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PULLBACK ATTRACTORS AND INVARIANT MEASURES FOR THE DISCRETE ZAKHAROV EQUATIONS*

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Abstract This article studies the probability distributions of solutions in the phase space for the discrete Zakharov equations. The authors first prove that the generated process of the solutions operators possesses a pullback-$\mathcal{D}$ attractor, and then they establish that there exists a unique family of invariant Borel probability measures supported by the pullback attractor.

Keywords Invariant measure, pullback attractor, discrete Zakharov equations.

MSC(2010) 35B41, 37Lxx, 35Qxx.

1. Introduction

This article studies the following non-autonomous system of discrete Zakharov equations

$$i\dot{\psi}_m + (A\psi)_m - h^2(D\psi)_m - u_m\psi_m + i\gamma\psi_m = f_m(t), \quad m \in \mathbb{Z},\ t > \tau, \tag{1.1}$$

$$\ddot{u}_m - (Au)_m + h^2(Du)_m - (A|\psi|^2)_m + \alpha\dot{u}_m + \mu u_m = g_m(t), \quad m \in \mathbb{Z},\ t > \tau, \tag{1.2}$$

with initial values

$$\psi_m(\tau) = \psi_{m,\tau}, \quad u_m(\tau) = u_{m,\tau}, \quad \dot{u}_m(\tau) = u_{1m,\tau}, \quad m \in \mathbb{Z},\ \tau \in \mathbb{R}, \tag{1.3}$$

where the unknown functions $\psi_m(\cdot) \in \mathbb{C}$, $u_m(\cdot) \in \mathbb{R}$, $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{Z}$ are the sets of complex, real and integer numbers, respectively. In addition, h, γ, α, μ are positive constants and i is the unit of the imaginary numbers such that $i^2 = -1$. In equations (1.1)-(1.2), $|\psi|^2 = (|\psi_m|^2)_{m\in\mathbb{Z}}$, A and D are linear operators defined as

$$(Au)_m = u_{m+1} - 2u_m + u_{m-1},\ \forall u = (u_m)_{m\in\mathbb{Z}},$$
$$(Du)_m = u_{m+2} - 4u_{m+1} + 6u_m - 4u_{m-1} + u_{m-2},\ \forall u = (u_m)_{m\in\mathbb{Z}}.$$

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Equations (1.1)-(1.2) can be regarded as a discrete analogue of the following non-autonomous Zakharov equations on $\mathbb{R}$:

$$\begin{cases} i\psi_t + \psi_{xx} - h^2\psi_{xxxx} - \psi u + i\gamma\psi = f(x,t), \\ u_{tt} - u_{xx} + h^2 u_{xxxx} - (|\psi|^2)_{xx} + \alpha u_t + \mu u = g(x,t), \end{cases}$$

where the complex function $\psi(x,t)$ denotes the envelope of the high-frequency electric field and the real function $u(x,t)$ represents the plasmas density measured from its equilibrium value (see [15,28]). The dissipative mechanism of the system is introduced by the terms $i\gamma\psi$, αu_t and μu. The external forces $f(x,t)$ and $g(x,t)$ are complex-valued and real-valued functions which are dependent of the time t, respectively. The quantum parameter h expresses the ratio between the ion plasmon energy and electron thermal energy.

There are some works studying the Cauchy problem and the initial boundary value problem of the continuous model of Zakharov equations or its related version, see [11,12,14,15] and the references therein. Also, there are some articles investigating the discrete Zakharov equations (see [24,25,34]). In [34], the authors proved the existence of the global attractor; in [24,25] the authors established the existence, finite dimensionality and upper semi-continuity of the kernel sections for the lattice system (1.1)-(1.2).

Lattice dynamical systems (LDSs for short) are spatiotemporal systems with discretization in some variables including coupled ordinary differential equations (ODEs for short) and coupled map lattices and cellular automata [9]. In some cases, LDSs occur as spatial discretizations of partial differential equations on unbounded (or bounded) domains. LDSs arise in a wide variety of applications, ranging from electrical engineering [7] to image processing and pattern recognition [8], laser systems [10], biology [20], chemical reaction theory [21], etc. There are many articles investigating the asymptotic behavior of LDSs, including the existence of various attractors (such as global attractor, uniform attractor, kernel sections, exponential attractor, uniform exponential attractor and random attractor), estimations of fractal dimension, Hausdorff dimension and Kolmogorov ε-entropy, see e. g. [1–5,16,17,29,31,38–43].

In this article, we are interested in the probability distributions of solutions in the phase space for equations (1.1)-(1.2). Precisely, we will investigate the invariant measure for the process generated by the solutions operators of equations (1.1)-(1.2). The invariant measures have been proven to be very useful in the understanding of turbulence (see Foias *et al.* [13]). The main reason is that the measurements of several aspects of turbulent flows are actually measurements of time-average quantities.

The invariant measures and statistical properties of dissipative systems were studied in a series of references. For instance, Wang investigated the upper semicontinuity of stationary statistical properties of dissipative systems in [30]. Łukaszewicz, Real and Robinson [26] used the notion of Generalized Banach limit to construct the invariant measures for general continuous dynamical systems on metric spaces. Later, Chekroun and Glatt-Holtz [6] improved the results of [30] and [26] to construct invariant measures for a broad class of dissipative autonomous dynamical systems. Recently, Łukaszewicz and Robinson [27] extended the result of [6] to construct invariant measures for dissipative non-autonomous dynamical systems.

Now, the ideas and approaches of [6, 27] have been successfully extended and applied to some concrete equations. For example, Zhao and Yang used the theory of [27] to construct the invariant Borel probability measures for the non-autonomous globally modified Navier-Stokes equations in [35] and for the regularized MHD equations in [44]; Zhao and Caraballo [36] extended the idea of [6] and constructed the trajectory statistical solutions for the globally modified Navier-Stokes equations; In [32, 37], Zhao and Łukaszewicz *et al.* used the approaches of [27] to investigate the invariant measures for the discrete long-wave-short-wave resonance equations and discrete Klein-Gordon-Schrödinger equations.

In this article, we will borrow the ideas of [27, 37] to prove the existence and uniqueness of the invariant Borel probability measures for the discrete Zakharov equations (1.1)-(1.2). To this end, we first prove the existence of the pullback-$\mathcal{D}$ attractor for the process $\{U(t,\tau)\}_{t\geqslant\tau}$ associated to the solutions operators in the phase space E_μ (see notation in Section 2). Then we verify the τ-continuity of $\{U(t,\tau)\}_{t\geqslant\tau}$ in the sense that for every fixed $z_*\in E_\mu$ and every fixed $t\in\mathbb{R}$, the E_μ-valued function $\tau\longmapsto U(t,\tau)z_*$ is continuous and bounded on $(-\infty,t]$. Combining these results, we can obtain the existence and uniqueness of a family of invariant Borel probability measures supported by the pullback-$\mathcal{D}$ attractor.

The key steps are proving the pullback-$\mathcal{D}$ asymptotic nullness and the τ-continuity of $\{U(t,\tau)\}_{t\geqslant\tau}$. The main difficulty comes from the nonlinear terms $(A|\psi|^2)_m$ and the higher-order derivative terms $(D\psi)_m$ and $(Du)_m$. This difficulty requires us to use some technical estimates and do some delicate computations. At the same time, recall that [24, 25] proved the existence and upper-semicontinuity of the compact kernel for the problem (1.1)-(1.3), with the conditions that the external forces such as $(f_m(t))_{m\in\mathbb{Z}}$ satisfies

$$\begin{cases} \text{For each } \ \tau\in\mathbb{R} \ \text{ and } \ \forall\,\varepsilon>0,\ \exists\, M(\varepsilon,\tau)\in\mathbb{N} \ \text{ such that} \\ \displaystyle\sum_{|m|\geqslant M(\varepsilon,\tau)} |f_m(s)|^2\leqslant\varepsilon \ \text{ for any } \ s\leqslant\tau \end{cases} \tag{1.4}$$

and

$$\sup_{t\in\mathbb{R}}\|f(t)\|^2=\sup_{t\in\mathbb{R}}\sum_{m\in\mathbb{Z}}|f_m(t)|^2<+\infty. \tag{1.5}$$

Compared to the kernel sections discussed in [24], on the one hand, the pullback-$\mathcal{D}$ attractor we will discuss here possesses more general basins of attraction, which are referred to a given universe $\mathcal{D}$ rather than only fixed bounded sets. Different universes provide different basins of attraction and will give rise to different pullback attractor, reflecting different aspects of the dynamics. Indeed, any fixed bounded sets lie in the universe $\mathcal{D}$. On the other hand, we remove conditions (1.4) and (1.5) imposed on the external forces. Condition (1.5) implies that $f(t)$ is continuous and uniformly bounded on $\mathbb{R}$. In this article we will just require that $f(t)$ is continuous on $\mathbb{R}$ and is unnecessary bounded. Indeed, we allow that $f(t)$ is unbounded and even can be exponentially growing with respect to time t. Without using the conditions (1.4) and (1.5), we shall also perform some technical estimates and delicate computations.

The rest of this article is arranged as follows. In the next section, we first introduce some notations, then we prove some estimates and show the global well-posedness of problem (1.1)-(1.3). In section 3, we establish the pullback-$\mathcal{D}$ asymptotic nullness of the process $\{U(t,\tau)\}_{t\geqslant\tau}$ and get the existence of the pullback-$\mathcal{D}$ attractor. In the last section, we prove the τ-continuity of $\{U(t,\tau)\}_{t\geqslant\tau}$ and obtain the existence of a unique family of invariant Borel probability measures supported by the pullback-$\mathcal{D}$ attractor.

2. Estimates and global well-posedness of solutions

In this section, we first introduce some notations and prove some estimates. Then we show the global well-posedness of solutions to problem (1.1)-(1.3), which implies its solutions operators generate a continuous process.

Set

$$\ell^2 = \Big\{ v = (v_m)_{m\in\mathbb{Z}},\ \ v_m \in \mathbb{C},\ \sum_{m\in\mathbb{Z}} |v_m|^2 < +\infty \Big\},$$
$$l^2 = \Big\{ v = (v_m)_{m\in\mathbb{Z}},\ \ v_m \in \mathbb{R},\ \sum_{m\in\mathbb{Z}} v_m^2 < +\infty \Big\}.$$

For brevity we denote $X = \ell^2$ or l^2 in the sequel and equip it with the inner product and norm as

$$(u,v) = \sum_{m\in\mathbb{Z}} u_m \bar{v}_m,\ \ \|u\|^2 = (u,u),\ \ u = (u_m)_{m\in\mathbb{Z}},\ \ v = (v_m)_{m\in\mathbb{Z}} \in X,$$

where $\bar{v}_m$ denotes the conjugate of v_m. We define two linear operators B and B^* from X to X as

$$(Bu)_m = u_{m+1} - u_m, \quad \forall\, m \in \mathbb{Z}, \quad \forall\, u = (u_m)_{m\in\mathbb{Z}} \in X,$$
$$(B^*u)_m = u_{m-1} - u_m, \quad \forall\, m \in \mathbb{Z}, \quad \forall\, u = (u_m)_{m\in\mathbb{Z}} \in X.$$

Also we define a linear form $(\cdot,\cdot)$ on l^2 by

$$(u,v)_\mu = (Bu,Bv) + \mu(u,v). \tag{2.1}$$

By some directly computations, we see both B and B^* are bounded from X to X, and B^* is the adjoint operator of B, moreover,

$$(Au,v) = -(Bu,Bv), \quad (Du,v) = (Au,Av), \quad \forall\, u,v \in X,$$
$$\|Au\|^2 \leqslant 16\|u\|^2,\ \ \|Du\|^2 \leqslant 256\|u\|^2, \quad \forall\, u \in X.$$

At the same time, we see from (2.1) that the bilinear form $(\cdot,\cdot)_\mu$ induces an inner product in l^2. In fact,

$$\mu\|u\|^2 \leqslant \|Bu\|^2 + \mu\|u\|^2 = \|u\|_\mu^2 \leqslant (4+\mu)\|u\|^2, \quad \forall\, u = (u_m)_{m\in\mathbb{Z}} \in l^2,$$

which implies that the norm $\|\cdot\|_\mu$ induced by $(\cdot,\cdot)_\mu$ is equivalent to the norm $\|\cdot\|$.

In this article, we denote by

$$\ell^2 = (\ell^2, (\cdot,\cdot), \|\cdot\|), \quad l_\mu^2 = (l^2, (\cdot,\cdot)_\mu, \|\cdot\|_\mu), \quad l^2 = (l^2, (\cdot,\cdot), \|\cdot\|).$$

Then ℓ^2, l_μ^2 and l^2 are all Hilbert spaces. Write

$$E_\mu = \ell^2 \times l_\mu^2 \times l^2,$$

and for $z^{(k)} = (\psi^{(k)}, u^{(k)}, \varphi^{(k)})^T \in E_\mu$, $k = 1, 2$, we define the inner product and norm of E_μ by

$$
\begin{aligned}
(z^{(1)}, z^{(2)})_{E_\mu} &= (\psi^{(1)}, \psi^{(2)}) + (u^{(1)}, u^{(2)})_\mu + (\varphi^{(1)}, \varphi^{(2)}) \\
&= \psi_m^{(1)} \overline{\psi_m^{(2)}} + \sum_{m\in\mathbb{Z}} \Big((Bu^{(1)})_m (Bu^{(2)})_m + \mu u_m^{(1)} u_m^{(2)} \Big) + \varphi_m^{(1)} \varphi_m^{(2)}, \\
\|z\|_{E_\mu} &= \sqrt{(z, z)_{E_\mu}}, \quad \forall\, z \in E_\mu.
\end{aligned}
$$

For the reason of technical computations, we shall choose $E_\mu = (E_\mu, (\cdot,\cdot)_{E_\mu}, \|\cdot\|_{E_\mu})$ as the phase space of problem (1.1)-(1.3).

To write equations (1.1)-(1.2) as an abstract first-order ODE with respect to time t in space E_μ, we denote

$$
\begin{aligned}
&\psi = (\psi_m)_{m\in\mathbb{Z}}, \;\; u = (u_m)_{m\in\mathbb{Z}}, \;\; f(t) = (f_m(t))_{m\in\mathbb{Z}}, \;\; g(t) = (g_m(t))_{m\in\mathbb{Z}}, \\
&\psi u = (\psi_m u_m)_{m\in\mathbb{Z}}, \quad A|\psi|^2 = \big((A|\psi|^2)_m \big)_{m\in\mathbb{Z}},
\end{aligned}
$$

and take

$$
\varphi = \dot{u} + \lambda u, \quad \text{where } \lambda = \frac{\mu\alpha}{\alpha^2 + 4\mu} \in (0, \frac{\alpha}{4}). \tag{2.2}
$$

Then equations (1.1)-(1.2) are equivalent to

$$
\begin{cases}
\dot{\psi} + (\gamma I - iA + ih^2 D)\psi = -i\psi u - if(t), \\
\dot{u} + \lambda u - \varphi = 0, \\
\dot{\varphi} + \Big(\lambda(\lambda - \alpha) I + \mu I - A + h^2 D \Big) u + (\alpha - \lambda)\varphi = A|\psi|^2 + g(t).
\end{cases}
$$

Further, we set $z = (\psi, u, \varphi)^T$, $F(z,t) = (-i\psi u - if(t), 0, g(t) + A|\psi|^2)^T$ and

$$
\Theta = \begin{pmatrix}
\gamma I - iA + ih^2 D & 0 & 0 \\
0 & \lambda I & -I \\
0 & \lambda(\lambda-\alpha) I + \mu I - A + h^2 D & (\alpha - \lambda) I
\end{pmatrix}, \tag{2.3}
$$

then problem (1.1)-(1.3) is equivalent to

$$
\begin{cases}
\dot{z} + \Theta z = F(z,t), \quad t > \tau, \\
z(\tau) = z_\tau = (\psi_\tau, u_\tau, \varphi_\tau)^T = (\psi_\tau, u_\tau, u_{1\tau} + \lambda u_\tau)^T,
\end{cases} \tag{2.4}
$$

where $\psi_\tau = (\psi_{m,\tau})_{m\in\mathbb{Z}}$, $u_\tau = (u_{m,\tau})_{m\in\mathbb{Z}}$, $u_{1\tau} = (u_{1m,\tau})_{m\in\mathbb{Z}}$, I is the identity operator and $(\cdot,\cdot,\cdot)^T$ the transposition of a vector.

We next investigate the well-posedness of problem (2.4). In this article we denote $\mathcal{C}(\mathbb{R}, l^2)$ and $\mathcal{C}(\mathbb{R}, \ell^2)$ the spaces of continuous functions from $\mathbb{R}$ into l^2 and ℓ^2, respectively. Then for a function $f(\cdot) \in \mathcal{C}(\mathbb{R}, l^2)$ (or $\mathcal{C}(\mathbb{R}, \ell^2)$) and $t \in \mathbb{R}$, we have

$$
\|f(t)\|^2 = \sum_{m\in\mathbb{Z}} |f_m(t)|^2 < +\infty. \tag{2.5}
$$

Obviously, the condition (2.5) is weaker than (1.5).

For the local existence and uniqueness of solutions to problem (2.4), we have

Lemma 2.1. *Let $f(t) = (f_m(t))_{m\in\mathbb{Z}} \in \mathcal{C}(\mathbb{R}, \ell^2)$ and $g(t) = (g_m(t))_{m\in\mathbb{Z}} \in \mathcal{C}(\mathbb{R}, l^2)$. Then for any initial value $z_\tau = (\psi_\tau, u_\tau, \varphi_\tau)^T \in E_\mu$, there is a unique local solution $z(t) = (\psi(t), u(t), \varphi(t))^T \in E_\mu$ of problem (2.4) such that $z(\cdot) \in \mathcal{C}([\tau, T_0), E_\mu) \cap \mathcal{C}^1((\tau, T_0), E_\mu)$ for some $T_0 > \tau$. Moreover, if $T_0 < +\infty$, then $\lim\limits_{t\to T_0^-} \|z(t)\|_{E_\mu} = +\infty$.*

The proof of above lemma is similar to that of [24, Lemma 2.2], and we omit the details here. Next, we estimate the solutions. For brevity, we will employ in the sequel the notation $a \lesssim b$ (similarly $\gtrsim$) to mean that $a \leqslant cb$ (similarly $a \geqslant cb$) for a universal constant $c > 0$ that only depends on the parameters coming from the problem.

Lemma 2.2. *Let $f(t) = (f_m(t))_{m\in\mathbb{Z}} \in \mathcal{C}(\mathbb{R}, \ell^2)$, $g(t) = (g_m(t))_{m\in\mathbb{Z}} \in \mathcal{C}(\mathbb{R}, l^2)$. Let $z(t) = (\psi(t), u(t), \varphi(t))^T \in E_\mu$ be the solution of problem (2.4) corresponding to the initial value $z_\tau = (\psi_\tau, u_\tau, \varphi_\tau)^T \in E_\mu$. Then*

$$\|\psi(t)\|^2 \lesssim \|\psi_\tau\|^2 e^{-\gamma(t-\tau)} + e^{-\gamma t}\int_\tau^t e^{\gamma s}\|f(s)\|^2 \mathrm{d}s, \quad t \geqslant \tau. \tag{2.6}$$

Proof. We write equation (1.1) as

$$i\dot{\psi} + A\psi - h^2 D\psi + i\gamma\psi - \psi u = f(t), \quad \forall\, t > \tau. \tag{2.7}$$

Taking the imaginary part of the inner product $(\ell^2, (\cdot,\cdot))$ of (2.7) with ψ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\psi(t)\|^2 + \gamma\|\psi(t)\|^2 \lesssim \|f(t)\|^2, \quad \forall\, t > \tau. \tag{2.8}$$

Applying Gronwall's inequality to (2.8), we obtain (2.6). The proof is completed. □

To estimate the solution $z(t) = (\psi(t), u(t), \varphi(t))^T$, we shall use the coercivity of the operator Θ defined by (2.3).

Lemma 2.3 ([24]). *For any $z = (\psi, u, \varphi)^T \in E_\mu$, there holds*

$$\mathbf{Re}(\Theta z, z)_{E_\mu} \geqslant \delta(\|u\|_\mu^2 + \|\varphi\|^2) + \frac{\alpha}{2}\|\varphi\|^2 + \gamma\|\psi\|^2, \tag{2.9}$$

where

$$\delta = \mu\alpha\big(\sqrt{\alpha^2 + 4\mu}(\sqrt{\alpha^2 + 4\mu} + \alpha)\big)^{-1}.$$

Lemma 2.4. *Let $f(t) = (f_m(t))_{m\in\mathbb{Z}} \in \mathcal{C}(\mathbb{R}, \ell^2)$ and $g(t) = (g_m(t))_{m\in\mathbb{Z}} \in \mathcal{C}(\mathbb{R}, l^2)$. Then the solution $z(t) = (\psi(t), u(t), v(t))^T \in E_\mu$ of problem (2.4) corresponding to the initial value $z_\tau = (\psi_\tau, u_\tau, v_\tau)^T \in E_\mu$ satisfies*

$$\begin{aligned}\|z(t)\|_{E_\mu}^2 \lesssim & \|z_\tau\|_{E_\mu}^2 e^{-\sigma(t-\tau)} + e^{-\sigma t}\int_\tau^t e^{\sigma s}\Big(\|f(s)\|^2 + \|g(s)\|^2\Big)\mathrm{d}s \\ & + e^{-\sigma t}\int_\tau^t e^{\sigma s}\|\psi(s)\|^4 \mathrm{d}s, \quad \forall\, t \geqslant \tau,\end{aligned} \tag{2.10}$$

where

$$\sigma = \min\{\delta, \gamma\}. \tag{2.11}$$

Proof. Taking the real part of the inner product $(\cdot,\cdot)_{E_\mu}$ of the equation in (2.4) with $z(t)$ gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|_{E_\mu}^2+\mathbf{Re}\big(\Theta z(t),z(t)\big)_{E_\mu}=\mathbf{Re}\big(F(z,t),z(t)\big)_{E_\mu},\quad t>\tau. \tag{2.12}$$

We need estimate the term $\mathbf{Re}\big(F(z,t),z(t)\big)_{E_\mu}$. In fact,

$$\mathbf{Re}\big(F(z,t),z(t)\big)_{E_\mu}=\mathbf{Re}\big(-if(t),\psi(t)\big)+\big(g(t),\varphi(t)\big)+\big(A|\psi(t)|^2,\varphi(t)\big). \tag{2.13}$$

By Cauchy inequality and some simple computations, we have

$$\mathbf{Re}\big(-if(t),\psi\big)\lesssim\frac{\gamma}{2}\|\psi\|^2+\|f(t)\|^2, \tag{2.14}$$

$$\big(g(t),\varphi(t)\big)\lesssim\frac{\alpha}{4}\|\varphi\|^2+\|g(t)\|^2, \tag{2.15}$$

$$\begin{aligned}\big(A|\psi|^2,\varphi\big)=&\big(B|\psi|^2,B\varphi\big)\leqslant\|B|\psi|^2\|\,\|B\varphi\|\leqslant4\||\psi|^2\|\|\varphi\|\\ \leqslant&4\left[\frac{\alpha}{16}\|\varphi(t)\|^2+\frac{4}{\alpha}\|\psi(t)\|^4\right]\lesssim\frac{\alpha}{4}\|\varphi\|^2+||\psi||^4.\end{aligned} \tag{2.16}$$

Inserting (2.9) and (2.13)-(2.16) into (2.12) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|_{E_\mu}^2+2\delta(\|u\|_\mu^2+\|\varphi\|^2)+\gamma\|\psi\|^2\lesssim\|g(t)\|^2+\|f(t)\|+\|\psi\|^4.$$

Let σ be chosen as in (2.11), then we get for any $t>\tau$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|_{E_\mu}^2+\sigma\|z(t)\|_{E_\mu}^2\lesssim\|f(t)\|^2+\|g(t)\|^2+\|\psi(t)\|^4. \tag{2.17}$$

Applying Gronwall's inequality to (2.17) gives (2.10). This ends the proof. $\square$

From estimates (2.6) and (2.10) we conclude that for any $z_\tau=(\psi_\tau,u_\tau,\varphi_\tau)^T\in E_\mu$, the corresponding solution $z(t)=(\psi(t),u(t),\varphi(t))^T\in E_\mu$ of problem (2.4) exists globally on $[\tau,+\infty)$. Moreover, from Lemma 2.1 we see that

$$z(\cdot)\in\mathcal{C}([\tau,+\infty),E_\mu)\cap\mathcal{C}^1((\tau,+\infty),E_\mu). \tag{2.18}$$

Therefore, in view of Lemma 2.1, the maps of solutions operators

$$U(t,\tau):z_\tau=(\psi_\tau,u_\tau,\varphi_\tau)^T\in E_\mu\longmapsto z(t)=(\psi(t),u(t),\varphi(t))^T\in E_\mu,\ \ \forall\,t\geqslant\tau,$$

generate a continuous process $\{U(t,\tau)\}_{t\geqslant\tau}$ on E_μ.

3. Existence of the pullback-$\mathcal{D}$ attractor

The purpose of this section is to prove the existence of the pullback-$\mathcal{D}$ attractor for the process $\{U(t,\tau)\}_{t\geqslant\tau}$ on E_μ. For this purpose, we need some natural assumptions on the initial values and on the external forces $f(t)$ and $g(t)$.

Firstly, we consider the basins of attraction which the initial values lie in. In this article we denote by $\mathcal{P}(E_\mu)$ the family of all nonempty subsets of E_μ and by $\mathcal{D}_\sigma$ the class of families of nonempty subsets $\widehat{D}=\{D(s)\,|\,s\in\mathbb{R}\}\subseteq\mathcal{P}(E_\mu)$ satisfying

$$\lim_{s\to-\infty}\big(e^{\frac{\sigma s}{2}}\sup_{z\in D(s)}\|z\|_{E_\mu}^2\big)=0.$$

Hereinafter, the constant σ comes from (2.11), the class $\mathcal{D}_\sigma$ will be called a universe in $\mathcal{P}(E_\mu)$. Obviously, all fixed bounded subsets of E_μ lie in $\mathcal{D}_\sigma$.

Secondly, we give the assumptions on the external forces $f(t)$ and $g(t)$ necessary to the existence of a bounded pullback absorbing set for the process $\{U(t,\tau)\}_{t\geqslant\tau}$.

(H) *Assume* $f(t)=(f_m(t))_{m\in\mathbb{Z}}\in\mathcal{C}(\mathbb{R},\ell^2)$ *and* $g(t)=(g_m(t))_{m\in\mathbb{Z}}\in\mathcal{C}(\mathbb{R},l^2)$. *Moreover, let*

$$\int_{-\infty}^{s} e^{\sigma\eta}\|f(\eta)\|^2\mathrm{d}\eta < e^{(\frac{\sigma}{2}+\omega)s}K(s), \tag{3.1}$$

for some continuous function $K(\cdot)$ *on the real line, bounded on intervals of the form* $(-\infty,t)$, *with* $0<\omega<\frac{\sigma}{2}$, *and let*

$$\int_{-\infty}^{s} e^{\sigma\eta}\|g(\eta)\|^2\mathrm{d}\eta < +\infty, \quad \text{for each } s\in\mathbb{R}. \tag{3.2}$$

The following example shows that the functions satisfying above assumption **(H)** do exist.

Example 3.1. Let $\|f(\eta)\|^2 \leqslant Me^{\varrho\eta}$ for all real η, with constant $M>0$ and $\varrho\geqslant\omega-\frac{\sigma}{2}$. Then

$$\int_{-\infty}^{s} e^{\sigma\eta}\|f(\eta)\|^2\mathrm{d}\eta < e^{(\frac{\sigma}{2}+\omega)s}K(s),$$

with $K(s)=\dfrac{M}{\sigma+\varrho}e^{(\frac{\sigma}{2}+\varrho-\omega)s}$. Thus, choosing $\varrho<0, \varrho=0$ or $\varrho>0$, we allow different behavior of f near infinities.

Lemma 3.1. *Let assumption* **(H)** *hold. Then the process* $\{U(t,\tau)\}_{t\geqslant\tau}$ *possesses a bounded pullback-*$\mathcal{D}_\sigma$ *absorbing set* $\widehat{\mathcal{B}}_0=\{\mathcal{B}_0(s)|s\in\mathbb{R}\}\subseteq\mathcal{P}(E_\mu)$ *in the sense that for any* $t\in\mathbb{R}$ *and any* $\widehat{D}=\{D(s)|\, s\in\mathbb{R}\}\in\mathcal{D}_\sigma$, *there exists a* $\tau_0=\tau_0(t,\widehat{D})\leqslant t$ *such that* $U(t,\tau)D(\tau)\subseteq\mathcal{B}_0(t)$ *for all* $\tau\leqslant\tau_0$, *where* $\mathcal{B}_0(s)=\mathcal{B}(0,R_\sigma(s))\subset E_\mu$ *is the closed ball of radius* $R_\sigma(s)$ *and centered at zero.*

Proof. For any $z_\tau\in D(\tau)$ with $\widehat{D}=\{D(s)|\, s\in\mathbb{R}\}\in\mathcal{D}_\sigma$, we have

$$\lim_{\tau\to-\infty} e^{\frac{\sigma}{2}\tau}\|z_\tau\|^2_{E_\mu}=0. \tag{3.3}$$

Also, (3.1) and (3.2) give

$$\int_{-\infty}^{t} e^{\sigma s}\Big(\|f(s)\|^2+\|g(s)\|^2\Big)\mathrm{d}s < +\infty, \quad \text{for each } t\in\mathbb{R}. \tag{3.4}$$

We shall estimate the third term on the right-hand side of (2.10). In fact, it follows from (2.6) that

$$\int_{\tau}^{t} e^{\sigma s}\|\psi(s)\|^4\mathrm{d}s \leqslant \varrho_1(t,\tau)+\varrho_2(t,\tau)+\varrho_3(t,\tau), \tag{3.5}$$

where

$$\begin{cases} \varrho_1(t,\tau) = \displaystyle\int_\tau^t e^{\sigma s}\|\psi_\tau\|^4 e^{-2\gamma(s-\tau)}\mathrm{d}s, \\ \varrho_2(t,\tau) \lesssim \displaystyle\int_\tau^t e^{\sigma s}\|\psi_\tau\|^2 e^{\gamma\tau}e^{-2\gamma s}\int_\tau^s e^{\gamma\eta}\|f(\eta)\|^2\mathrm{d}\eta\mathrm{d}s, \\ \varrho_3(t,\tau) \lesssim \displaystyle\int_\tau^t e^{(\sigma-2\gamma)s}\Big(\int_\tau^s e^{\gamma\eta}\|f(\eta)\|^2\mathrm{d}\eta\Big)^2\mathrm{d}s. \end{cases}$$

By (3.3), we have $\lim\limits_{\tau\to-\infty}\|\psi_\tau\|^4 e^{\sigma\tau} = 0$ and then by the Lebesgue's Dominated Convergence Theorem,

$$\lim_{\tau\to-\infty}\varrho_1(t,\tau) = 0. \tag{3.6}$$

We write the bound of $\varrho_2(t,\tau)$ in the form

$$\|\psi_\tau\|^2 e^{\frac{\sigma}{2}\tau}\int_\tau^t \Big(e^{\sigma s}e^{(\gamma-\frac{\sigma}{2})\tau}e^{-2\gamma s}\int_\tau^s e^{\gamma\eta}\|f(\eta)\|^2\mathrm{d}\eta\Big)\mathrm{d}s, \tag{3.7}$$

and then by assumption **(H)**, $e^{(\frac{\sigma}{2}-\gamma)s}\displaystyle\int_{-\infty}^s e^{\gamma\eta}\|f(\eta)\|^2\mathrm{d}\eta \leqslant K(s)$, where $K(s)$ is a continuous function on the real line which is bounded on every interval of the form $(-\infty,t)$. Hence,

$$\lim_{\tau\to-\infty}\varrho_2(t,\tau) = 0, \tag{3.8}$$

since the integral in (3.7) stays bounded as $\tau\to-\infty$. Similarly, we obtain from (3.1)

$$\int_{-\infty}^s e^{\gamma\eta}\|f(\eta)\|^2\mathrm{d}\eta \leqslant e^{(\gamma-\sigma)s}\int_{-\infty}^s e^{\sigma\eta}\|f(\eta)\|^2\mathrm{d}s \leqslant e^{(\gamma-\frac{\sigma}{2}+\omega)s}K(s), \tag{3.9}$$

for some $\omega\in(0,\sigma/2)$. Thus $e^{(\frac{\sigma}{2}-\gamma-\omega)s}\displaystyle\int_{-\infty}^s e^{\gamma\eta}\|f(\eta)\|^2\mathrm{d}\eta \leqslant \tilde{K}(s)$ for some function $\tilde{K}(\cdot)$ possessing the same properties as the function $K(\cdot)$ above. Since $\tilde{K}(s)$ is bounded on every interval of the form $(-\infty,t)$, we can assume that $\tilde{K}^2(s)\leqslant\mathcal{B}(t)$ for some quantity $\mathcal{B}(t)$ depending only on t. Hence

$$\varrho_3(t,\tau) \lesssim \int_\tau^t e^{(\sigma-2\gamma)s}\Big(\int_\tau^s e^{\gamma\eta}\|f(\eta)\|^2\mathrm{d}\eta\Big)^2\mathrm{d}s \leqslant B(t)\int_\tau^t e^{\omega s}\mathrm{d}s \lesssim \mathcal{B}(t)e^{\omega t}.$$

Also by the Lebesgue's Dominated Convergence Theorem, we have

$$\lim_{\tau\to-\infty}\varrho_3(t,\tau) \lesssim \int_{-\infty}^t e^{(\sigma-2\gamma)s}\Big(\int_{-\infty}^s e^{\gamma\eta}\|f(\eta)\|^2\mathrm{d}\eta\Big)^2\mathrm{d}s, \tag{3.10}$$

and the right-hand side of (3.10) is bounded by a quantity depending only on t. By (3.5), (3.6), (3.8) and (3.10) we see that if $f(t) = (f_m(t))_{m\in\mathbb{Z}}$ satisfies the conditions in assumption **(H)**, then the third term on the right-hand side of (2.10) is bounded as

$$\lim_{\tau\to-\infty}e^{-\sigma t}\int_\tau^t e^{\sigma s}\|\psi(s)\|^4\mathrm{d}s \lesssim e^{-\sigma t}\int_{-\infty}^t e^{(\sigma-2\gamma)s}\Big(\int_{-\infty}^s e^{\gamma\eta}\|f(\eta)\|^2\mathrm{d}\eta\Big)^2\mathrm{d}s. \tag{3.11}$$

Now set

$$
\begin{aligned}
R_\sigma^2(t) \lesssim & 1+e^{-\sigma t}\int_{-\infty}^{t} e^{\sigma s}\Big(\|f(s)\|^2+\|g(s)\|^2\Big)\mathrm{d}s \\
& + e^{-\sigma t}\int_{-\infty}^{t} e^{(\sigma-2\gamma)s}\Big(\int_{-\infty}^{s} e^{\gamma\eta}\|f(\eta)\|^2\mathrm{d}\eta\Big)^2\mathrm{d}s, \quad t\in\mathbb{R}.
\end{aligned} \tag{3.12}
$$

Then from (2.10), (3.3), (3.4) and (3.11), we see that the family $\widehat{\mathcal{B}}_0=\{\mathcal{B}(0,R_\sigma(s))\,|\,s\in\mathbb{R}\}$ is the desired bounded pullback-$\mathcal{D}_\sigma$ absorbing set for $\{U(t,\tau)\}_{t\geqslant\tau}$ in E_μ. The proof is completed. $\square$

Next, we are going to investigate the pullback-$\mathcal{D}_\sigma$ asymptotic nullness of the process $\{U(t,\tau)\}_{t\geqslant\tau}$ in E_μ.

Define a smooth function $\chi(x)\in\mathcal{C}^1(\mathbb{R}_+,\mathbb{R}_+)$ such that

$$
\begin{cases}
\chi(x)=0 & \text{for } 0\leqslant x\leqslant 1,\\
0\leqslant\chi(x)\leqslant 1 & \text{for } 1\leqslant x\leqslant 2,\\
\chi(x)=1 & \text{for } x\geqslant 2,\\
|\chi'(x)|\leqslant\chi_0 \ \text{(positive constant)}, & \text{for } x\geqslant 0.
\end{cases} \tag{3.13}
$$

Let $\widehat{D}=\{D(s)|\,s\in\mathbb{R}\}\in\mathcal{D}_\sigma$ and $t,\tau\in\mathbb{R}$ with $t\geqslant\tau$. Let

$$
z(t)=z(t;\tau,z_\tau)=U(t,\tau)z_\tau=(\psi(t),u(t),\varphi(t))^T=(\psi_m(t),u_m(t),\varphi_m(t))^T_{m\in\mathbb{Z}}\in E_\mu
$$

be a solution of problem (2.4) with initial value $z_\tau\in D(\tau)$. Let M be some positive integer and define

$$
\begin{cases}
\xi_m(t)=\chi(\dfrac{|m|}{M})\psi_m(t),\quad v_m(t)=\chi(\dfrac{|m|}{M})u_m(t),\quad w_m(t)=\chi(\dfrac{|m|}{M})\varphi_m(t),\\
y(t)=(y_m(t))_{m\in\mathbb{Z}}\quad\text{with}\quad y_m(t)=\big(\xi_m(t),v_m(t),w_m(t)\big)^T.
\end{cases}
$$

Taking the real part of the inner product $(\cdot,\cdot)_{E_\mu}$ of the equation in (2.3) with $y(t)$ yields

$$
\mathbf{Re}\big(\dot z(t),y(t)\big)_{E_\mu}+\mathbf{Re}\big(\Theta z(t),y(t)\big)_{E_\mu}=\mathbf{Re}\big(F(z,t),y(t)\big)_{E_\mu},\quad\forall\, t>\tau. \tag{3.14}
$$

We next estimate the three terms of (3.14) in three auxiliary lemmas.

Lemma 3.2. *The term* $\mathbf{Re}\big(\dot z(t),y(t)\big)_{E_\mu}$ *in* (3.14) *satisfies*

$$
\mathbf{Re}\big(\dot z(t),y(t)\big)_{E_\mu}-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{m\in\mathbb{Z}}\chi(\frac{|m|}{M})|z_m(t)|^2_{E_\mu}\gtrsim-\frac{R_\sigma(t)}{M},\quad\forall\,\tau\leqslant\tau_0\leqslant t, \tag{3.15}
$$

hereinafter $R_\sigma(t)$ is the radius of the bounded pullback-$\mathcal{D}_\sigma$ absorbing set obtained in Lemma 3.1, $\tau_0=\tau_0(t,\widehat{D})$ is the pullback absorbing time and

$$
|z_m|^2_{E_\mu}=|\psi_m|^2+|(Bu)_m|^2+\mu u_m^2+\varphi_m^2,\ z=(z_m)_{m\in\mathbb{Z}}=(\psi_m,u_m,\varphi_m)^T_{m\in\mathbb{Z}}. \tag{3.16}
$$

Proof. By direct computations, we have

$$
\begin{aligned}
\mathbf{Re}\big(\dot z(t), y(t)\big)_{E_\mu} =& \sum_{m\in\mathbb{Z}} (B\dot u)_m (Bv)_m + \mu(\dot u, v) + (\dot\psi, \xi) + (\dot\varphi, w) \\
=& \sum_{m\in\mathbb{Z}} (B\dot u)_m \Big[\chi(\frac{|m+1|}{M})u_{m+1} - \chi(\frac{|m|}{M})u_m\Big] \\
&+ \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}} \sum_{m\in\mathbb{Z}} \chi(\frac{|m|}{M})\Big[\mu u_m^2 + \varphi_m^2 + |\psi_m|^2\Big] \\
=& \sum_{m\in\mathbb{Z}} \chi(\frac{|m|}{M})(B\dot u)_m (Bu)_m + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}} \sum_{m\in\mathbb{Z}} \chi(\frac{|m|}{M})\Big[\mu u_m^2 + \varphi_m^2 + |\psi_m|^2\Big] \\
&+ \sum_{m\in\mathbb{Z}} \Big[\chi(\frac{|m+1|}{M}) - \chi(\frac{|m|}{M})\Big](\dot u_{m+1} - \dot u_m)u_{m+1} \\
=& \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}} \sum_{m\in\mathbb{Z}} \chi(\frac{|m|}{M})|z_m|^2_{E_\mu} \\
&+ \sum_{m\in\mathbb{Z}} \Big[\chi(\frac{|m+1|}{M}) - \chi(\frac{|m|}{M})\Big](\varphi_{m+1} - \lambda u_{m+1} - \varphi_m + \lambda u_m)u_{m+1},
\end{aligned}
$$

where $|z_m|^2_{E\mu}$ is defined by (3.16). Thus, combining Lemma 3.1, the Mean Value Theorem and (3.13), we obtain

$$
\begin{aligned}
&\mathbf{Re}\big(\dot z(t), y(t)\big)_{E_\mu} - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}} \sum_{m\in\mathbb{Z}} \chi(\frac{|m|}{M})|z_m(t)|^2_{E_\mu} \\
=& \sum_{m\in\mathbb{Z}} \Big[\chi(\frac{|m+1|}{M}) - \chi(\frac{|m|}{M})\Big](\varphi_{m+1} - \lambda u_{m+1} - \varphi_m + \lambda u_m)u_{m+1} \\
=& \sum_{m\in\mathbb{Z}} \chi'(\frac{\tilde m}{M})\frac{1}{M}(\varphi_{m+1} - \lambda u_{m+1} - \varphi_m + \lambda u_m)u_{m+1} \\
\gtrsim& -\frac{R_\sigma(t)}{M}, \quad \forall\, \tau \leqslant \tau_0 \leqslant t,
\end{aligned}
$$

hereinafter $\tilde m$ is a positive constant locating between $|m+1|$ and $|m|$. □

Lemma 3.3. *The term* $\mathbf{Re}\big(\Theta z(t), y(t)\big)_{E_\mu}$ *in* (3.14) *satisfies*

$$
\begin{aligned}
&\mathbf{Re}\big(\Theta z(t), y(t)\big)_{E_\mu} \\
&- \sum_{m\in\mathbb{Z}} \chi(\frac{|m|}{M})\Big[\delta\left((Bu)_m^2 + \mu u_m^2 + \varphi_m^2\right) + \frac{\alpha}{2}\varphi_m^2 + \gamma|\psi_m|^2\Big] \\
&- \frac{h^2}{2}\frac{\mathrm{d}}{\mathrm{dt}} \sum_{m\in\mathbb{Z}} \chi(\frac{|m|}{M})(Au)_m^2 - \frac{\lambda h^2}{2} \sum_{m\in\mathbb{Z}} \chi(\frac{|m|}{M})(Au)_m^2 \\
\gtrsim& -\frac{R_\sigma(t)}{M}, \quad \forall\, \tau \leqslant \tau_0 \leqslant t.
\end{aligned} \tag{3.17}
$$

Proof. We first perform some computations. In fact,

$$
\mathbf{Re}(\Theta z, y)_{E_\mu} = \gamma(\psi, \xi) + \lambda(Bu, Bv) + \mu\lambda(u, v) + \lambda(\lambda - \alpha)(u, w)
$$

$$
\begin{aligned}
&+ (Bu, Bw) - (B\varphi, Bv) + h^2(Du, w) + (\alpha - \lambda)(\varphi, w) \\
&+ \mathbf{Im}(A\psi, \xi) - \mathbf{Im}(h^2 D\psi, \xi),
\end{aligned} \tag{3.18}
$$

$$
(\psi, \xi) = \sum_{m\in\mathbb{Z}} \chi(\frac{|m|}{M})|\psi_m|^2, \qquad (u, v) = \sum_{m\in\mathbb{Z}} \chi(\frac{|m|}{M})u_m^2, \tag{3.19}
$$

$$
(u, w) = \sum_{m\in\mathbb{Z}} \chi(\frac{|m|}{M})u_m\varphi_m, \qquad (\varphi, w) = \sum_{m\in\mathbb{Z}} \chi(\frac{|m|}{M})\varphi_m^2, \tag{3.20}
$$

$$
\begin{aligned}
(Bu, Bv) &= \sum_{m\in\mathbb{Z}} (Bu)_m (Bv)_m = \sum_{m\in\mathbb{Z}} (Bu)_m \Big[\chi(\frac{|m+1|}{M})u_{m+1} - \chi(\frac{|m|}{M})u_m\Big] \\
&= \sum_{m\in\mathbb{Z}} (Bu)_m \Big[(Bu)_m \chi(\frac{|m|}{M}) + \Big(\chi(\frac{|m+1|}{M} - \chi(\frac{|m|}{M})\Big)u_{m+1}\Big] \\
&= \sum_{m\in\mathbb{Z}} \chi(\frac{|m|}{M})(Bu)_m^2 + \sum_{m\in\mathbb{Z}} \chi'(\frac{\tilde{m}}{M})\frac{1}{M}(u_{m+1} - u_m)u_{m+1} \\
&\gtrsim \sum_{m\in\mathbb{Z}} \chi(\frac{|m|}{M})(Bu)_m^2 - \frac{R_\sigma(t)}{M}, \quad \forall\, \tau \leqslant \tau(t, \widehat{D}),
\end{aligned} \tag{3.21}
$$

and also by the Mean Value Theorem and (3.13), we have

$$
\begin{aligned}
(Bu, Bw) - (B\varphi, Bv) &= \sum_{m\in\mathbb{Z}} (Bu)_m (Bw)_m - \sum_{m\in\mathbb{Z}} (B\varphi)_m (Bv)_m \\
&= \sum_{m\in\mathbb{Z}} (u_{m+1} - u_m)\Big(\chi(\frac{|m+1|}{M})\varphi_{m+1} - \chi(\frac{|m|}{M})\varphi_m\Big) \\
&\quad - \sum_{m\in\mathbb{Z}} (\varphi_{m+1} - \varphi_m)\Big(\chi(\frac{|m+1|}{M})u_{m+1} - \chi(\frac{|m|}{M})u_m\Big) \\
&= \sum_{m\in\mathbb{Z}} \Big(\chi(\frac{|m+1|}{M}) - \chi(\frac{|m|}{M})\Big)(u_{m+1}\varphi_m - u_m\varphi_{m+1}) \\
&= \sum_{m\in\mathbb{Z}} \chi'(\frac{\tilde{m}}{M})\frac{1}{M}(u_{m+1}\varphi_m - u_m\varphi_{m+1}) \\
&\gtrsim -\frac{R_\sigma(t)}{M}, \quad \forall\, \tau \leqslant \tau(t, \widehat{D}),
\end{aligned} \tag{3.22}
$$

$$
\begin{aligned}
\mathbf{Im}(A\psi, \xi) &= -\,\mathbf{Im}(B\psi, B\xi) \\
&= -\,\mathbf{Im}\Big(\sum_{m\in\mathbb{Z}} (\psi_{m+1} - \psi_m)\big(\chi(\frac{|m+1|}{M})\bar{\psi}_{m+1} - \chi(\frac{|m|}{M})\bar{\psi}_m\big)\Big) \\
&= \mathbf{Im}\Big(\sum_{m\in\mathbb{Z}} \big(\chi(\frac{|m|}{M})\psi_{m+1}\bar{\psi}_m + \chi(\frac{|m+1|}{M})\bar{\psi}_{m+1}\psi_m\big)\Big) \\
&\geqslant -\sum_{m\in\mathbb{Z}} \Big|\chi(\frac{|m+1|}{M}) - \chi(\frac{|m|}{M})\Big||\psi_{m+1}||\psi_m| \\
&\gtrsim -\frac{R_\sigma(t)}{M}, \quad \forall\, \tau \leqslant \tau_0 \leqslant t.
\end{aligned} \tag{3.23}
$$

For the term $h^2(Du, w)$ in (3.18), we have

$$
(Du, w) = (Du, \dot{v}) + \lambda(Du, v), \tag{3.24}
$$

and

$$
\begin{aligned}
(Du,\dot v) =&(Au,A\dot v)\\
=&\sum_{m\in\mathbb{Z}}(Au)_m(A\dot v)_m\\
=&\sum_{m\in\mathbb{Z}}(Au)_m\Big[\chi(\frac{|m|}{M})(A\dot u)_m+(A\dot v)_m-\chi(\frac{|m|}{M})(A\dot u)_m\Big]\\
=&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{m\in\mathbb{Z}}\chi(\frac{|m|}{M})(Au)_m^2+\sum_{m\in\mathbb{Z}}(Au)_m\Big[(A\dot v)_m-\chi(\frac{|m|}{M})(A\dot u)_m\Big].
\end{aligned}
$$

Since

$$
\begin{aligned}
&\sum_{m\in\mathbb{Z}}(Au)_m\Big[(A\dot v)_m-\chi(\frac{|m|}{M})(A\dot u)_m\Big]\\
=&\sum_{m\in\mathbb{Z}}(Au)_m\Big[\Big(\chi(\frac{|m+1|}{M})-\chi(\frac{|m|}{M})\Big)\dot u_{m+1}+\Big(\chi(\frac{|m-1|}{M})-\chi(\frac{|m|}{M})\Big)\dot u_{m-1}\Big]\\
\gtrsim&-\frac{1}{M}\sum_{m\in\mathbb{Z}}|(Au)_m|(|\dot u_{m+1}|+|\dot u_{m-1}|)\\
\gtrsim&-\frac{1}{M}\sum_{m\in\mathbb{Z}}|u_{m+1}(t)-2u_m+u_{m-1}|\big(|\varphi_{m+1}-\lambda u_{m+1}|+|\varphi_{m-1}-\lambda u_{m-1}|\big)\\
\gtrsim&-\frac{R_\sigma(t)}{M},\quad\forall\,\tau\leqslant\tau_0\leqslant t,
\end{aligned}
$$

we have

$$
(Du,\dot v)-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{m\in\mathbb{Z}}\chi(\frac{|m|}{M})(Au)_m^2\gtrsim-\frac{R_\sigma(t)}{M},\quad\forall\,\tau\leqslant\tau_0\leqslant t. \tag{3.25}
$$

Similarly,

$$
\begin{aligned}
(Du,v) =&(Au,Av)\\
=&\sum_{m\in\mathbb{Z}}(Au)_m\left(\chi(\frac{|m|}{M})(Au)_m+(Av)_m-\chi(\frac{|m|}{M})(Au)_m\right)\\
\gtrsim&\frac{1}{2}\sum_{m\in\mathbb{Z}}\chi(\frac{|m|}{M})(Au)_m^2-\frac{R_\sigma(t)}{M},\quad\forall\,\tau\leqslant\tau_0\leqslant t.
\end{aligned}
\tag{3.26}
$$

From (3.24)-(3.26), we obtain

$$
(Du,w)\gtrsim\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{m\in\mathbb{Z}}\chi(\frac{|m|}{M})(Au)_m^2+\frac{\lambda}{2}\sum_{m\in\mathbb{Z}}\chi(\frac{|m|}{M})(Au)_m^2\frac{R_\sigma(t)}{M},\quad\forall\,\tau\leqslant\tau_0\leqslant t. \tag{3.27}
$$

For the term $\mathbf{Im}(D\psi,\xi)$ in (3.18), we have by Lemma 3.1, the Mean Value Theorem and (3.13) that

$$
-\mathbf{Im}(D\psi,\xi)=-\mathbf{Im}(A\psi,A\xi)
$$

$$
\begin{aligned}
&= \mathbf{Im}\left(\sum_{m\in\mathbb{Z}} 2\left(\chi(\frac{|m+1|}{M})-\chi(\frac{|m|}{M})\right)\bar{\psi}_{m+1}\psi_m + 2\left(\chi(\frac{|m-1|}{M})-\chi(\frac{|m|}{M})\right)\bar{\psi}_{m-1}\psi_m\right.\\
&\quad \left. + \left(\chi(\frac{|m+1|}{M}) - \chi(\frac{|m-1|}{M})\right)\psi_{m+1}\bar{\psi}_{m-1}\right)\\
&\geqslant -2\sum_{m\in\mathbb{Z}}\left|\chi(\frac{|m+1|}{M})-\chi(\frac{|m|}{M})\right||\bar{\psi}_{m+1}\psi_m| - 2\sum_{m\in\mathbb{Z}}\left|\chi(\frac{|m-1|}{M})-\chi(\frac{|m|}{M})\right||\bar{\psi}_{m-1}\psi_m|\\
&\quad - \sum_{m\in\mathbb{Z}}\left|\chi(\frac{|m+1|}{M}) - \chi(\frac{|m-1|}{M})\right||\psi_{m+1}\bar{\psi}_{m-1}|\\
&\gtrsim \frac{-R_\sigma(t)}{M}, \quad \forall\, \tau \leqslant \tau_0 \leqslant t.
\end{aligned}
\tag{3.28}
$$

Taking (3.18)-(3.23) and (3.26)-(3.28) into account, we get

$$
\begin{aligned}
&\mathbf{Re}(\Theta z, y)_{E_\mu} - \frac{h^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{m\in\mathbb{Z}}\chi(\frac{|m|}{M})(Au)_m^2 - \frac{\lambda h^2}{2}\sum_{m\in\mathbb{Z}}\chi(\frac{|m|}{M})(Au)_m^2\\
&\quad - \sum_{m\in\mathbb{Z}}\chi(\frac{|m|}{M})\left[\delta\left((Bu)_m^2 + \mu u_m^2 + \varphi_m^2\right) + \frac{\alpha}{2}\varphi_m^2 + \gamma|\psi_m|^2\right]\\
&\gtrsim \sum_{m\in\mathbb{Z}}\chi(\frac{|m|}{M})\left[(\lambda-\delta)\left((Bu)_m^2 + \mu u_m^2\right) + \left(\frac{\alpha}{2} - \lambda - \delta\right)\varphi_m^2 + \lambda(\lambda-\alpha)u_m\varphi_m\right]\\
&\quad - \frac{R_\sigma(t)}{M}, \quad \forall\, \tau \leqslant \tau_0 \leqslant t.
\end{aligned}
\tag{3.29}
$$

At the same time, by some computations we find

$$
\begin{aligned}
&\sum_{m\in\mathbb{Z}}\chi(\frac{|m|}{M})\left[(\lambda-\delta)\left((Bu)_m^2 + \mu u_m^2\right) + \left(\frac{\alpha}{2} - \lambda - \delta\right)\varphi_m^2 + \lambda(\lambda-\alpha)u_m\varphi_m\right]\\
&\geqslant \sum_{m\in\mathbb{Z}}\chi(\frac{|m|}{M})\left[\mu(\lambda-\delta)u_m^2 + \left(\frac{\alpha}{2} - \lambda - \delta\right)\varphi_m^2 - \lambda\alpha|u_m||\varphi_m|\right]\\
&= \sum_{m\in\mathbb{Z}}\chi(\frac{|m|}{M})\left(\sqrt{\mu(\lambda-\delta)}|u_m| - \sqrt{\frac{\alpha}{2} - \lambda - \delta}|\varphi_m|\right)^2 \geqslant 0.
\end{aligned}
\tag{3.30}
$$

Inserting (3.30) into (3.29) gives (3.17). □

Lemma 3.4. *The term* $\mathbf{Re}\big(F(z,t), y(t)\big)_{E_\mu}$ *in* (3.14) *satisfies*

$$
\begin{aligned}
&\mathbf{Re}\big(F(z,t), y(t)\big)_{E_\mu}\\
&\lesssim \frac{\gamma}{2}\sum_{|m|\geqslant M}\chi(\frac{|m|}{M})|\psi_m|^2 + \frac{\alpha}{2}\sum_{|m|\geqslant M}\chi(\frac{|m|}{M})\varphi_m^2 + \sum_{|m|\geqslant M}|g_m(t)|^2\\
&\quad + \sum_{|m|\geqslant M}|f_m(t)|^2 + R_\sigma(t)e^{-\gamma(t-s)}\sum_{|m|\geqslant M}\chi(\frac{|m|}{M})\big(|\psi_{m+1}(\tau)|^2 + |\psi_m(\tau)|^2 + |\psi_{m-1}(\tau)|^2\big)\\
&\quad + R_\sigma(t)\int_\tau^t\sum_{|m|\geqslant M}\big(|f_{m+1}(s)|^2 + |f_m(s)|^2 + |f_{m-1}(s)|^2\big)\mathrm{d}s\\
&\quad + \frac{R_\sigma(t)}{M}\int_\tau^t R_\sigma(s)e^{-\gamma(t-s)}\mathrm{d}s, \quad \forall \tau \leqslant \tau_0 \leqslant t.
\end{aligned}
\tag{3.31}
$$

Proof. Direct computations give

$$\mathbf{Re}\big(F(z,t),y(t)\big)_{E_\mu}=\mathbf{Im}\big(f(t),\xi\big)+\big(g(t),w(t)\big)+\big(A|\psi|^2(t),w(t)\big). \tag{3.32}$$

At the same time, for each $\tau\leqslant\tau_0\leqslant t$, we have

$$\mathbf{Im}(f(t),\xi)\lesssim\frac{\gamma}{2}\sum_{|m|\geqslant M}\chi(\frac{|m|}{M})|\psi_m|^2+\sum_{|m|\geqslant M}|f_m(t)|^2, \tag{3.33}$$

$$\big(g(t),w\big)\lesssim\frac{\alpha}{4}\sum_{|m|\geqslant M}\chi(\frac{|m|}{M})\varphi_m^2+\sum_{|m|\geqslant M}|g_m(t)|^2, \tag{3.34}$$

and

$$\begin{aligned}\big(A|\psi|^2,w\big)=&\sum_{m\in\mathbb{Z}}\big(|\psi_{m+1}|^2-2|\psi_m|^2+|\psi_{m-1}|^2\big)\,\chi(\frac{|m|}{M})\varphi_m\\ \lesssim&\frac{\alpha}{4}\sum_{|m|\geqslant M}\chi(\frac{|m|}{M})\varphi_m^2+R_\sigma(t)(I_1+I_2+I_3),\end{aligned} \tag{3.35}$$

where

$$\begin{cases}I_1=\sum\limits_{|m|\geqslant M}\chi(\frac{|m|}{M})|\psi_{m+1}|^2,\\ I_2=\sum\limits_{|m|\geqslant M}\chi(\frac{|m|}{M})|\psi_m|^2,\\ I_3=\sum\limits_{|m|\geqslant M}\chi(\frac{|m|}{M})|\psi_{m-1}|^2.\end{cases}$$

Next we need estimate the terms I_1, I_2 and I_3 in (3.35). To this end, we let

$$\zeta=(\zeta_{m+1})_{m\in\mathbb{Z}}=\big(\chi(\frac{|m|}{M})\psi_{m+1}\big)_{m\in\mathbb{Z}}.$$

Taking the imaginary part of the inner product $(\cdot,\cdot)$ of equation (2.7) with ζ gives

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{|m|\geqslant M}\chi(\frac{|m|}{M})|\psi_{m+1}|^2+\gamma\sum_{|m|\geqslant M}\chi(\frac{|m|}{M})|\psi_{m+1}|^2\\ =&\mathbf{Im}\sum_{|m|\geqslant M}\chi(\frac{|m|}{M})\bar{\psi}_{m+1}f_{m+1}(t)-\mathbf{Im}(A\psi,\zeta)+h^2\mathbf{Im}(D\psi,\zeta)\\ \lesssim&\frac{\gamma}{2}\sum_{|m|\geqslant M}\chi(\frac{|m|}{M})|\psi_{m+1}|^2+\sum_{|m|\geqslant M}|f_{m+1}(t)|^2-\mathbf{Im}(A\psi,\zeta)\\ &+\mathbf{Im}(D\psi,\zeta),\quad\forall\,\tau\leqslant\tau_0\leqslant t.\end{aligned} \tag{3.36}$$

At the same time, by Lemma 3.1 and some computations, we have

$$\begin{aligned}-\mathbf{Im}(A\psi,\zeta)=&\mathbf{Im}(B\psi,B\zeta)\lesssim\frac{R_\sigma(t)}{M},\quad\forall\,\tau\leqslant\tau_0\leqslant t,\\ \mathbf{Im}(D\psi,\zeta)=&\mathbf{Im}(A\psi,A\zeta)\end{aligned} \tag{3.37}$$

$$
\begin{aligned}
&\leqslant 2\sum_{m\in\mathbb{Z}}\Big|\chi(\frac{|m|}{M})-\chi(\frac{|m+1|}{M})\Big||\psi_{m+1}||\psi_m| \\
&\quad+\sum_{m\in\mathbb{Z}}\Big|\chi(\frac{|m-1|}{M})-\chi(\frac{|m+1|}{M})\Big||\psi_{m-1}||\psi_{m+1}| \\
&\quad+2\sum_{m\in\mathbb{Z}}\Big|\chi(\frac{|m|}{M})-\chi(\frac{|m-1|}{M})\Big||\psi_m||\psi_{m-1}| \\
&\lesssim \frac{R_\sigma(t)}{M},\ \ \forall\,\tau\leqslant\tau_0\leqslant t.
\end{aligned}\tag{3.38}
$$

It then follows from (3.36)-(3.38) that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{|m|\geqslant M}\chi(\frac{|m|}{M})|\psi_{m+1}|^2+\gamma\sum_{|m|\geqslant M}\chi(\frac{|m|}{M})|\psi_{m+1}|^2 \\
\lesssim&\sum_{|m|\geqslant M}|f_{m+1}(t)|^2+\frac{R_\sigma(t)}{M},\ \ \forall\,\tau\leqslant\tau_0\leqslant t.
\end{aligned}\tag{3.39}
$$

Applying Gronwall inequality to (3.39) yields

$$
\begin{aligned}
I_1\lesssim&\int_\tau^t\Big(\sum_{|m|\geqslant M}|f_{m+1}(s)|^2+\frac{R_\sigma(s)}{M}\Big)e^{-\gamma(t-s)}\mathrm{d}s \\
&+e^{-\gamma(t-\tau)}\Big(\sum_{|m|\geqslant M}\chi(\frac{|m|}{M})|\psi_{m+1}(\tau)|^2\Big),\ \ \forall\,\tau\leqslant\tau_0\leqslant t.
\end{aligned}\tag{3.40}
$$

Similarly, set

$$
p=(p_m)_{m\in\mathbb{Z}}=\chi(\frac{|m|}{M})\psi_m,\qquad q=(q_{m-1})_{m\in\mathbb{Z}}=\chi\left(\frac{|m|}{M}\right)\psi_{m-1}.
$$

Taking the imaginary part of the inner product $(\cdot,\cdot)$ of equation (2.7), respectively with p and q yields

$$
\begin{aligned}
I_2\lesssim&\int_\tau^t\Big(\sum_{|m|\geqslant M}|f_m(s)|^2+\frac{R_\sigma(s)}{M}\Big)e^{-\gamma(t-s)}\mathrm{d}s \\
&+e^{-\gamma(t-\tau)}\Big(\sum_{|m|\geqslant M}\chi(\frac{|m|}{M})|\psi_m(\tau)|^2\Big),\ \ \forall\,\tau\leqslant\tau_0\leqslant t,
\end{aligned}\tag{3.41}
$$

and

$$
\begin{aligned}
I_3\lesssim&\int_\tau^t\Big(\sum_{|m|\geqslant M}|f_{m-1}(s)|^2+\frac{R_\sigma(s)}{M}\Big)e^{-\gamma(t-s)}\mathrm{d}s \\
&+e^{-\gamma(t-\tau)}\Big(\sum_{|m|\geqslant M}\chi(\frac{|m|}{M})|\psi_{m-1}(\tau)|^2\Big),\ \ \forall\,\tau\leqslant\tau_0\leqslant t.
\end{aligned}\tag{3.42}
$$

Combining (3.32)-(3.35) and (3.40)-(3.42), we obtain (3.31). □

Now we begin to prove the pullback-$\mathcal{D}_\sigma$ asymptotic nullness of $\{U(t,\tau)\}_{t\geqslant\tau}$ in E_μ.

Lemma 3.5. *Let assumption* **(H)** *hold. Then for any given* $t\in\mathbb{R}$, $\forall\epsilon>0$ *and* $\widehat{D}=\{D(s)|s\in\mathbb{R}\}\in\mathcal{D}_\sigma$, *there exist some* $M_*=M_*(t,\varepsilon,\widehat{D})\in\mathbb{N}$ *and* $\tau_*=\tau_*(t,\varepsilon,\widehat{D})\leqslant t$ *such that*

$$\sup_{z(\tau)\in D(\tau)}\sum_{|m|\geqslant M_*}\left|(U(t,\tau)z(\tau))_m\right|^2_{E_\mu}\leqslant\epsilon^2,\quad\forall\tau\leqslant\tau_*. \tag{3.43}$$

Proof. We conclude from (3.14) and Lemma 3.2-Lemma 3.4 that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{m\in\mathbb{Z}}\chi(\frac{|m|}{M})\big[|z_m|^2_{E_\mu}+h^2(Au)^2_m\big]+\sigma\sum_{m\in\mathbb{Z}}\chi(\frac{|m|}{M})\big[|z_m|^2_{E_\mu}+h^2(Au)^2_m\big]\\
\lesssim&\frac{R_\sigma(t)}{M}+\sum_{|m|\geqslant M}|f_m(t)|^2+\sum_{|m|\geqslant M}|g_m(t)|^2\\
&+R_\sigma(t)e^{-\gamma(t-\tau)}\sum_{|m|\geqslant M}\chi(\frac{|m|}{M})\Big(|\psi_{m+1}(\tau)|^2+|\psi_m(\tau)|^2+|\psi_{m-1}(\tau)|^2\Big)\\
&+R_\sigma(t)\int_\tau^t\sum_{|m|\geqslant M}\big(|f_{m+1}(s)|^2+|f_m(s)|^2+|f_{m-1}(s)|^2\big)\\
&+\frac{R_\sigma(t)}{M}\int_\tau^t R_\sigma(s)\mathrm{e}^{-\gamma(t-s)}\mathrm{d}s,\quad\forall\,\tau\leqslant\tau_0\leqslant t.
\end{aligned}\tag{3.44}$$

Now, for each given $\epsilon>0$ and given $t\in\mathbb{R}$, there exists obviously a positive number $M_1=M_1(t,\epsilon)\in\mathbb{N}$ such that

$$\frac{R_\sigma(t)}{M}\leqslant\frac{\sigma\epsilon^2}{12},\quad\forall\,M>M_1. \tag{3.45}$$

Also we can pick some $\tau_1=\tau_1(t,\epsilon)\in\mathbb{N}$ such that

$$\begin{aligned}
&R_\sigma(t)e^{-\gamma(t-\tau)}\sum_{|m|\geqslant M}\chi(\frac{|m|}{M})\Big(|\psi_{m+1}(\tau)|^2+|\psi_m(\tau)|^2+|\psi_{m-1}(\tau)|^2\Big)\\
\lesssim&R_\sigma(t)e^{-\gamma(t-\tau)}\|\psi_\tau\|^2\leqslant R_\sigma(t)e^{-\gamma(t-\tau)}\|z_\tau\|^2_{E_\mu}\leqslant\frac{\sigma\epsilon^2}{12},\quad\tau\leqslant\tau_1\leqslant\tau_0\leqslant t.
\end{aligned}\tag{3.46}$$

At the same time,

$$R_\sigma(t)\int_\tau^t\Big(\sum_{|m|\geqslant M}|f_m(s)|^2\Big)e^{-\gamma(t-s)}\mathrm{d}s=R_\sigma(t)e^{-\gamma t}\int_\tau^t e^{\gamma s}\Big(\sum_{|m|\geqslant M}|f_m(s)|^2\Big)\mathrm{d}s.$$

By (3.4), (3.12) and the assumption **(H)**, we see $R_\sigma(t)e^{-\gamma t}$ is a constant depending only on t. We also see from (3.9) that

$$\int_{-\infty}^t e^{\gamma\eta}\|f(\eta)\|^2\mathrm{d}\eta<+\infty,\quad\text{for each }\ t\in\mathbb{R}.$$

Consequently there is some $M_2=M_2(t,\epsilon)\in\mathbb{N}$ such that

$$R_\sigma(t)\int_\tau^t\Big(\sum_{|m|\geqslant M}|f_m(s)|^2\Big)e^{-\gamma(t-s)}\mathrm{d}s\leqslant R_\sigma(t)e^{-\gamma t}\sum_{|m|\geqslant M}\int_{-\infty}^t e^{\gamma s}|f_m(s)|^2\mathrm{d}s$$

$$\leqslant \frac{\sigma\epsilon^2}{12}, \quad M > M_2. \tag{3.47}$$

Now, by (3.12), we have

$$\begin{aligned}
&\int_\tau^t \frac{R_\sigma(s)}{M} e^{-\gamma(t-s)}\mathrm{d}s\\
=&\frac{1}{M}\int_\tau^t e^{-\gamma(t-s)}\mathrm{d}s + \frac{1}{M}\int_\tau^t e^{-\gamma(t-s)}\int_{-\infty}^s e^{(\sigma-2\gamma)\rho}(\int_\tau^\rho e^{\gamma\eta}\|f(\eta)\|^2\mathrm{d}\eta)^2\mathrm{d}\rho\mathrm{d}s\\
&+\frac{1}{M}\int_\tau^t e^{-\gamma(t-s)}e^{-\sigma s}\int_{-\infty}^s e^{\sigma\theta}(\|f(\theta)\|^2+\|g(\theta)\|^2)\mathrm{d}\theta\mathrm{d}s\\
\leqslant&\frac{1}{M}+\frac{1}{M}\int_\tau^t e^{-\gamma(t-s)}\mathrm{d}s\int_{-\infty}^t e^{(\sigma-2\gamma)\rho}\mathrm{d}\rho(\int_\tau^t e^{\gamma\eta}\|f(\eta)\|^2\mathrm{d}\eta)^2\\
&+\frac{e^{-\gamma t}}{M}\int_\tau^t e^{(\gamma-\sigma)s}\mathrm{d}s\int_{-\infty}^t e^{\sigma\theta}(\|f(\theta)\|^2+\|g(\theta)\|^2)\mathrm{d}\theta\\
\lesssim&\frac{1}{M}+\frac{e^{(\sigma-2\gamma)t}}{M}\int_{-\infty}^t e^{\gamma\eta}\|f(\eta)\|^2\mathrm{d}\eta+\frac{e^{-\sigma t}}{M}\int_{-\infty}^t e^{\sigma\theta}(\|f(\theta)\|^2+\|g(\theta)\|^2)\mathrm{d}\theta.
\end{aligned}$$

Hence, by (3.1) and (3.2), for above t and ϵ, we see that there exists some $M_3 = M_3(t,\epsilon)\in\mathbb{N}$ such that

$$\frac{R_\sigma(t)}{M}\int_\tau^t R_\sigma(s)e^{-\gamma(t-s)}\mathrm{d}s \leqslant \frac{\sigma\epsilon^2}{12}, \quad \forall M > M_3. \tag{3.48}$$

At this stage, we take (3.44), (3.45), (3.47), (3.48) into account and obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{m\in\mathbb{Z}}\chi(\frac{|m|}{M})\left[|z_m(t)|^2_{E_\mu}+h^2\left(A\left(u(t)\right)\right)^2_m\right]\\
&\quad+\sigma\sum_{m\in\mathbb{Z}}\chi(\frac{|m|}{M})\left[|z_m(t)|^2_{E_\mu}+h^2\left(A\left(u(t)\right)\right)^2_m\right]\\
\lesssim&\sum_{|m|\geqslant M}|g_m(t)|^2+\sum_{|m|\geqslant M}|f_m(t)|^2+\frac{\sigma\epsilon^2}{3}.
\end{aligned} \tag{3.49}$$

Applying Gronwall inequality to (3.49) and using the fact

$$h^2\sum_{m\in\mathbb{Z}}\chi(\frac{|m|}{M})((Au(\tau))_m)^2 \lesssim \|u_\tau\|^2 \leqslant \|z_\tau\|^2_{E_\mu},$$

we have

$$\begin{aligned}
&\sum_{m\in\mathbb{Z}}\chi(\frac{|m|}{M})\Big(|z_m(t)|^2_{E_\mu}+h^2\left(A\left(u(t)\right)\right)^2_m\Big)\\
\lesssim&\|z_\tau\|^2_{E_\mu}e^{-\sigma(t-\tau)}+e^{-\sigma t}\int_\tau^t e^{\sigma s}\sum_{|m|\geqslant M}\big(|f_m(s)|^2+|g_m(s)|^2\big)\mathrm{d}s+\frac{\epsilon^2}{3}.
\end{aligned} \tag{3.50}$$

From (3.4) we see that there exists some $M_4 = M_4(t,\epsilon)\in\mathbb{N}$ such that

$$e^{-\sigma t}\int_\tau^t e^{\sigma s}\sum_{|m|\geqslant M}|(g_m(s)|^2+|f_m(s)|^2)\mathrm{d}s$$

$$
\begin{aligned}
&\leqslant e^{-\sigma t} \sum_{|m| \geqslant M} \int_{-\infty}^{t} e^{\sigma s}(|g_m(s)|^2 + |f_m(s)|^2) \mathrm{d}s \\
&\leqslant \frac{\epsilon^2}{3}, \quad \forall M > M_4. && (3.51)
\end{aligned}
$$

By (3.3), there is some $\tau_2 = \tau_2(t, \epsilon, \widehat{D})$ such that,

$$
e^{-\sigma t} \cdot e^{\sigma \tau} \sup_{z_\tau \in D(\tau)} \|z_\tau\|_{E_\mu}^2 \leqslant \frac{\epsilon^2}{3}, \quad \forall \tau \leqslant \tau_2. \tag{3.52}
$$

Now we choose

$$
M^* = \max\{M_1, M_2, M_3, M_4\}, \quad \tau^* = \min\{\tau_0, \tau_1, \tau_2\},
$$

and then follows from (3.50)-(3.52) that

$$
\sum_{|m| \geqslant 2M^*} |z_m(t)|_{E_\mu}^2 \leqslant \sum_{m \in \mathbb{Z}} \chi(\frac{|m|}{M}) \left[|z_m(t)|_{E_\mu}^2 + h^2\big(A\big(u(t)\big)\big)_m^2 \right] \leqslant \epsilon^2.
$$

Hence

$$
\sup_{z_\tau \in D(\tau)} \sum_{|m| \geqslant 2M^*} |(U(t,\tau) z_\tau)_m|_{E_\mu}^2 = \sup_{z_\tau \in D(\tau)} \sum_{|m| \geqslant 2M^*} |z_m(t)|_{E_\mu}^2 \leqslant \epsilon^2, \ \ \forall \tau \leqslant \tau^*.
$$

The proof of Lemma 3.5 is completed. □

Combining Lemma 3.1 and Lemma 3.5, we can obtain, using [37, Theorem 2.1], the main result of this section as follows.

Theorem 3.1. *Let assumption* **(H)** *hold. Then the process* $\{U(t,\tau)\}_{t \geqslant \tau}$ *possesses a pullback-*$\mathcal{D}_\sigma$ *attractor* $\widehat{\mathcal{A}}_{\mathcal{D}_\sigma} = \{\mathcal{A}_{\mathcal{D}_\sigma}(t) | t \in \mathbb{R}\}$ *satisfying*

(a) *Compactness: for each* $t \in \mathbb{R}$, $\mathcal{A}_{\mathcal{D}_\sigma}(t)$ *is a nonempty compact subset of* E_μ*;*

(b) *Invariance:* $U(t,\tau)\mathcal{A}_{\mathcal{D}_\sigma}(\tau) = \mathcal{A}_{\mathcal{D}_\sigma}(t), \ \ \forall \tau \leqslant t$*;*

(c) *Pullback attracting:* $\widehat{\mathcal{A}}_{\mathcal{D}_\sigma}(t)$ *is pullback-*$\mathcal{D}_\sigma$ *attracting in the following sense*

$$
\lim_{\tau \to -\infty} \mathrm{dist}_{E_\mu}\big(U(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}_\sigma}(t)\big) = 0, \ \ \forall \widehat{D} = \{D(t) | t \in \mathbb{R}\} \in \mathcal{D}_\sigma, \quad t \in \mathbb{R}.
$$

4. Invariant measures supported by the pullback attractor

In this section, we will apply the theory of Łukaszewicz and Robinson [27] to prove the existence of a unique family of invariant Borel probability measures supported by the pullback-$\mathcal{D}_\sigma$ attractor $\widehat{\mathcal{A}}_{\mathcal{D}_\sigma}$ obtained in Theorem 3.1.

We first introduce two definitions.

Definition 4.1 ([13]). A generalized Banach limit is any linear functional, which we denote by $\mathrm{LIM}_{T\to\infty}$, defined on the space of all bounded real-valued functions on $[0, +\infty)$ that satisfies

(i) $\mathrm{LIM}_{T\to\infty}\phi(T) \geqslant 0$ for nonnegative functions $\phi(\cdot)$;

(ii) $\mathrm{LIM}_{T\to\infty}\phi(T) = \lim\limits_{T\to\infty}\phi(T)$ if the usual limit $\lim\limits_{T\to\infty}\phi(T)$ exists.

Definition 4.2 ([27]). A process $\{\mathcal{U}(t,\tau)\}_{t\geqslant\tau}$ is said to be τ-continuous on a metric space X if for every fixed $x_0 \in X$ and every fixed $t \in \mathbb{R}$, the X-valued function $\tau \longmapsto \mathcal{U}(t,\tau)x_0$ is continuous and bounded on $(-\infty, t]$.

Remark 4.1. Notice that we consider the "pullback" asymptotic behavior and we require generalized limits as $\tau \to -\infty$. For a given real-valued function ϕ defined on $(-\infty, 0]$ and a given Banach limit $\mathrm{LIM}_{T\to\infty}$, we define

$$\mathrm{LIM}_{t\to-\infty}\phi(t) = \mathrm{LIM}_{t\to\infty}\phi(-t).$$

The following result was proved by Łukaszewicz and Robinson in [27].

Proposition 4.1 ([27]). *Let $\{\mathcal{U}(t,\tau)\}_{t\geqslant\tau}$ be a τ-continuous process in a complete metric space X that has a pullback-$\mathcal{D}$ attractor $\mathcal{A}(\cdot)$. Fix a generalized Banach limit $\mathrm{LIM}_{T\to\infty}$ and let $\kappa : \mathbb{R} \longmapsto X$ be a continuous map such that $\kappa(\cdot) \in \mathcal{D}$. Then there exists a unique family of Borel probability measures $\{\mu_t\}_{t\in\mathbb{R}}$ in X such the support of the measure μ_t is contained in $\mathcal{A}(t)$ and*

$$\mathrm{LIM}_{\tau\to-\infty}\frac{1}{t-\tau}\int_\tau^t \phi(\mathcal{U}(t,s)\kappa(s))\mathrm{d}s = \int_{\mathcal{A}(t)}\phi(v)\mathrm{d}\mu_t(v),$$

for any real-valued continuous functional ϕ on X. In addition, μ_t is invariant in the sense that

$$\int_{\mathcal{A}(t)}\phi(v)\mathrm{d}\mu_t(v) = \int_{\mathcal{A}(\tau)}\phi(\mathcal{U}(t,\tau)v)\mathrm{d}\mu_\tau(v), \quad t \geqslant \tau.$$

In order to apply the above result to the pullback-$\mathcal{D}_\sigma$ attractor $\widehat{\mathcal{A}}_{\mathcal{D}_\sigma}$ obtained in Theorem 3.1, we shall prove the τ-continuous property of the process $\{U(t,\tau)\}_{t\geqslant\tau}$ in the space E_μ. We begin with the following estimate.

Lemma 4.1. *Let $z^{(k)}(t) = z^{(k)}(t;\tau,z_\tau) = (\psi^{(k)}(t), u^{(k)}(t), \varphi^{(k)}(t))^T$ $(k = 1,2)$ be two solutions of problem (2.4) with initial values $z_\tau^{(k)} \in E_\mu$, respectively. Then*

$$\begin{aligned}
&\|z^{(1)}(t) - z^{(2)}(t)\|_{E_\mu}^2 \\
\lesssim&\|z_\tau^{(1)} - z_\tau^{(2)}\|_{E_\mu}^2 \exp\Big\{\int_\tau^t \big(\|\psi^{(1)}(s)\|^2 + \|\psi^{(2)}(s)\|^2 + \|u^{(2)}(s)\|^2\big)\mathrm{d}s\Big\}.
\end{aligned} \tag{4.1}$$

Proof. Let $z^{(k)}(t) = z^{(k)}(t;\tau,z_\tau^{(k)}) = (\psi^{(k)}(t), u^{(k)}(t), \varphi^{(k)}(t))^T$, $k = 1,2$, be two solutions of problem (2.4) corresponding to initial data $z_\tau^{(1)}$, $z_\tau^{(2)} \in E_\mu$, respectively. Set

$$\begin{cases}
\tilde{\psi}(t) = \psi^{(1)}(t) - \psi^{(2)}(t),\\
\tilde{u}(t) = u^{(1)}(t) - u^{(2)}(t),\\
\tilde{\varphi}(t) = \varphi^{(1)}(t) - \varphi^{(2)}(t),\\
\tilde{z}(t) = z^{(1)}(t) - z^{(2)}(t).
\end{cases}$$

It is easy to see that $\tilde{z}(t)$ satisfies

$$\begin{cases}
\frac{\mathrm{d}}{\mathrm{d}t}\tilde{z}(t) + \Theta\tilde{z}(t) = F(z^{(1)}(t),t) - F(z^{(2)}(t),t), \ \ \forall\, t > \tau,\\
\tilde{z}|_{t=\tau} = \tilde{z}(\tau) = z_\tau^{(1)} - z_\tau^{(2)}.
\end{cases} \tag{4.2}$$

From (2.9), we deduce that

$$\mathbf{Re}(\Theta\tilde{z},\tilde{z})_{E_\mu} \geqslant \delta(\|\tilde{u}\|_\mu^2+\|\tilde{\varphi}\|^2)+\frac{\alpha}{2}\|\tilde{\varphi}\|^2+\gamma\|\tilde{\psi}\|^2. \tag{4.3}$$

At the same time, direct computations show that

$$\begin{aligned}
&\|F(z^{(1)},t)-F(z^{(2)},t)\|_{E_\mu}^2\\
=&\left\|\Big(-i(\psi^{(1)}u^{(1)}-\psi^{(2)}u^{(2)}),0,A|\psi^{(1)}|^2-A|\psi^{(2)}|^2\Big)^T\right\|_{E_\mu}^2\\
\leqslant&\|\psi^{(1)}(u^{(1)}-u^{(2)})+u^{(2)}(\psi^{(1)}-\psi^{(2)})\|^2+16\big\||\psi^{(1)}|+|\psi^{(2)}|\big\|^2\big\||\psi^{(1)}|-|\psi^{(2)}|\big\|^2\\
\lesssim&\big(\|\psi^{(1)}\|^2+\|\psi^{(2)}\|^2+\|u^{(2)}\|^2\big)\|\tilde{z}\|_{E_\mu}^2.
\end{aligned} \tag{4.4}$$

Taking the real part of the inner product $(\cdot,\cdot)_{E_\mu}$ of the equation in (4.2) with $\tilde{z}$ and then using (4.3)-(4.4), we find

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\|\tilde{z}(s)\|_{E_\mu}^2+\sigma\|\tilde{z}(s)\|_{E_\mu}^2\\
\lesssim&\big(\|\psi^{(1)}(s)\|^2+\|\psi^{(2)}(s)\|^2+\|u^{(2)}(s)\|^2\big)\|\tilde{z}(s)\|_{E_\mu}^2,\ \ \forall\, s>\tau.
\end{aligned} \tag{4.5}$$

Integrating (4.5) over $[\tau,t]$ yields

$$\|\tilde{z}(t)\|_{E_\mu}^2 \lesssim \|\tilde{z}(\tau)\|_{E_\mu}^2+\int_\tau^t\big(\|\psi^{(1)}(s)\|^2+\|\psi^{(2)}(s)\|^2+\|u^{(2)}(s)\|^2\big)\|\tilde{z}(s)\|_{E_\mu}^2\mathrm{d}s. \tag{4.6}$$

Applying Gronwall inequality to (4.6) gives

$$\|\tilde{z}(t)\|_{E_\mu}^2 \lesssim \|\tilde{z}(\tau)\|_{E_\mu}^2\exp\Big\{\int_\tau^t\Big(\|\psi^{(1)}(s)\|^2+\|\psi^{(2)}(s)\|^2+\|u^{(2)}(s)\|^2\Big)\mathrm{d}s\Big\}.$$

The proof of Lemma 4.1 is completed. □

Lemma 4.2. *Let the assumption* **(H)** *hold. Then for every fixed $z_*\in E_\mu$ and every fixed $t\in\mathbb{R}$, the E_μ-valued function $\tau\longmapsto U(t,\tau)z_*$ is continuous and bounded on $(-\infty,t]$.*

Proof. Let $z_*=(\psi_*,u_*,\varphi_*)^T\in E_\mu$ and $t\in\mathbb{R}$ be given. First we shall prove that for each fixed $s_*\in(-\infty,t]$, the E_μ-valued function $\tau\longmapsto U(t,\tau)z_*$ is continuous at $\tau=s_*$. To this end, we will show that for any $\epsilon>0$, there exists some $\delta=\delta(\epsilon,s_*)>0$, such that if $r<t$, $s_*<t$ and $|r-s_*|<\delta$, then $\|U(t,r)z_*-U(t,s_*)z_*\|_{E_\mu}<\epsilon$. Without loss of generality, we assume $r<s_*$. Set

$$\begin{cases}
U(\cdot,s_*)U(s_*,r)z_*=(\psi_*^{(1)}(\cdot),u_*^{(1)}(\cdot),\varphi_*^{(1)}(\cdot))^T,\\
U(\cdot,s_*)U(r,r)z_*=(\psi_*^{(2)}(\cdot),u_*^{(2)}(\cdot),\varphi_*^{(2)}(\cdot))^T,
\end{cases}$$

then we get from (4.1) that

$$\begin{aligned}
&\|U(t,r)z_*-U(t,s_*)z_*\|_{E_\mu}^2\\
=&\|U(t,s_*)U(s_*,r)z_*-U(t,s_*)U(r,r)z_*\|_{E_\mu}^2
\end{aligned}$$

$$\lesssim \|U(s_*,r)z_* - U(r,r)z_*\|^2_{E_\mu} \exp\Big\{\int_{s_*}^t \big(\|\psi_*^{(1)}(\theta)\|^2 + \|\psi_*^{(2)}(\theta)\|^2 + \|u_*^{(2)}(\theta)\|^2\big)\mathrm{d}\theta\Big\}. \tag{4.7}$$

Now, from (2.18) find that the solutions of problem (2.4) belong to $\mathcal{C}([\tau,+\infty), E_\mu)$. Thus for above s_*, we have

$$\exp\Big\{\int_{s_*}^t \Big(\|\psi_*^{(1)}(\theta)\|^2 + \|\psi_*^{(2)}(\theta)\|^2 + \|u_*^{(2)}(\theta)\|^2\Big)\mathrm{d}\theta\Big\} < \mathcal{B}(t,s_*), \tag{4.8}$$

where the bound $\mathcal{B}(t,s_*)$ is independent of r. Therefore, we conclude from (2.18) and (4.8) that if $|r-s|$ is sufficiently small, then the right hand side of (4.7) is as small as needed. Thus, the E_μ-valued function $\tau \longmapsto U(t,\tau)z_*$ is continuous at $\tau = s_*$.

Secondly, for above $z_* \in E_\mu$ and $t \in \mathbb{R}$, we deduce from (2.10), (3.1), (3.3) and (3.11) that

$$\begin{aligned}
&\lim_{\tau\to-\infty} \|U(t,\tau)z_*\|^2_{E_\mu}\\
\lesssim &\lim_{\tau\to-\infty} \|z_\tau\|^2_{E_\mu} e^{-\sigma(t-\tau)} + \lim_{\tau\to-\infty} e^{-\sigma t}\int_\tau^t e^{\sigma\theta}\Big(\|f(\theta)\|^2 + \|g(\theta)\|^2\Big)\mathrm{d}\theta\\
&+ \lim_{\tau\to-\infty} e^{-\sigma t}\int_\tau^t e^{\sigma\theta}\|\psi(\theta)\|^4\mathrm{d}\theta\\
= &e^{-\sigma t}\int_{-\infty}^t e^{\sigma\theta}\Big(\|f(\theta)\|^2 + \|g(\theta)\|^2\Big)\mathrm{d}\theta + \int_{-\infty}^t e^{(\sigma-2\gamma)s}\Big(\int_\tau^s e^{\gamma\eta}\|f(\eta)\|^2\mathrm{d}\eta\Big)^2\mathrm{d}s \qquad (4.9)\\
< &+\infty,
\end{aligned}$$

and the expression of (4.9) is independent of τ. Remember that we have proved that the E_μ-valued function $\tau \longmapsto U(t,\tau)z_*$ is continuous with respect to $\tau \in (-\infty,t]$ in the space E_μ. Therefore the E_μ-valued function $\tau \longmapsto U(t,\tau)z_*$ is bounded on $(-\infty,t]$. The proof of this lemma is completed. $\square$

At this stage, we conclude from Theorem 3.1, Proposition 4.1 and Lemma 4.2 the main result of this section.

Theorem 4.1. *Let the assumption* **(H)** *hold. Let $\{U(t,\tau)\}_{t\geqslant\tau}$ be the process generated by the solutions operators of problem (2.4), and $\hat{\mathcal{A}}_{\mathcal{D}_\sigma} = \{\mathcal{A}_{\mathcal{D}_\sigma}(t)\ |t \in \mathbb{R}\}$ the pullback-$\mathcal{D}_\sigma$ attractor obtained in Theorem 3.1. Fix a generalized Banach limit $\mathrm{LIM}_{T\to\infty}$ and let $\phi : \mathbb{R} \mapsto E_\mu$ be a given continuous map with $\phi(\cdot) \in \mathcal{D}_\sigma$. Then there exists a unique family of Borel probability measures $\{\mathfrak{m}_t\}_{t\in\mathbb{R}}$ in the space E_μ such that the support of the measure $\mathfrak{m}_t$ is contained in $\mathcal{A}_{\mathcal{D}_\sigma}(t)$ and*

$$\mathrm{LIM}_{\tau\to-\infty}\frac{1}{t-\tau}\int_\tau^t \kappa(U(t,s)\phi(s))\mathrm{d}s = \int_{\mathcal{A}_{\mathcal{D}_\sigma}(t)} \kappa(z)\mathrm{d}\mathfrak{m}_t(z),$$

for any real-valued continuous functional κ on E_μ. Moreover, $\mathfrak{m}_t$ is invariant in the sense that

$$\int_{\mathcal{A}_{\mathcal{D}_\sigma}(t)} \kappa(z)\mathrm{d}\mathfrak{m}_t(z) = \int_{\mathcal{A}_{\mathcal{D}_\sigma}(\tau)} \kappa(U(t,\tau)z)\mathrm{d}\mathfrak{m}_\tau(z), \quad t \geqslant \tau.$$

We end this article with two remarks.

Remark 4.2. The obtained invariant measure in Theorem 4.1 depends merely on the specific generalized limit and the given continuous map $\kappa(\cdot) \in \mathcal{D}_\sigma$. It is unique in this sense. We can not prove that the invariant measure is the same for different generalized Banach limit and differential continuous map $\kappa(\cdot) \in \mathcal{D}_\sigma$. The support of the invariant measure is contained in the pullback attractor $\mathcal{A}_\mathcal{D}(t)$ but we can not establish that the support is the whole pullback attractor.

Remark 4.3. If we rewrite the equation in (2.4) as

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \mathcal{G}(z,t),$$

where $\mathcal{G}(z,t) = -\Theta z + F(z,t)$, then we can prove the following Liouville-type equation in Statistical Mechanics

$$\int_{\mathcal{A}_\mathcal{D}(t)} \Phi(z)\mathrm{d}\mathfrak{m}_t(z) - \int_{\mathcal{A}_\mathcal{D}(\tau)} \Phi(z)\mathrm{d}\mathfrak{m}_\tau(z) = \int_\tau^t \int_{\mathcal{A}_\mathcal{D}(s)} \big(\mathcal{G}(z,s), \Phi'(z)\big)\mathrm{d}\mathfrak{m}_s(z)\mathrm{d}s,$$

for all "test" functions Φ (cf. [13, P_{178}, Definition 1.2]). We will investigate this problem in another paper.

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