_j}dx \\ &= -\frac{\lambda}{p}\frac{d}{dt}\sum_{j=1}^n\int_\Omega|\frac{\partial u}{\partial x_j}|^p dx, \end{aligned} \quad (2.7)$$

$$\begin{aligned} -\alpha Re(\Delta|u|^2u, u_t) &= \frac{\alpha}{2}Re\int_\Omega \nabla|u|^2\cdot\frac{d}{dt}\nabla|u|^2dx - \frac{\alpha}{2}Re\int_{\partial\Omega}\frac{\partial|u|^2}{\partial n}\cdot\frac{d}{dt}|u|^2dx \\ &= \frac{\alpha}{4}\frac{d}{dt}\|\nabla|u|^2\|^2, \end{aligned} \quad (2.8)$$

$$\begin{aligned} Re(\beta(x)f(|u|^2)u, u_t) &= \frac{1}{2}\int_\Omega\beta(x)f(|u|^2)\frac{d}{dt}|u|^2dx \\ &= \frac{1}{2}\frac{d}{dt}\int_\Omega\beta(x)F(|u|^2)dx. \end{aligned} \quad (2.9)$$

From (2.6)-(2.9), we deduce that

$$\frac{d}{dt}\Big[\frac{1}{2}\|\Delta u\|^2 - \frac{\lambda}{p}\sum_{j=1}^n\int_\Omega|\frac{\partial u}{\partial x_j}|^p dx + \frac{\alpha}{4}\|\nabla|u|^2\|^2 + \frac{1}{2}\int_\Omega\beta(x)F(|u(t)|^2)dx\Big] = 0.$$

The proof is complete. □

Lemma 2.3. *Assume that* $|f(s)| \le As^q + B,\ A, B > 0$. *Under the assumptions in Lemma 2.2, if either*
case 1: *when* $\lambda < 0$ *and* $0 \le q < \min\{\frac{4}{n}, \frac{p}{2} - 1 + \frac{p}{n}\},\ p \ge 2$, *or*
case 2: *when* $\lambda > 0,\ 0 \le q < \frac{4}{n}$, *and* $2 \le p < \frac{2n+8}{n+2}$,
then we have

$$\|\Delta u\|^2 + \sum_{j=1}^n\int_\Omega|\frac{\partial u}{\partial x_j}|^p dx \le C, \quad \int_\Omega\beta(x)F(|u(t)|^2)dx \le C.$$

Proof. From the assumed conditions, we have

$$\big|\int_\Omega \beta(x)F(|u(t)|^2)dx\big| \leq C\big(\int_\Omega |u(t)|^{2q+2}dx + \int_\Omega |u(t)|^2dx\big),$$

where $F(s) = \int_0^s f(\tau)d\tau$.

According to Lemma 2.1, the second term on the right-hand side of above inequality is bounded. Then

$$\big|\int_\Omega \beta(x)F(|u(t)|^2)dx\big| \leq C(\int_\Omega |u(t)|^{2q+2}dx + 1). \tag{2.10}$$

By the Gagliardo-Nirenberg inequality, we get

$$\|u\|^{2q+2}_{L^{2q+2}} \leq C\|u\|^{\frac{4+q(4-n)}{2}}\|D^2u\|^{\frac{qn}{2}}.$$

Noticing that $nq < 4$, combining Lemma 2.1 and the ϵ-Young inequality, we have

$$\|u\|^{2q+2}_{L^{2q+2}} \leq C\|\Delta u\|^{\frac{qn}{2}} \leq \frac{1}{4}\|\Delta u\|^2 + C. \tag{2.11}$$

Using the Gagliardo-Nirenberg inequality again,

$$\|u\|^{2q+2}_{L^{2q+2}} \leq C\|Du\|^{\frac{2npq}{np-2n+2p}}_{L^p}\|u\|^{\frac{4q(p-n)+2(np-2n+2p)}{np-2n+2p}}.$$

Combing with Lemma 2.1, when $q < \frac{p}{2} - 1 + \frac{p}{n}$, we have

$$\|u\|^{2q+2}_{L^{2q+2}} \leq C\|Du\|^{\frac{2npq}{np-2n+2p}}_{L^p} \leq \frac{|\lambda|}{2p}\|Du\|^p_{L^p} + C. \tag{2.12}$$

case 1: $\lambda < 0$, the second term in $E(t)$ is positive. Then from Lemma 2.2, we have

$$\frac{1}{4}\|\Delta u\|^2 + \frac{\alpha}{4}\|\nabla|u|^2\|^2 - \frac{\lambda}{p}\sum_{j=1}^n\int_\Omega |\frac{\partial u}{\partial x_j}|^p dx \leq C.$$

Then

$$\|\Delta u\|^2 + \sum_{j=1}^n\int_\Omega |\frac{\partial u}{\partial x_j}|^p dx \leq C.$$

By (2.11), we have

$$\int_\Omega \beta(x)F(|u(t)|^2)dx \leq C.$$

In conclusion, when $0 \leq q < \min\{\frac{4}{n}, \frac{p}{2} - 1 + \frac{p}{n}\}$ and $p \geq 2$,

$$\|\Delta u\|^2 + \sum_{j=1}^n\int_\Omega |\frac{\partial u}{\partial x_j}|^p dx \leq C,$$

$$\int_\Omega \beta(x)F(|u(x,t)|^2)dx \leq C.$$

case 2: $\lambda > 0$, and $(n-2)p < 2n$, if $\frac{2p-2n+np}{4} < 2$. Then by using the Gagliardo-Nirenberg inequality and the ϵ-Young inequality, we have

$$\begin{aligned}\frac{\lambda}{p}\sum_{j=1}^{n}\int_\Omega |\frac{\partial u}{\partial x_j}|^p dx \le \frac{\lambda}{p}C\|Du\|_{L^p}^p &\le \frac{\lambda}{p}C\|u\|^{\frac{np+2p-2n}{4}}\|\Delta u\|^{\frac{np+2p-2n}{4}}\\ &\le \frac{\lambda}{p}C\|\Delta u\|^{\frac{2p-2n+np}{4}}\\ &\le \frac{1}{8}\|\Delta u\|^2 + C. \end{aligned}\tag{2.13}$$

In order to satisfy the conditions $(n-2)p < 2n$ and $(n+2)p < 8+2n$, we only need to have $p < \frac{2n+8}{n+2}$. In particular, when $2 \le p < \frac{2n+8}{n+2}$, from (2.11), we have

$$\frac{1}{8}\|\Delta u\|^2 + \frac{\alpha}{4}\|\nabla |u|^2\|^2 \le C, \quad \int_\Omega \beta(x)F(|u(t)|^2)dx \le C.$$

The proof is complete. □

Lemma 2.4. *Under the assumptions in Lemma 2.3, if $n < 6$, then the following estimate holds*

$$\|u_t\|_{H^{-2}(\Omega)\cap W^{-1,p'}(\Omega)} \le C.$$

Proof. $\forall v \in H_0^2(\Omega)\cap W_0^{1,p}(\Omega)$, taking the inner product of the equation (1.1) with v, we have

$$\begin{aligned}i(u_t, v) + (\Delta^2 u, v) + \lambda(\sum_{j=1}^{n}\frac{\partial}{\partial x_j}(|\frac{\partial u}{\partial x_j}|^{p-2}\frac{\partial u}{\partial x_j}), v)&\\ - \alpha(\Delta|u|^2 u, v) + (\beta(x)f(|u|^2)u, v) = 0.&\end{aligned}\tag{2.14}$$

For the second term on the left-hand side of (2.14), we have

$$(\Delta^2 u, v) = \int_\Omega \Delta^2 u\overline{v}dx = \int_\Omega \Delta u\Delta\overline{v}dx + \int_{\partial\Omega}(\frac{\partial\Delta u}{\partial n}\cdot\overline{v} - \Delta u\cdot\frac{\partial\overline{v}}{\partial n})dS \le \|\Delta u\|\|\Delta v\| \le C\|v\|_2.$$

For the third term, combing the conclusion of Lemma 2.3 and the Hölder inequality, we obtain

$$\begin{aligned}|\lambda\sum_{j=1}^{n}\frac{\partial}{\partial x_j}(|\frac{\partial u}{\partial x_j}|^{p-2}\frac{\partial u}{\partial x_j})\overline{v}dx| &\le |\lambda|\sum_{j=1}^{n}|\frac{\partial u}{\partial x_j}|^{p-1}|\frac{\partial\overline{v}}{\partial x_j}|dx\\ &\le |\lambda|\sum_{j=1}^{n}\Big[\big(\int_\Omega|\frac{\partial u}{\partial x_j}|^p dx\big)^{\frac{p-1}{p}}\big(\int_\Omega|\frac{\partial\overline{v}}{\partial x_j}|^p dx\big)^{\frac{1}{p}}\Big]\\ &\le C\|v\|_{W_0^{1,p}}.\end{aligned}\tag{2.15}$$

By the integration by parts, we deduce

$$\begin{aligned}|(\Delta|u|^2 u, v)| &= |\int_\Omega(\Delta|u|^2 u)\cdot\overline{v}dx|\\ &\le 2\Big[\int_\Omega |u||\nabla u|^2|\overline{v}|dx + \int_\Omega |u|^2|\nabla u||\nabla\overline{v}|dx\Big].\end{aligned}\tag{2.16}$$

We estimate the terms in (2.16) respectively. Using the Hölder inequality

$$\int_\Omega |u||\nabla u|^2|\overline{v}|dx \le \|u\|_{L^6}\|\nabla u\|_{L^3}^2\|v\|_{L^6},$$

and the Gagliardo-Nirenberg inequality for $n<6$,

$$\|u\|_{L^6} \le C\|u\|^{\frac{6-n}{6}}\|\Delta u\|^{\frac{n}{6}},$$

$$\|\nabla u\|_{L^3}^2 \le C\|u\|^{\frac{6-n}{6}}\|\Delta u\|^{\frac{6+n}{6}}.$$

Thus, for the first term in (2.16), we have

$$\int_\Omega |u||\nabla u|^2|\overline{v}|dx \le C\|u\|^{\frac{6-n}{3}}\|\Delta u\|^{\frac{n+3}{3}}\|v\|^{\frac{6-n}{6}}\|\Delta v\|^{\frac{n}{6}} \le C.$$

Similarly, by the Hölder inequality, we obtain

$$\int_\Omega |u|^2|\nabla u||\nabla \overline{v}|dx \le C\|u\|_{L^6}^2\|\nabla u\|_{L^3}\|\nabla v\|_{L^3}.$$

Using the Gagliardo-Nirenberg inequality for $n<6$,

$$\|u\|_{L^6}^2 \le C\|u\|^{\frac{6-n}{3}}\|\Delta u\|^{\frac{n}{3}},$$

$$\|\nabla u\|_{L^3} \le C\|u\|^{\frac{6-n}{12}}\|\Delta u\|^{\frac{6+n}{12}}.$$

Then we have

$$\int_\Omega |u|^2|\nabla u||\nabla \overline{v}|dx \le C\|u\|^{\frac{30-5n}{12}}\|\Delta u\|^{\frac{6+5n}{12}}\|v\|^{\frac{6-n}{12}}\|\Delta v\|^{\frac{6+n}{12}} \le C.$$

Finally, we need to estimate the term $|(\beta(x)f(|u|^2)u,v)|$. By the assumed conditions, we get

$$\begin{aligned}|(\beta(x)f(|u|^2)u,v)| &\le C|\int_\Omega (A|u|^{2q}+B)u\overline{v}dx| \\ &\le C\int_\Omega |u|^{2q+1}|\overline{v}|dx + C\|u\|\|v\|.\end{aligned}$$

By the Hölder inequality, the estimates (2.11) and (2.12), combining with Lemma 2.3, we have

$$\int_\Omega |u|^{2q+1}|\overline{v}|dx \le \|u\|_{L^{2q+2}}^{2q+1}\|v\|_{L^{2q+2}} \le C. \tag{2.17}$$

In conclusion, from (2.14)-(2.17), we obtain $\|u_t\|_{H^{-2}(\Omega)\cap W^{-1,p'}(\Omega)} \le C$. □

3. Existence of global solutions

In this section, we give the proof of Theorem 1.1.

The proof of Theorem 1.1. Utilizing the estimates from Lemmas 2.2−2.4, we have

$$\|\Delta u\| + \|u\|_{W_0^{1,p}(\Omega)} + \|u_t\|_{H^{-2}(\Omega)\cap W^{-1,p'}(\Omega)} \le C.$$

Next we will use the Galerkin method to prove the existence of solutions. Assume $\{w_j\}_{j=1}^{\infty}$ is a complete basis in $H_0^2(\Omega)$, and let

$$u_m(x,t)=\sum_{j=1}^{m}(u,w_j)w_j.$$

Then $u_m(x,t)$ is the solution of approximate problems corresponding to equations (1.1)-(1.3). Considering the initial problem of the following ordinary differential equations:

$$\begin{aligned}i(u_{mt},w_i)+(\Delta u_m,\Delta w_i)-\lambda\int_\Omega\sum_{j=1}^{n}|\frac{\partial u_m}{\partial x_j}|^{p-2}\frac{\partial u_m}{\partial x_j}\frac{\partial \overline{w}_i}{\partial x_j}dx\\ -\alpha(\Delta|u_m|^2u_m,w_i)+(\beta(x)f(|u_m|^2)u_m,w_i)=0,\\ u_m(0)=u_{0m}(x).\end{aligned}$$

Assuming $u_{0m}(x)\to u_0(x)$ in $H_0^2\cap W_0^{1,p}$ as $m\to\infty$, combining the estimates of Lemma 2.2 to Lemma 2.4, we have

$$\|\Delta u_m\|+\|u_m\|_{W_0^{1,p}(\Omega)}+\|u_{mt}\|_{H^{-2}(\Omega)\cap W^{-1,p'}(\Omega)}\le C.$$

Consequently, there exits a subsequence of $\{u_m(x,t)\}$ (still denoted by $\{u_m(x,t)\}$) such that

$$u_m(x,t)\to u(x,t)\quad \text{weakly-}*\text{ in } L^\infty(0,T;H_0^2(\Omega));$$

$$u_m(x,t)\to u(x,t)\quad \text{weakly-}*\text{ in } L^\infty(0,T;W_0^{1,p}(\Omega));$$

$$u_{mt}(x,t)\to u_t(x,t)\quad \text{weakly-}*\text{ in } L^\infty(0,T;W^{-1,p'}(\Omega)\cap H^{-2}(\Omega)).$$

We obtain from [10] that there exists a subsequence of $\{u_m(x,t)\}$ (not relabeled) such that $u_m(x,t)\to u(x,t)$ strongly in $L^2(0,T;L^2(\Omega)), Du_m(x,t)\to Du(x,t)$ strongly in $L^2(0,T;L^2(\Omega))$. Then there exits a further subsequence of $\{u_m(x,t)\}$ (not relabeled again) such that $u_m(x,t)\to u(x,t)$, $Du_m(x,t)\to Du(x,t)$, for almost every $(x,t)\in\Omega\times[0,T]$.

Next, we need to prove that $u(x,t)$ satisfies equation (1.4). The main difficulty is the convergence of nonlinear terms. First, since $\|u_m\|_{W_0^{1,p}(\Omega)}\le C$, $\sum_{j=1}^{n}|\frac{\partial u_m}{\partial x_j}|^{p-2}|\frac{\partial u_m}{\partial x_j}|$ is uniformly bounded in $L^\infty(0,T;L^{\frac{p}{p-1}}(\Omega))$. Notice that $Du_m(x,t)\to Du(x,t)$ a.e. in $\Omega\times[0,T]$, we have the following result from the Lemma in [9]

$$\sum_{j=1}^{n}|\frac{\partial u_m}{\partial x_j}|^{p-2}\frac{\partial u_m}{\partial x_j}\to\sum_{j=1}^{n}|\frac{\partial u}{\partial x_j}|^{p-2}\frac{\partial u}{\partial x_j}\quad\text{weakly in }\Omega\times[0,T].$$

Furthermore,

$$\lambda\int_0^T\int_\Omega\sum_{j=1}^{n}|\frac{\partial u_m}{\partial x_j}|^{p-2}\frac{\partial u_m}{\partial x_j}\frac{\partial\overline{v}}{\partial x_j}dxdt\to\lambda\int_0^T\int_\Omega\sum_{j=1}^{n}|\frac{\partial u}{\partial x_j}|^{p-2}\frac{\partial u}{\partial x_j}\frac{\partial\overline{v}}{\partial x_j}dxdt.$$

Next we prove $\int_0^T(\Delta|u_m|^2u_m,v)dt\to\int_0^T(\Delta|u|^2u,v)dt$, and

$$(\Delta|u_m|^2u_m,v)-(\Delta|u|^2u,v)=(\Delta(|u_m|^2-|u|^2)u_m,v)+(\Delta|u|^2(u_m-u),v).$$

We consider the convergence one by one.

$$
\begin{aligned}
&-(\Delta(|u_m|^2-|u|^2)u_m, v) = -\int_\Omega \Delta(|u_m|^2-|u|^2)u_m\overline{v}dx\\
=&\int_\Omega \nabla(u_m\overline{u_m}-u\overline{u})(\nabla u_m\overline{v}+u_m\nabla\overline{v})dx\\
=&\int_\Omega \nabla\big[(u_m-u)\overline{u_m}+u(\overline{u_m}-\overline{u})\big](\nabla u_m\overline{v}+u_m\nabla\overline{v})dx\\
=&\int_\Omega \nabla(u_m-u)\overline{u_m}(\nabla u_m\overline{v}+u_m\nabla\overline{v})dx+\int_\Omega(u_m-u)\nabla\overline{u_m}(\nabla u_m\overline{v}+u_m\nabla\overline{v})dx\\
&+\int_\Omega \nabla u(\overline{u_m}-\overline{u})(\nabla u_m\overline{v}+u_m\nabla\overline{v})dx+\int_\Omega u\nabla(\overline{u_m}-\overline{u})(\nabla u_m\overline{v}+u_m\nabla\overline{v})dx\\
=&I_1+I_2+I_3+I_4.
\end{aligned}
$$

By the Hölder inequality and Gagliardo-Nirenberg inequality, we get

$$I_1 \le \|\nabla(u_m-u)\|_{L^3}\|u_m\|_{L^6}(\|v\|_{L^6}\|\nabla u_m\|_{L^3}+\|\nabla v\|_{L^3}\|u_m\|_{L^6});$$

$$I_2 \le \|u_m-u\|_{L^6}\|\nabla u_m\|_{L^3}(\|v\|_{L^6}\|\nabla u_m\|_{L^3}+\|\nabla v\|_{L^3}\|u_m\|_{L^6});$$

$$I_3 \le \|\nabla u\|_{L^3}\|u_m-u\|_{L^6}(\|\nabla v\|_{L^3}\|u_m\|_{L^6}+\|v\|_{L^6}\|\nabla u_m\|_{L^3});$$

$$I_4 \le \|u\|_{L^6}\|\nabla(u_m-u)\|_{L^3}(\|\nabla v\|_{L^3}\|u_m\|_{L^6}+\|v\|_{L^6}\|\nabla u_m\|_{L^3}).$$

Furthermore, by the Gagliardo-Nirenberg inequality

$$\|\nabla(u_m-u)\|_{L^3} \le C\|u_m-u\|^{\frac{6-n}{12}}\|\Delta(u_m-u)\|^{\frac{6+n}{12}},$$

$$\|u_m-u\|_{L^6} \le C\|u_m-u\|^{\frac{6-n}{6}}\|\Delta(u_m-u)\|^{\frac{n}{6}},$$

we have for $n<6$, $\int_0^T(\Delta(|u_m|^2-|u|^2)u_m, v)dt \to 0$, as $m\to\infty$.

$$
\begin{aligned}
(\Delta|u|^2(u_m-u), v) &= \int_\Omega \nabla|u|^2\nabla((u_m-u)\overline{v})dx\\
&\le 2\int_\Omega \big(|u||\nabla u||\nabla(u_m-u)\overline{v}|+|u||\nabla u||(u_m-u)\nabla\overline{v}|\big)dx.
\end{aligned}
$$

Similarly, we have $\int_0^T(\Delta|u|^2(u_m-u), v)dt\to 0,\ m\to\infty$. Then we get

$$\int_0^T(\Delta|u_m|^2u_m, v)dt \to \int_0^T(\Delta|u|^2u, v)dt.$$

At last, we show that

$$\int_0^T(\beta(x)f(|u_m|^2)u_m, v)dt \to \int_0^T(\beta(x)f(|u|^2)u, v)dt.$$

Since $\|u_m\|_{L^{2q+2}}$ is bounded, and

$$|\beta(x)f(|u_m|^2)u_m| \le C(|u_m|^{2q+1}+|u_m|),$$

then we can obtain that $\beta(x)f(|u_m|^2)u_m$ is uniformly bounded in $L^\infty(0,T;L^{2q+2/2q+1})$, and combining the result $u_m(x,t) \to u(x,t)$ a.e.. Thus we have

$$\beta(x)f(|u_m|^2)u_m \to \beta(x)f(|u|^2)u \quad \text{weakly.}$$

Finally, we obtain

$$\int_0^T \int_\Omega \beta(x)f(|u_m|^2)u_m\overline{v}dxdt \to \int_0^T \int_\Omega \beta(x)f(|u|^2)u\overline{v}dxdt.$$

The proof is complete. □

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