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ON THE UPSS METHOD FOR NON-HERMITIAN SINGULAR SADDLE POINT PROBLEMS*

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Abstract Recently, a new Uzawa-type method, referred as the UPSS method, is proposed for solving the non-Hermitian nonsingular saddle point problems, see Dou, Yang and Wu (2017). In this paper, we give the semi-convergence analysis of the UPSS method when it is used to solve non-Hermitian singular saddle point problems. An example is given to verify the effectiveness of this method for solving non-Hermitian singular saddle point problems.

Keywords Singular saddle-point problem, UPSS method, semi-convergence.

MSC(2010) 65F10, 65F15.

1. Introduction

Block structure linear systems, especially the 2×2 block structure linear systems, arising from a variety of scientific and engineering applications, for example, finite element or finite difference methods discretization of some partial differential equations [6, 8, 20], numerical methods for solving weighted least squares problems [24], augmented immersed interface method for Stokes and Darcy or Navier-stokes and Darcy coupling equatios [17] and so on. In this paper, we consider the 2×2 block structure linear systems arising from mixed or hybrid finite element discretization Navier-Stokes equations [6], it is called the saddle point problem and has the form

$$\mathcal{A}\mathbf{x} = \begin{bmatrix} A & B \\ -B^* & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ -g \end{bmatrix} = b, \tag{1.1}$$

where $A \in \mathbb{C}^{n\times n}$ is a non-Hermitian positive definite matrix, $B \in \mathbb{C}^{n\times m}$ is a matrix with $\text{rank}(B) = r$, here and in the sequence, $\text{rank}(\cdot)$ is the rank of a given matrix, $f \in \mathbb{C}^n$ and $g \in \mathbb{C}^m$ are given vectors, with $m \le n$.

Usually, the block matrices A and B are large and sparse, (1.1) is suitable for being solved by the iterative methods. When $r = m$, (1.1) is the nonsingular saddle point problem [2], and when $r < m$, (1.1) is the singular saddle point problem. Moreover, in this case, we suppose that the singular saddle point problem(1.1)is consistent, i.e.,$b \in \text{range}(\mathcal{A})$, the range of $\mathcal{A}$. Efficient numerical methods for solving

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nonsingular and singular saddle point problems have been studied in the literatures, see [12] and the references therein. The Uzawa method [1], one of the most important iteration methods, for solving (1.1) received wide attention and obtained considerable achievements in recent years. There are variant forms of the Uzawa method, see [14, 18, 23, 25, 26] for examples. The semi-convergence of these Uzawa-types methods for singular saddle point problems are studied in [11, 14, 18, 21, 22]. Recently, based on the shift-splitting iteration method [5] and the preconditioning techniques, Dou, Yang and Wu proposed a new Uzawa-type method, named the UPSS method, for solving non-Hermitian nonsingular saddle point problems (1.1), see [12].

Method 1.1 (The UPSS Method)**.** *Given initial vectors* $x_0 \in \mathbb{C}^n, y_0 \in \mathbb{C}^m$*, and two relaxation parameters* $\alpha, \tau > 0$*. For* $k = 0, 1, 2, \cdots$*, until the iteration sequence converges, compute*

$$\begin{cases} x_{k+1} = x_k + 2(\alpha P + A)^{-1}(f - Ax_k - By_k), \\ y_{k+1} = y_k + \tau Q^{-1}(B^* x_{k+1} - g), \end{cases} \tag{1.2}$$

where $P \in \mathbb{C}^{n\times n}$ *and* $Q \in \mathbb{C}^{m\times m}$ *are Hermitian positive definite matrices.*

The iteration scheme of UPSS method (1.2) can be rewritten as

$$\mathbf{x}_{k+1} = \Gamma \mathbf{x}_k + M^{-1}b, \tag{1.3}$$

where

$$\Gamma = \begin{bmatrix} \frac{1}{2}(\alpha P + A) & 0 \\ -B^* & \frac{1}{\tau}Q \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{2}(\alpha P - A) & -B \\ 0 & \frac{1}{\tau}Q \end{bmatrix}$$

is the iteration matrix and

$$M = \begin{bmatrix} \frac{1}{2}(\alpha P + A) & 0 \\ -B^* & \frac{1}{\tau}Q \end{bmatrix}.$$

Theoretical as well as numerical results demonstrated that the UPSS method is a more efficient method for solving non-Hermitian nonsingular saddle point problems.

In this paper, we will show that the UPSS method proposed in [12] can be used to solve the non-Hermitian singular saddle point problem (1.1).

2. Semi-convergence of the UPSS method

In this section, we will study the semi-convergence of the UPSS method when it is used to solve non-Hermitian singular saddle point problems (1.1). $\sigma(E)$ and $\rho(E)$ denote the spectral set and the spectral radius of a square matrix E, respectively. The smallest nonnegative integer i such that rank(E^i)=rank(E^{i+1}) is called the index of E, and is denoted by index(E). We denote the range and the null spaces of E by $R(E)$ and $N(E)$, respectively.

For the singular saddle point matrix $\mathcal{A}$, one can require only that the iterative scheme (1.3) is semi-convergent to a solution $\mathbf{x}_\star$ of the linear system $\mathcal{A}\mathbf{x} = b$ for any initial vector $\mathbf{x}_0$, see [7].

Definition 2.1 ([7]). The iteration method (1.3) is semi-convergent if for any initial guess $[x_0^*,\ y_0^*]^*$, the iteration sequence $[x_k^*,\ y_k^*]^*$ produced by (1.3) converges to a solution $[x_\star^*,\ y_\star^*]^*$ of linear systems $\mathcal{A}\mathbf{x} = b$. Moreover, it holds

$$\begin{bmatrix} x_\star \\ y_\star \end{bmatrix} = (I-\Gamma)^D c + (I-E)\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \quad \text{with} \quad E = (I-\Gamma)(I-\Gamma)^D,$$

where I is the identity matrix and $(I-\Gamma)^D$ denotes the Drazin inverse of $I-\Gamma$.

Following lemma describes the sufficient and necessary semi-convergence conditions of the iteration scheme (1.3).

Lemma 2.1 ([7]). *The iteration scheme (1.3) is semi-convergent if and only if*

$$\text{index}(I-\Gamma) = 1 \quad \textit{and} \quad \vartheta(\Gamma) < 1,$$

where $\vartheta(\Gamma) = \max\{|\lambda|, \lambda \in \sigma(\Gamma), \lambda \neq 1\} < 1$ *is called the pseudo-spectral radius of the iteration matrix* Γ.

To study the semi-convergence of the UPSS method for solving non-Hermitian singular saddle point problems (1.1), we only need to verify that the iteration scheme (1.3) satisfies the two conditions in Lemma 2.1.

First, we consider the condition $\text{index}(I-\Gamma) = 1$. The sufficient and necessary condition for $\text{index}(I-\Gamma) = 1$ is precisely described in the following lemma.

Lemma 2.2 ([28]). Index$(I-\Gamma) = 1$ *holds if and only if, for any* $0 \neq Y \in R(\mathcal{A})$, $Y \notin N(\mathcal{A}M^{-1})$.

Based on Lema 2.2, we obtain the following result about iteration scheme (1.3), the proof is similar to that in [27].

Theorem 2.1. *Let* $A \in \mathbb{C}^{n\times n}$ *be non-Hermitian positive definite and* $B \in \mathbb{C}^{m\times n}$ *be rank deficient. Assume that* α, $\tau > 0$ *and* Γ *is the iteration matrix of the UPSS method. Then index*$(I-\Gamma) = 1$.

Proof. Let $Z = (\xi^*, \eta^*)^*$ with $\xi \in \mathbb{C}^n$, $\eta \in \mathbb{C}^m$ such that $Y = \mathcal{A}Z = \begin{bmatrix} A\xi + B\eta \\ -B^*\xi \end{bmatrix} \neq 0$. Then,

$$\mathcal{A}M^{-1}Y = \begin{bmatrix} (2A + 2\tau BQ^{-1}B^*)(\alpha P + A)^{-1}(A\xi + B\eta) - \tau BQ^{-1}B^*\xi \\ -2B^*(\alpha P + A)^{-1}(A\xi + B\eta) \end{bmatrix}. \tag{2.1}$$

In order to prove $\text{index}(I-\Gamma) = 1$, by Lemma 2.2, it is sufficient to prove $Y \notin N(\mathcal{A}M^{-1})$. We consider it according to three cases:
Case I. $A\xi + B\eta = 0$. Since $Y \neq 0$, it follows from (2.1) that

$$B^*\xi \neq 0 \quad \text{and} \quad \mathcal{A}M^{-1}Y = \begin{bmatrix} -\tau BQ^{-1}B^*\xi \\ 0 \end{bmatrix}.$$

Note that Q is Hermitian positive definite and $\tau > 0$, we get that $\tau BQ^{-1}B^*\xi \neq 0$. Thus, $Y \notin N(\mathcal{A}M^{-1})$.

Case II. $A\xi + B\eta \neq 0$ and $B^*(\alpha P + A)^{-1}(A\xi + B\eta) = 0$. Now (2.1) becomes

$$\mathcal{A}M^{-1}Y = \begin{bmatrix} (2A + 2\tau BQ^{-1}B^*)(\alpha P + A)^{-1}(A\xi + B\eta) - \tau BQ^{-1}B^*\xi \\ 0 \end{bmatrix}.$$

Suppose to the contrary that $(2A+2\tau BQ^{-1}B^*)(\alpha P+A)^{-1}(A\xi+B\eta)-\tau BQ^{-1}B^*\xi=0$.

On one hand, from the positive definitiveness of A, we have

$$(\alpha P + A)^{-1}(A\xi + B\eta) = \tau(2A + 2\tau BQ^{-1}B^*)^{-1}BQ^{-1}B^*\xi.$$

Thus, $\tau(2A + 2\tau BQ^{-1}B^*)^{-1}BQ^{-1}B^*\xi \neq 0$ as $A\xi + B\eta \neq 0$, which leads to

$$B^*\xi \notin N\left((2A + 2\tau BQ^{-1}B^*)^{-1}BQ^{-1}\right). \tag{2.2}$$

Note that

$$\tau B^*(2A + 2\tau BQ^{-1}B^*)^{-1}BQ^{-1}B^*\xi = B^*(\alpha P + A)^{-1}(A\xi + B\eta) = 0.$$

This implies

$$B^*\xi \in N\left(B^*(2A + 2\tau BQ^{-1}B^*)^{-1}BQ^{-1}\right). \tag{2.3}$$

It follows from (2.2) and (2.3) that

$$\operatorname{rank}\left((2A + 2\tau BQ^{-1}B^*)^{-1}BQ^{-1}\right) > \operatorname{rank}\left(B^*(2A + 2\tau BQ^{-1}B^*)^{-1}BQ^{-1}\right). \tag{2.4}$$

On the other hand, it is easy to verify that

$$\begin{aligned}\operatorname{rank}\left(B^*(2A + 2\tau BQ^{-1}B^*)^{-1}BQ^{-1}\right) &= \operatorname{rank}\left(B^*(2A+2\tau BQ^{-1}B^*)^{-1}B\right) = \operatorname{rank}(B) \\ &= \operatorname{rank}\left((2A + 2\tau BQ^{-1}B^*)^{-1}BQ^{-1}\right),\end{aligned}$$

which is in contradiction with (2.4). Thus, $(2A + 2\tau BQ^{-1}B^*)(\alpha P + A)^{-1}(A\xi + B\eta) - \tau BQ^{-1}B^*\xi \neq 0$. That is to say $Y \notin N(\mathcal{A}M^{-1})$.

Case III. $A\xi + B\eta \neq 0$ and $B^*(\alpha P + A)^{-1}(A\xi + B\eta) \neq 0$. In this case, it is obvious that $Y \notin N(\mathcal{A}M^{-1})$.

In summary, for any $0 \neq Y \in R(\mathcal{A})$, we have $Y \notin N(\mathcal{A}M^{-1})$. So $\operatorname{index}(I-\Gamma) = 1$ by Lemma 2.1. □

To verify the condition $\vartheta(\Gamma) < 1$ of Lemma 2.1, we need the following result.

Lemma 2.3 ([19]). *Both roots of the complex quadratic equation $x^2 - bx + c = 0$ are less than one in modulus if and only if $|b - \bar{b}c| + c^2 < 1$, where $\bar{b}$ is the conjugate complex number of b.*

Let λ be an eigenvalue of the UPSS iteration matrix Γ and $(u^*, v^*)^* \in \mathbb{C}^{m+n}$ be the corresponding eigenvector, in terms of the expression of Γ, we have

$$\begin{bmatrix} \frac{1}{2}(\alpha P - A) & -B \\ 0 & \frac{1}{\tau}Q \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \lambda \begin{bmatrix} \frac{1}{2}(\alpha P + A) & 0 \\ -B^* & \frac{1}{\tau}Q \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}. \tag{2.5}$$

Theorem 2.2. *Assume that $A \in \mathbb{C}^{n\times n}$ is non-Hermitian positive definite and $B \in \mathbb{C}^{m\times n}$ is rank deficient, α, $\tau > 0$. If λ is the eigenvalue of the iteration matrix Γ and $(u^*, v^*)^* \in \mathbb{C}^{m+n}$ is the corresponding eigenvector, then $\lambda = 1$ if and only if $u = 0$.*

Proof. If $\lambda = 1$, then (2.5) becomes

$$\begin{bmatrix} \frac{1}{2}(\alpha P - A) & -B \\ 0 & \frac{1}{\tau}Q \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(\alpha P + A) & 0 \\ -B^* & \frac{1}{\tau}Q \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix},$$

which is equivalent to

$$\begin{cases} Au = -Bv, \\ -B^*u = 0. \end{cases} \tag{2.6}$$

The first equation in (2.6) gives $u = -A^{-1}Bv$. Substituting it into the second equation in (2.6), we have $B^*A^{-1}Bv = 0$. Now the positive definitiveness of A imply that $Bv = 0$. Therefore, $u = 0$.

Conversely, if $u = 0$, then we have

$$\begin{bmatrix} \frac{1}{2}(\alpha P - A) & -B \\ 0 & \frac{1}{\tau}Q \end{bmatrix} \begin{bmatrix} 0 \\ v \end{bmatrix} = \lambda \begin{bmatrix} \frac{1}{2}(\alpha P + A) & 0 \\ -B^* & \frac{1}{\tau}Q \end{bmatrix} \begin{bmatrix} 0 \\ v \end{bmatrix}.$$

Note that Q is Hermitian positive definite,$\tau > 0$ and $v \neq 0$, so $\lambda = 1$. □

Theorem 2.3. *Assume that $A \in \mathbb{C}^{n\times n}$ is non-Hermitian positive definite and $B \in \mathbb{C}^{m\times n}$ is rank deficient, $\alpha,\ \tau > 0$. If λ is the eigenvalue of the iteration matrix Γ and $(u^*, v^*)^* \in \mathbb{C}^{m+n}$ is the corresponding eigenvector, then both (i) and (ii) below are true.*

(i). If $u \in N(B^*)$, then $\vartheta(T) < 1$.

(ii). If $u \notin N(B^*)$, then $\vartheta(T) < 1$ if and only if

$$0 < \tau < \frac{2\alpha p}{\lambda_{max}(Q^{-1}B^*P^{-1}B)}.$$

where $p = \frac{u^*Pu}{u^*u}$, $\quad m + ni = \frac{u^*Au}{u^*u}$, $\quad s = \frac{u^*BQ^{-1}B^*u}{u^*u}$.

Proof. Note that (2.5) can be rewritten as

$$\begin{cases} ((\alpha P - A) - \lambda(\alpha P + A))u = 2Bv, \\ \lambda\tau B^*u = (\lambda - 1)Qv. \end{cases} \tag{2.7}$$

(i). If $u \in N(B^*)$, i.e., $B^*u = 0$, it follows from the second equation in (2.7) that $(\lambda - 1)Qv = 0$. Suppose $\lambda \neq 1$, then $v = 0$. Since $(u^*, v^*)^* \in \mathbb{C}^{m+n}$ is the corresponding eigenvector, then $u \neq 0$. Now the first equation in (2.7) becomes

$$((\alpha P - A) - \lambda(\alpha P + A))u = 0.$$

Multiplying the above equation by$\frac{u^*}{u^*u}$, and let $p = \frac{u^*Pu}{u^*u}$, $\quad m + ni = \frac{u^*Au}{u^*u}$, $\quad s = \frac{u^*BQ^{-1}B^*u}{u^*u}$, we have

$$(\alpha p + m + ni)\lambda - (\alpha p - m - ni) = 0.$$

Hence

$$|\lambda|^2 = \frac{(\alpha p - m)^2 + n^2}{(\alpha p + m)^2 + n^2}. \tag{2.8}$$

Note that $p,\ m > 0$, $n \neq 0$ and $\alpha > 0$, we have $|\lambda| < 1$.

(ii). If $u \notin N(B^*)$, from [12], we know that the λ satisfies the quadratic equation

$$\lambda^2 - \frac{2\alpha p - 2\tau s}{\alpha p + m + ni}\lambda + \frac{\alpha p - m - ni}{\alpha p + m + ni} = 0. \tag{2.9}$$

It follows from Lemma 2.3 that $|\lambda| < 1$ if and only if $\frac{|4(\alpha p - \tau s)m| + (\alpha p - m)^2 + n^2}{(\alpha p + m)^2 + n^2} < 1$, which is equivalent to

$$|(\alpha p - \tau s)m| < \alpha p m. \tag{2.10}$$

Since matrix A is positive definite, matrices P and Q are Hermitian positive definite and $u \notin N(B^*)$, so $m > 0,\ s > 0$. Then the inequality (2.10) holds for any $\alpha > 0$ and $0 < \tau < \frac{2\alpha p}{s}$. From [12], we know that $s \leq \lambda_{max}(Q^{-1}B^*P^{-1}B)$, here $\lambda_{max}(Q^{-1}B^*P^{-1}B)$ is the largest eigenvalue of $Q^{-1}B^*P^{-1}B$. Hence (2.10) holds for any $\alpha > 0$ and $0 < \tau < \frac{2\alpha p}{\lambda_{max}(Q^{-1}B^*P^{-1}B)}$.

The proof is completed. □

3. Numerical results

In this section, an example is given to illustrate the effectiveness of the UPSS method for solving the non-Hermitian singular saddle point problem (1.1). The modified local HSS (MLHSS) [15], the Uzawa-HSS [22] and the Uzawa-PSS [10] methods are compared with the UPSS method from aspects of the number of iteration steps (denoted by 'IT ') and the elapsed CPU times in seconds (denoted by 'CPU ').

The iteration scheme of the MLHSS method [15] is

$$\begin{cases} x_{k+1} = x_k + (Q_1 + H)^{-1}(f - Ax_k - By_k), \\ y_{k+1} = y_k + Q_2^{-1}(B^* x_{k+1} - g), \end{cases}$$

where $Q_1 \in \mathbb{C}^{n\times n}$ and $Q_2 \in \mathbb{C}^{m\times m}$ are Hermitian positive definite matrices. The iteration scheme of the Uzawa-HSS method [22, 23] is defined as follows

$$\begin{cases} x_{k+1} = x_k + 2\alpha(\alpha I + S)^{-1}(\alpha I + H)^{-1}(f - Ax_k - By_k), \\ y_{k+1} = y_k + \tau Q^{-1}(B^* x_{k+1} + g), \end{cases}$$

where α and τ are two positive constants. Splitting matrix A into its positive definite and skew-Hermitian parts as $A_P + A_s$, then the iteration scheme of the Uzawa-PSS method [10] can be defined as

$$\begin{cases} x_{k+1} = x_k + 2\alpha(\alpha I + A_s)^{-1}(\alpha I + A_p)^{-1}(f - Ax_k - By_k), \\ y_{k+1} = y_k + \tau Q^{-1}(B^* x_{k+1} - g), \end{cases}$$

where $A_p = D_H + 2L_H, A_S = L_H^* - L_H + S$, D_H and L_H being the diagonal part and strictly lower triangular part of H.

In the implementation, we choose $Q_1 = \alpha I$ and $Q_2 = \frac{1}{\tau}Q$ in the MLHSS method. For the preconditioning matrices P and Q of the tested methods, we choose $P = H$ and $Q = \text{diag}(B^* D^{-1} B)$, where $D = \text{diag}(A)$. In addition, all the involved sub-linear system are solved by Cholesky or LU factorization in combination with AMD reordering. The involved parameters of all methods are choosing to be the experimentally found optimal ones, which are resulting in the least iteration step. The inner iteration is terminated when the relative residual satisfies $res = |r_k|/|r_0| < 10^{-3}$. All the tested iteration methods are stared from zero vector and terminated when the current iteration solution $\mathbf{x}_k$ satisfies

$$\text{RES} = \frac{||b - \mathcal{A}\mathbf{x}_k||}{||b||} < 10^{-6}$$

or the iteration steps exceed $k_{max} = 1500$. In addition, All runs are performed in MATLAB 2010 on a person computer with Intel Core (4G RAM) Windows 7 system.

Example 3.1. Let us consider the singular saddle-point problem (1.1) has the following coefficient sub-matrices:

$$A = \begin{bmatrix} I \otimes T + T \otimes I & 0 \\ 0 & I \otimes T + T \otimes I \end{bmatrix} \in \mathbb{R}^{2q^2 \times 2q^2},\ B = \begin{bmatrix} \hat{B} \ b_1 \ b_2 \end{bmatrix} \in \mathbb{R}^{2q^2 \times (q^2+2)},$$

with

$$\hat{B} = \begin{bmatrix} I \otimes F \\ F \otimes I \end{bmatrix} \in \mathbb{R}^{2q^2 \times q^2},\ b_1 = \hat{B}^T \begin{bmatrix} e \\ 0 \end{bmatrix},\ b_2 = \hat{B}^T \begin{bmatrix} 0 \\ e \end{bmatrix},\ e = [1, 1, \cdots, 1] \in \mathbb{R}^{q^2/2}$$

and

$$T = \frac{\nu}{h^2} \cdot tridiag(-1, 2, -1) + \frac{1}{2h} \cdot tridiag(-1, 0, 1) \in \mathbb{R}^{q \times q}, F = \frac{1}{h} \cdot tridiag(-1, 1, 0) \in \mathbb{R}^{q \times q}.$$

Here, $\otimes$ denotes the Kronecker product, ν is a parameter and $h = \frac{1}{q+1}$ is the discretization meshsize, see [22, 29].

The matrix B is an augmentation of the full rank matrix $\hat{B}$ with two linearly independent vectors b_1 and b_2. As b_1 and b_2 are linear combinations of the columns of the matrix $\hat{B}$, B is a rank-deficient matrix. In the test problems, we choose $\nu = 1$ and $\nu = 0.1$. For each ν, we take three meshsizes, i.e., $q = 16,\ 32,\ 64$.

In Table 1, we list the numerical results of the UPSS method, the Uzawa-HSS method, the Uzawa-PSS method and the MLHSS method for $\nu = 1$. The same items are listed in Table 2 for $\nu = 0.1$.

From the numerical results, we can see that all of the testing methods can converge to the approximate solution of singular saddle point problems (1.1). However, the Uzawa-HSS, the Uzawa-PSS and the MLHSS methods converges very slowly. The UPSS method is the most efficient one, which uses the least IT and CPU times than the Uzawa-HSS, the Uzawa-PSS and the MLHSS methods to achieve stopping criterion.

Theoretical analysis state that the UPSS method can be used to solve non-Hermitian singular saddle point problems (1.1), and the numerical results, which

Table 1. Numerical results of iteration methods for $\nu = 1$

Method		α	τ	IT	CPU	RES
$q = 16$	UPSS	2.6	0.44	36	0.0036	7.1422e-7
	Uzawa–HSS	260	0.14	129	0.0312	9.7418e-7
	Uzawa–PSS	586	0.67	208	0.0156	9.6298e-7
	MLHSS	0.0019	0.17	64	0.0046	9.9726e-7
$q = 32$	UPSS	3.8	0.35	54	0.0156	9.0487 e-7
	Uzawa–HSS	636	0.095	249	0.1560	9.8928e-7
	Uzawa–PSS	510	0.08	280	1.7628	9.7976e-7
	MLHSS	34	0.21	83	0.0468	9.8990e-7
$q = 64$	UPSS	6.2	0.32	81	0.3900	8.3985e-7
	Uzawa–HSS	390	0.022	623	3.6192	9.8546e-7
	Uzawa–PSS	900	0.04	687	51.7455	9.8382e-7
	MLHSS	28	0.11	128	0.5928	9.8579e-7

Table 2. Numerical results of iteration methods for $\nu = 0.1$

Method		α	τ	IT	CPU	RES
$q = 16$	UPSS	2.8	0.5	62	0.0156	9.7737e-7
	Uzawa–HSS	10	0.11	249	0.0468	9.8527e-7
	Uzawa–PSS	56	0.68	208	0.0312	9.8396e-7
	MLHSS	5.3	0.35	109	0.0236	9.0993e-7
$q = 32$	UPSS	4.4	0.44	83	0.0312	9.1085e-7
	Uzawa–HSS	98	0.03	337	0.1716	9.8347e-7
	Uzawa–PSS	65	0.17	347	0.9288	9.7499e-7
	MLHSS	4.8	0.27	121	0.0468	9.2534e-7
$q = 64$	UPSS	6.5	0.35	114	0.3744	9.5576e-7
	Uzawa–HSS	100	0.08	502	3.0888	9.8524e-7
	Uzawa–PSS	100	0.05	765	53.4147	9.9754e-7
	MLHSS	4.5	0.15	171	0.8580	9.3958e-7

confirm the theoretical results, demonstrate that the UPSS method is more efficient than the Uzawa-HSS, the Uzawa-PSS and the MLHSS methods for solving non-Hermitian singular saddle point problems.

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