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DYNAMICAL BEHAVIOR OF A STOCHASTIC FOOD CHAIN CHEMOSTAT MODEL WITH MONOD RESPONSE FUNCTIONS

Miaomiao Gao[1], Daqing Jiang[1,2,3,†], Kai Qi[1], Tasawar Hayat[3,4], Ahmed Alsaedi[3] and Bashir Ahmad[3]

Abstract This paper studies a food chain chemostat model with Monod response functions, which is perturbed by white noise. Firstly, we prove the existence and uniqueness of the global positive solution. Then sufficient conditions for the existence of a unique ergodic stationary distribution are established by constructing suitable Lyapunov functions. Moreover, we consider the extinction of microbes in two cases. In the first case, both the predator and prey species are extinct. In the second case, only the predator species is extinct, and the prey species survives. Finally, numerical simulations are carried out to illustrate the theoretical results.

Keywords Stochastic food chain chemostat model, Monod response function, stationary distribution, extinction.

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1. Introduction

The chemostat is a continuous culture device mainly used for growing microorganisms. It has the advantage that the parameters are readily measurable, and plays an important role in mathematical biology and theoretical ecology [2, 17]. In recent years, the dynamics of chemostat models has been extensively studied, see e.g. [1,4,8,11,12,16,18]. The deterministic food chain chemostat model with Monod

†the corresponding author. Email address:daqingjiang2010@hotmail.com(D. Jiang)
[1]College of Science, China University of Petroleum (East China), Qingdao 266580, China
[2]Key Laboratory of Unconventional Oil and Gas Development, China University of Petroleum (East China), Ministry of Education, Qingdao 266580, China
[3]Nonlinear Analysis and Applied Mathematics(NAAM)-Research Group, Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia
[4]Department of Mathematics, Quaid-I-Azam University 45320, Islamabad 44000, Pakistan

response functions takes the following form [2]

$$\begin{cases} \frac{ds(t)}{dt} = Q(s_0 - s(t)) - \frac{1}{\delta}\frac{\mu s(t)x_1(t)}{k+s(t)}, \\ \frac{dx_1(t)}{dt} = \frac{\mu s(t)x_1(t)}{k+s(t)} - Qx_1(t) - \alpha\frac{\gamma x_1(t)x_2(t)}{l+x_1(t)}, \\ \frac{dx_2(t)}{dt} = \frac{\gamma x_1(t)x_2(t)}{l+x_1(t)} - Qx_2(t), \end{cases} \tag{1.1}$$

where the predator x_2 completely feeds on the prey x_1, and the prey x_1 consumes a single nutrient s. $s(t)$, $x_1(t)$ and $x_2(t)$ stand for the concentrations of nutrient, prey and predator at time t, respectively. s_0 is the original input concentration of nutrient s and Q is the common dilution rate. $\frac{1}{\delta}$ and α represent the consumption coefficients. The terms $\frac{\mu s(t)}{k+s(t)}$ and $\frac{\gamma x_1(t)}{l+x_1(t)}$ denote the Monod growth functional responses. μ and γ are the maximum growth rates of x_1 and x_2, respectively. k and l are the corresponding half-saturation constants.

However, the chemostat is inevitably subject to environmental noise at the micro level. The deterministic chemostat model has some limitations to accurately predict the future dynamics. As May [14] pointed out, some parameters involved in the system exhibit random fluctuations to a greater or lesser extent due to various environmental noises. Therefore, some authors [6,19,21–24,26–28] have investigated the classical and competitive chemostat systems with stochastic disturbance. For a single-species chemostat model in which the maximum growth rate is influenced by white noise, Xu and Yuan [23] obtained an analogue of break-even concentration and proved that large noise can make the microorganism go extinct. Sun etc [19] considered a stochastic two-species Monod competition chemostat model, which is subject to environmental noise. They studied the asymptotic behavior and the steady state distribution. Zhang and Jiang [28] discovered sufficient conditions which guarantee that the principle of competitive exclusion holds for a stochastic chemostat model with Holling type II functional response.

To the best of our knowledge, there is little amount of work on the food chain chemostat model with random perturbation. For a Lotka-Volterra food chain chemostat model in which the dilution rate is influenced by white noise, Sun etc [20] established sufficient conditions for some population dynamical properties and proved the existence of stationary distribution. In addition to the dilution rate, some papers [3, 23, 25] indicate that the maximum growth rate is also one of the parameters that are very sensitive to white noise in chemostat model. Motivated by the related work, in this article, we consider that the maximum growth rates of microorganisms in model (1.1) are effected by white noise, i.e.,

$$\mu \to \mu + \beta_1 \dot{W}_1(t), \quad \gamma \to \gamma + \beta_2 \dot{W}_2(t).$$

For the sake of simplicity, we use s, x_1 and x_2 to represent $s(t)$, $x_1(t)$ and $x_2(t)$, respectively. The model of interest is

$$\begin{cases} ds = \left[Q(s_0 - s) - \frac{1}{\delta}\frac{\mu s x_1}{k+s}\right]dt - \frac{1}{\delta}\frac{\beta_1 s x_1}{k+s}dW_1(t), \\ dx_1 = \left[\frac{\mu s x_1}{k+s} - Qx_1 - \alpha\frac{\gamma x_1 x_2}{l+x_1}\right]dt + \frac{\beta_1 s x_1}{k+s}dW_1(t) - \alpha\frac{\beta_2 x_1 x_2}{l+x_1}dW_2(t), \\ dx_2 = \left[\frac{\gamma x_1 x_2}{l+x_1} - Qx_2\right]dt + \frac{\beta_2 x_1 x_2}{l+x_1}dW_2(t), \end{cases} \tag{1.2}$$

where $W_i(t), i = 1, 2$ are standard one-dimensional independent Brownian motions and β_i^2 represents the noise intensity. Other parameters are used as in model (1.1).

To study above system, we make the following dimensionless transformation

$$\bar{s} = \frac{s}{s_0},\ \tau = Qt,\ x = \frac{x_1}{\delta s_0},\ y = \frac{\alpha x_2}{\delta s_0},\ m_1 = \frac{\mu}{Q},\ m_2 = \frac{\gamma}{Q},$$

$$a_1 = \frac{k}{s_0},\ a_2 = \frac{l}{\delta s_0},\ \sigma_i = \beta_i\sqrt{\frac{1}{Q}},\ B_i(\tau) = \frac{W_i(\frac{\tau}{Q})}{\sqrt{\frac{1}{Q}}},\ i = 1, 2.$$

Then system (1.2) is transformed into the following equations (replacing $\bar{s}$, τ with s, t)

$$\begin{cases} ds = \left(1 - s - \frac{m_1 sx}{a_1+s}\right) dt - \frac{\sigma_1 sx}{a_1+s} dB_1(t), \\ dx = \left(\frac{m_1 sx}{a_1+s} - x - \frac{m_2 xy}{a_2+x}\right) dt + \frac{\sigma_1 sx}{a_1+s} dB_1(t) - \frac{\sigma_2 xy}{a_2+x} dB_2(t), \\ dy = \left(\frac{m_2 xy}{a_2+x} - y\right) dt + \frac{\sigma_2 xy}{a_2+x} dB_2(t). \end{cases} \tag{1.3}$$

The corresponding deterministic system to (1.3) is

$$\begin{cases} \frac{ds}{dt} = 1 - s - \frac{m_1 sx}{a_1+s}, \\ \frac{dx}{dt} = \frac{m_1 sx}{a_1+s} - x - \frac{m_2 xy}{a_2+x}, \\ \frac{dy}{dt} = \frac{m_2 xy}{a_2+x} - y. \end{cases} \tag{1.4}$$

Let $\lambda_i = \frac{a_i}{m_i - 1}$, $i = 1, 2$ and according to the theory in [2, 8], system (1.4) has the following properties

- If $m_1 \le 1$, or $m_1 > 1$ and $\lambda_1 \ge 1$, then $\lim_{t\to\infty} x(t) = 0$ (obviously, $\lim_{t\to\infty} y(t) = 0$);
- If $m_2 \le 1$, or $m_2 > 1$ and $\lambda_1 + \lambda_2 > 1$, then $\lim_{t\to\infty} y(t) = 0$;
- If $m_i > 1$ and $\lambda_1 + \lambda_2 < 1$, $i = 1, 2$, then predator y is surviving.

The rest of this paper is arranged as follows. In Section 2, we analyze model (1.3) and review the basic theories, which are necessary for later discussion. In Section 3, we prove the solution of system (1.3) is positive and global. For the equivalent system (2.1) of model (1.3), Section 4 gives sufficient conditions for the existence of a unique ergodic stationary distribution. In Section 5, sufficient conditions for extinction of microbes are established in two cases. In Section 6, we validate our theoretical results by some examples and make a further discussion.

2. Model analysis and preliminaries

By system (1.3), one yields

$$d(s + x + y) = (1 - (s + x + y))dt.$$

Then we can obtain the region

$$\Gamma_0 = \{(s, x, y) \in \mathbb{R}^3_+ : s + x + y = 1\}$$

is a positively invariant set of system (1.3). So we can analyze the dynamical properties of system (1.3) by studying the following system

$$\begin{cases} dx = (\frac{m_1 sx}{a_1+s} - x - \frac{m_2 xy}{a_2+x})dt + \frac{\sigma_1 sx}{a_1+s} dB_1(t) - \frac{\sigma_2 xy}{a_2+x} dB_2(t), \\ dy = (\frac{m_2 xy}{a_2+x} - y)dt + \frac{\sigma_2 xy}{a_2+x} dB_2(t), \end{cases} \tag{2.1}$$

where $s = 1 - x - y$ and

$$(x, y) \in \Gamma_* := \{(x, y) \in \mathbb{R}^2_+ : x + y < 1\}.$$

System (2.1) presents a stochastically perturbed version of the following deterministic system

$$\begin{cases} \frac{dx}{dt} = \frac{m_1 s x}{a_1 + s} - x - \frac{m_2 x y}{a_2 + x}, \\ \frac{dy}{dt} = \frac{m_2 x y}{a_2 + x} - y. \end{cases} \tag{2.2}$$

This model has three equilibria $E_0 : (0, 0)$, $E_1 : (1 - \lambda_1, 0)$, $E_2 : (x^*, y^*)$, where x^* and y^* satisfy $\frac{m_1(1-x^*-y^*)}{1+a_1-x^*-y^*} - \frac{m_2 y^*}{a_2+x^*} = 1$ and $\frac{m_2 x^*}{a_2+x^*} = 1$. About the properties of E_0, E_1 and E_2, the reader can refer to [2,8].

Next, we present some basic theories in stochastic differential equations which are introduced in [15] and give a lemma [9] which provides a criterion for the existence of a unique ergodic stationary distribution.

Throughout this paper, let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathscr{F}_0$ contains all $\mathbb{P}-$null sets). Denote

$$\mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x_i > 0 \text{ for all } 1 \leq i \leq d\},\ \overline{\mathbb{R}}^d_+ = \{x \in \mathbb{R}^d : x_i \geq 0 \text{ for all } 1 \leq i \leq d\}.$$

In general, consider the d-dimensional stochastic differential equation

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t), \text{ for } t \geq t_0, \tag{2.3}$$

with initial value $x(t_0) = x_0 \in \mathbb{R}^d$ and $B(t)$ denotes d-dimensional standard Brownian motion defined on the above probability space. Define the differential operator L associated with Eq. (2.3) by

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{d} f_i(x, t)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{d}[g^T(x, t)g(x, t)]_{ij}\frac{\partial^2}{\partial x_i \partial x_j}.$$

If L acts on a function $V \in C^{2,1}(\mathbb{R}^d \times \overline{\mathbb{R}}_+; \overline{\mathbb{R}}_+)$, then

$$LV(x, t) = V_t(x, t) + V_x(x, t)f(x, t) + \frac{1}{2}trac[g^T(x, t)V_{xx}(x, t)g(x, t)],$$

where $V_t = \frac{\partial V}{\partial t}, V_x = (\frac{\partial V}{\partial x_1}, \cdots, \frac{\partial V}{\partial x_d})$ and $V_{xx} = (\frac{\partial^2 V}{\partial x_i \partial x_j})_{d\times d}$. By Itô formula, if $x(t)$ is a solution of Eq. (2.3), then

$$dV(x(t), t) = LV(x(t), t)dt + V_x(x(t), t)g(x(t), t)dB(t).$$

Let $X(t)$ be a homogeneous Markov process in R^l (R^l represents euclidean l-space) satisfying the stochastic equation

$$dX(t) = h(X)dt + \sum_{m=1}^{k} g_m(X)dB_m(t).$$

The diffusion matrix is

$$A(x) = (a_{ij}(x)), \quad a_{ij}(x) = \sum_{m=1}^{k} g_m^{(i)}(x)g_m^{(j)}(x).$$

Lemma 2.1. *Assume there exists a bounded open domain $G \subset R^l$ with regular boundary Γ, having the following properties*

(A1) In the domain G and some neighborhood thereof, the smallest eigenvalue of the diffusion matrix $A(x)$ is bounded away from zero;

(A2) If $x \in R^l \setminus G$, the mean time τ at which a path issuing from x reaches the set G is finite, and $sup_{x\in K} E^x \tau < \infty$ for every compact subset $K \in R^l$.

Then the Markov process $X(t)$ has a unique stationary distribution $\pi(\cdot)$. Let $f(x)$ be a function integrable with respect to the measure π. For all $x \in R^l$, the following formula holds

$$\mathbb{P}\left\{\lim_{t\to\infty} \frac{1}{t}\int_0^t f(X(s))ds = \int_{R^l} f(x)\pi(dx)\right\} = 1.$$

For simplicity, we define $\langle h\rangle_t = \frac{1}{t}\int_0^t h(s)ds$, where $h(t)$ is an integrable function on $[0,\infty)$.

3. Existence and uniqueness of positive solution

In this section, we shall show that system (1.3) has a unique global positive solution with any given initial value by making use of the Lyapunov function method as mentioned in [15].

Theorem 3.1. *For any initial value $(s(0), x(0), y(0)) \in \mathbb{R}^3_+$, model (1.3) has a unique solution $(s(t), x(t), y(t))$ on $t \geq 0$ and the solution will remain in $\mathbb{R}^3_+$ with probability one, that is to say, $(s(t), x(t), y(t)) \in \mathbb{R}^3_+$ for all $t \geq 0$ almost surely (a.s.).*

Proof. Since the coefficients of model (1.3) satisfy the local Lipschitz condition, there exists a unique local solution $(s(t), x(t), y(t))$ on $t \in [0, \tau_e)$ for any given initial value $(s(0), x(0), y(0)) \in \mathbb{R}^3_+$, where τ_e represents the explosion time. To illustrate this solution is global, we only need to prove $\tau_e = \infty$ a.s. To this end, let $m_0 \geq 1$ be sufficiently large such that $s(0), x(0), y(0)$ all lie within the interval $[\frac{1}{m_0}, m_0]$. For each integer $m \geq m_0$, define the stopping time as follows

$$\tau_m = \inf\left\{t \in [0, \tau_e) : \min\{s(t), x(t), y(t)\} \leq \frac{1}{m} \text{ or } \max\{s(t), x(t), y(t)\} \geq m\right\},$$

where throughout this paper, we set $\inf \emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). Obviously, τ_m is increasing as $m \to \infty$. Set $\tau_\infty = \lim_{m\to+\infty} \tau_m$, whence $\tau_\infty \leq \tau_e$ a.s. If $\tau_\infty = \infty$ a.s. is true, then $\tau_e = \infty$ and $(s(t), x(t), y(t)) \in \mathbb{R}^3_+$ a.s. for all $t \geq 0$. Namely, in order to complete the proof we only need to prove $\tau_\infty = \infty$ a.s. If this statement is incorrect, then there is a pair of constants $T > 0$ and $\epsilon \in (0, 1)$ such that

$$\mathbb{P}\{\tau_\infty \leq T\} > \epsilon.$$

Hence, there is an integer $m_1 \geq m_0$ such that

$$\mathbb{P}\{\tau_m \leq T\} \geq \epsilon \text{ for all } m \geq m_1.$$

In addition, for $t \leq \tau_m$, we can easily see

$$d[s(t) + x(t) + y(t)] = [1 - (s(t) + x(t) + y(t))]dt.$$

This implies that

$$\lim_{t\to\infty} s(t) + x(t) + y(t) = 1.$$

Therefore, there exists a positive constant K such that

$$s(t) + x(t) + y(t) \leq K.$$

Define a non-negative $\mathcal{C}^2$-function $\overline{V} : \mathbb{R}^3_+ \to \overline{\mathbb{R}}_+$ by

$$\overline{V}(s, x, y) = (s - 1 - \log s) + (x - 1 - \log x) + (y - 1 - \log y).$$

Let $m \geq m_1$ and $T > 0$ be arbitrary. For any $0 \leq t \leq \tau_m \wedge T = \min\{\tau_m, T\}$, making use of Itô formula to $\overline{V}$, we obtain

$$d\overline{V}(s, x, y) = L\overline{V}(s, x, y)dt + \frac{\sigma_1(x - s)}{a_1 + s}dB_1(t) + \frac{\sigma_2(y - x)}{a_2 + x}dB_2(t),$$

where $L\overline{V} : \mathbb{R}^3_+ \to \mathbb{R}$ is defined by

$$\begin{aligned}
L\overline{V}(s, x, y) =& \left(1 - \frac{1}{s}\right)\left(1 - s - \frac{m_1 sx}{a_1 + s}\right) + \frac{1}{2s^2}\left(\frac{\sigma_1 sx}{a_1 + s}\right)^2 \\
&+ \left(1-\frac{1}{x}\right)\left(\frac{m_1 sx}{a_1+s}-x-\frac{m_2 xy}{a_2+x}\right)+\frac{1}{2x^2}\left[\left(\frac{\sigma_1 sx}{a_1+s}\right)^2+\left(\frac{\sigma_2 xy}{a_2 + x}\right)^2\right] \\
&+ \left(1 - \frac{1}{y}\right)\left(\frac{m_2 xy}{a_2 + x} - y\right) + \frac{1}{2y^2}\left(\frac{\sigma_2 xy}{a_2 + x}\right)^2 \\
=& 1 - s - \frac{1}{s} + 1 + \frac{m_1 x}{a_1 + s} + \frac{1}{2}\left(\frac{\sigma_1 x}{a_1 + s}\right)^2 \\
&- x - \frac{m_1 s}{a_1 + s} + 1 + \frac{m_2 y}{a_2 + x} + \frac{1}{2}\left(\frac{\sigma_1 s}{a_1 + s}\right)^2 + \frac{1}{2}\left(\frac{\sigma_2 y}{a_2 + x}\right)^2 \\
&- y - \frac{m_2 x}{a_2 + x} + 1 + \frac{1}{2}\left(\frac{\sigma_2 x}{a_2 + x}\right)^2 \\
\leq& 4 + \frac{m_1 x}{a_1 + s} + \frac{m_2 y}{a_2 + x} \\
&+ \frac{1}{2}\left(\left(\frac{\sigma_1 x}{a_1 + s}\right)^2 + \left(\frac{\sigma_1 s}{a_1 + s}\right)^2 + \left(\frac{\sigma_2 y}{a_2 + x}\right)^2 + \left(\frac{\sigma_2 x}{a_2 + x}\right)^2\right) \\
\leq& 4 + \left(\frac{m_1}{a_1} + \frac{m_2}{a_2}\right)K + \left(\frac{\sigma_1^2}{a_1^2} + \frac{\sigma_2^2}{a_2^2}\right)K^2 := H,
\end{aligned}$$

where H is a positive constant. The rest of the proof is similar to the statement of Theorem 2.1 in [7], so we omit it. This completes the proof. □

4. Existence of ergodic stationary distribution

Since stationary distribution can enrich the dynamical behavior of stochastic chemostat systems [13], the aim of this section is to investigate sufficient conditions for the existence of a unique ergodic stationary distribution of system (2.1).

Theorem 4.1. *If there exists a constant* c_2 *satisfying* $c_2 > \left(\frac{m_2}{{a_2}^2} + \frac{c_1 m_1(a_1+1)}{{a_1}^2}\right)\frac{a_1+1}{m_1-1}$, *where* $c_1 = \frac{m_2 a_2(a_1+1-\bar{x})^2}{m_1 a_1 \bar{x}(a_2+\bar{x})^2}$, *such that the following condition holds*

$$\lambda := \frac{m_2\bar{x}}{a_2+\bar{x}} - 1 - \frac{c_2\bar{x}\sigma_1^2}{2}\left(\frac{1}{a_1+1}\right)^2 - \frac{\sigma_2^2}{2}\left(\frac{1}{a_2+1}\right)^2 > 0,$$

where $\bar{x} = 1-\lambda_1$, *then for any initial value* $(x(0), y(0)) \in \Gamma_*$, *system* (2.1) *admits a unique stationary distribution and it has the ergodic property.*

Proof. To prove Theorem 4.1, we need to validate conditions (A1) and (A2) of Lemma 2.1. The diffusion matrix of system (2.1) is given by

$$A(x,y) = \begin{pmatrix} (\frac{\sigma_1 x(1-x-y)}{a_1+1-x-y})^2 + (\frac{\sigma_2 xy}{a_2+x})^2 & -(\frac{\sigma_2 xy}{a_2+x})^2 \\ -(\frac{\sigma_2 xy}{a_2+x})^2 & (\frac{\sigma_2 xy}{a_2+x})^2 \end{pmatrix},$$

which is positive definite. This implies (A1) in Lemma 2.1 holds.

Next, we check the condition (A2) in Lemma 2.1. According to the theory developed by Zhu and Yin [29], we need to verify there exists a non-negative $\mathcal{C}^2$-function V and a neighborhood D_ρ such that LV is negative for any $(x,y) \in \Gamma_* \backslash D_\rho$.

Define a non-negative $\mathcal{C}^2$-function $V : \Gamma_* \to \overline{\mathbb{R}}_+$ by

$$\begin{aligned} V(x,y) =& M\Bigg[-\log y - c_1 x + c_2\left(x - \bar{x} - \bar{x}\log\frac{x}{\bar{x}}\right) \\ &+ \left(\frac{c_2\bar{x}(2(m_1-1)a_2^2 + 2m_2 a_1 a_2 + a_1\sigma_2^2)}{2a_1 a_2^2}\right) y\Bigg] \\ &- \log(1-x-y) + Mc_1 \\ :=& M\left[U + \left(\frac{c_2\bar{x}(2(m_1-1)a_2^2 + 2m_2 a_1 a_2 + a_1\sigma_2^2)}{2a_1{a_2}^2}\right) y\right] + V_2 \\ :=& MV_1 + V_2, \end{aligned}$$

where $M > 0$ is a constant satisfying $1 + \frac{m_1}{a_1} + \frac{\sigma_1^2}{2a_1^2} - M\lambda \le -2$, $c_1 = \frac{m_2 a_2(a_1+1-\bar{x})^2}{m_1 a_1\bar{x}(a_2+\bar{x})^2}$, $c_2 > \left(\frac{m_2}{a_2^2} + \frac{c_1 m_1(a_1+1)}{a_1^2}\right)\frac{a_1+1}{m_1-1}$, $U = -\log y - c_1 x + c_2\left(x - \bar{x} - \bar{x}\log\frac{x}{\bar{x}}\right)$, $V_1 = U + \left(\frac{c_2\bar{x}(2(m_1-1)a_2^2 + 2m_2 a_1 a_2 + a_1\sigma_2^2)}{2a_1 a_2^2}\right) y$, $V_2 = -\log(1-x-y) + Mc_1$.

By Itô formula, one derives

$$\begin{aligned} L(-\log y) &= -\frac{m_2 x}{a_2+x} + 1 + \frac{1}{2}\left(\frac{\sigma_2 x}{a_2+x}\right)^2 \\ &\le -\frac{m_2 x}{a_2+x} + 1 + \frac{\sigma_2^2}{2}\left(\frac{1}{a_2+1}\right)^2. \end{aligned} \tag{4.1}$$

$$\begin{aligned} L(-x) &= -\frac{m_1 x(1-x-y)}{a_1+1-x-y} + x + \frac{m_2 xy}{a_2+x} \\ &= -\frac{m_1 x(1-x-y)}{a_1+1-x-y} + \frac{m_1 x(1-x)}{a_1+1-x} - \frac{m_1 x(1-x)}{a_1+1-x} + x + \frac{m_2 xy}{a_2+x} \end{aligned}$$

$$
\begin{aligned}
&=\frac{a_1 m_1 xy}{(a_1+1-x-y)(a_1+1-x)}-\frac{m_1 x(1-x)}{a_1+1-x}+x+\frac{m_2 xy}{a_2+x}\\
&\leq -\frac{m_1 x(1-x)}{a_1+1-x}+x+\left(\frac{m_1}{a_1}+\frac{m_2}{a_2}\right)xy. \qquad (4.2)
\end{aligned}
$$

$$
\begin{aligned}
&L\left(x-\bar{x}-\bar{x}\log\frac{x}{\bar{x}}\right)\\
=&\frac{x-\bar{x}}{x}\left(\frac{m_1 x(1-x-y)}{a_1+1-x-y}-x-\frac{m_2 xy}{a_2+x}\right)+\frac{\bar{x}}{2}\left(\frac{\sigma_1 s}{a_1+s}\right)^2+\frac{\bar{x}}{2}\left(\frac{\sigma_2 y}{a_2+x}\right)^2\\
\leq&-\frac{(m_1-1)(x-\bar{x})^2}{a_1+1-x-y}+\frac{(1-m_1)(x-\bar{x})y}{a_1+1-x-y}-\frac{m_2(x-\bar{x})y}{a_2+x}\\
&+\frac{\bar{x}\sigma_1^2}{2}\left(\frac{1}{a_1+1}\right)^2+\frac{\bar{x}\sigma_2^2}{2a_2^2}y^2\\
\leq&-\frac{(m_1-1)(x-\bar{x})^2}{a_1+1}+\left[\frac{m_1-1}{a_1}+\frac{m_2}{a_2}+\frac{\sigma_2^2}{2a_2^2}\right]\bar{x}y+\frac{\bar{x}\sigma_1^2}{2}\left(\frac{1}{a_1+1}\right)^2. \qquad (4.3)
\end{aligned}
$$

According to (4.1)-(4.3), it follows that

$$
\begin{aligned}
LU\leq&-\left(\frac{m_2\bar{x}}{a_2+\bar{x}}-1-\frac{c_2\bar{x}\sigma_1^2}{2}\left(\frac{1}{a_1+1}\right)^2-\frac{\sigma_2^2}{2}\left(\frac{1}{a_2+1}\right)^2\right)+\frac{m_2\bar{x}}{a_2+\bar{x}}-\frac{m_2 x}{a_2+x}\\
&+c_1\left(-\frac{m_1 x(1-x)}{a_1+1-x}+x\right)-\frac{c_2(m_1-1)(x-\bar{x})^2}{a_1+1}\\
&+c_1\left(\frac{m_1}{a_1}+\frac{m_2}{a_2}\right)xy+c_2\left(\frac{m_1-1}{a_1}+\frac{m_2}{a_2}+\frac{\sigma_2^2}{2a_2^2}\right)\bar{x}y.
\end{aligned}
$$

Let

$$
F(x)=\frac{m_2\bar{x}}{a_2+\bar{x}}-\frac{m_2 x}{a_2+x}+c_1\left(-\frac{m_1 x(1-x)}{a_1+1-x}+x\right)-\frac{c_2(m_1-1)(x-\bar{x})^2}{a_1+1}.
$$

Direct calculations result in

$$
F'(x)=-\frac{m_2 a_2}{(a_2+x)^2}+c_1\left[-\frac{m_1(1-x)}{a_1+1-x}+1+\frac{m_1 a_1 x}{(a_1+1-x)^2}\right]-\frac{2c_2(m_1-1)(x-\bar{x})}{a_1+1},
$$

and

$$
F'(\bar{x})=0.
$$

Moreover

$$
\begin{aligned}
F''(x)=&\frac{2m_2 a_2}{(a_2+x)^3}+\frac{2c_1 m_1 a_1(a_1+1)}{(a_1+1-x)^3}-\frac{2c_2(m_1-1)}{a_1+1}\\
\leq&\frac{2m_2}{a_2^2}+\frac{2c_1 m_1(a_1+1)}{a_1^2}-\frac{2c_2(m_1-1)}{a_1+1}<0.
\end{aligned}
$$

Then

$$
F(x)\leq F(\bar{x})=0.
$$

Thus, we get

$$
LU\leq-\lambda+c_1\left(\frac{m_1}{a_1}+\frac{m_2}{a_2}\right)xy+c_2\left(\frac{m_1-1}{a_1}+\frac{m_2}{a_2}+\frac{\sigma_2^2}{2a_2^2}\right)\bar{x}y,
$$

and

$$
\begin{aligned}
LV_1 \leq& -\lambda + c_1\left(\frac{m_1}{a_1}+\frac{m_2}{a_2}\right)xy \\
&+\frac{c_2\bar{x}(2(m_1-1)a_2^2+2m_2a_1a_2+a_1\sigma_2^2)}{2a_1a_2^2}\frac{m_2xy}{a_2+x} \\
\leq& -\lambda + \left[c_1\left(\frac{m_1}{a_1}+\frac{m_2}{a_2}\right)\right. \\
&\left.+\frac{c_2m_2\bar{x}(2(m_1-1)a_2^2+2m_2a_1a_2+a_1\sigma_2^2)}{2a_1a_2^3}\right]xy. \qquad (4.4)
\end{aligned}
$$

Applying Itô formula to V_2 yields

$$
\begin{aligned}
LV_2 = L(-\log s) =& -\frac{1}{s}+1+\frac{m_1x}{a_1+s}+\frac{1}{2}\left(\frac{\sigma_1x}{a_1+s}\right)^2 \\
\leq& -\frac{1}{1-x-y}+1+\frac{m_1}{a_1}+\frac{\sigma_1^2}{2a_1^2}. \qquad (4.5)
\end{aligned}
$$

One can obtain from (4.4) and (4.5)

$$
\begin{aligned}
LV(x,y) \leq& M\left\{-\lambda+\left[c_1\left(\frac{m_1}{a_1}+\frac{m_2}{a_2}\right)\right.\right. \\
&\left.\left.+\frac{c_2m_2\bar{x}(2(m_1-1)a_2^2+2m_2a_1a_2+a_1\sigma_2^2)}{2a_1a_2^3}\right]xy\right\} \\
&-\frac{1}{1-x-y}+1+\frac{m_1}{a_1}+\frac{\sigma_1^2}{2a_1^2}.
\end{aligned}
$$

It can be seen from above formula, if $x \to 0^+$ or $y \to 0^+$, then

$$
LV(x,y) \leq -M\lambda + 1 + \frac{m_1}{a_1}+\frac{\sigma_1^2}{2a_1^2} \leq -2; \qquad (4.6)
$$

if $x+y \to 1^-$, then

$$
LV(x,y) < -\infty. \qquad (4.7)
$$

By (4.6) and (4.7), we can conclude that

$$
LV(x,y) \leq -1, \quad \text{for any} \quad (x,y) \in \Gamma_* \setminus D_\rho,
$$

where $D_\rho = \{(x,y) \in \Gamma_* : x \geq \rho, y \geq \rho, x+y \leq 1-\rho\}$ and $0 < \rho < \frac{1}{2}$ is a sufficiently small constant. This completes the proof. □

Remark 4.1. In the proof of Theorem 4.1, we use the equilibrium E_1 of model (2.2) to construct the Lyapunov function $V(x,y)$ which consists of two parts, V_1 and V_2. The item $\left(\frac{c_2\bar{x}(2(m_1-1)a_2^2+2m_2a_1a_2+a_1\sigma_2^2)}{2a_1a_2^2}\right)y$ in V_1 is constructed to eliminate the term $c_2\left(\frac{m_1-1}{a_1}+\frac{m_2}{a_2}+\frac{\sigma_2^2}{2a_2^2}\right)\bar{x}y$ in LU, leaving only the item containing xy. The term Mc_1 in V_2 guarantees V is non-negative.

5. Extinction

In this section, We shall establish sufficient conditions for extinction of microbes in two cases, one of which is that both the prey and predator species are extinct, the other of which is the prey species is surviving and the predator species is extinct.

Theorem 5.1. *Let $(x(t), y(t))$ be solution of model (2.1) with any given initial value $(x(0), y(0)) \in \Gamma_*$. If one of the following conditions holds*

(a) $\sigma_1^2 > \frac{m_1^2}{2}$, *or*

(b) $\sigma_1^2 \leq m_1(a_1+1)$ *and* $\frac{m_1}{a_1+1} - \frac{\sigma_1^2}{2(a_1+1)^2} - 1 < 0$, *then*

$$\limsup_{t\to\infty} \frac{\log x(t)}{t} \leq \frac{m_1^2}{2\sigma_1^2} - 1 < 0 \ a.s. \quad \text{If (a) holds;}$$

$$\limsup_{t\to\infty} \frac{\log x(t)}{t} \leq \frac{m_1}{a_1+1} - \frac{\sigma_1^2}{2(a_1+1)^2} - 1 < 0 \ a.s. \quad \text{If (b) holds.}$$

In other words, the prey will become extinct exponentially with probability one. Naturally, predator will also be extinct.

Proof. Applying Itô formula to $\log x$ leads to

$$d\log x = \left[\frac{m_1 s}{a_1+s} - 1 - \frac{m_1 y}{a_2+x} - \frac{\sigma_1^2}{2}\left(\frac{s}{a_1+s}\right)^2 - \frac{\sigma_2^2}{2}\left(\frac{y}{a_2+x}\right)^2\right] dt + \frac{\sigma_1 s}{a_1+s} dB_1(t) - \frac{\sigma_2 y}{a_2+x} dB_2(t).$$

Integrating this equation from 0 to t and dividing by t on both sides, we obtain

$$\begin{aligned}
&\frac{\log x(t) - \log x(0)}{t} \\
=& m_1\left\langle \frac{s}{a_1+s}\right\rangle - 1 - m_2\left\langle \frac{y}{a_2+x}\right\rangle - \frac{\sigma_1^2}{2}\left\langle \left(\frac{s}{a_1+s}\right)^2\right\rangle - \frac{\sigma_2^2}{2}\left\langle \left(\frac{y}{a_2+x}\right)^2\right\rangle \\
&+ \frac{\sigma_1}{t}\int_0^t \frac{s}{a_1+s} dB_1(\tau) - \frac{\sigma_2}{t}\int_0^t \frac{y}{a_2+x} dB_2(\tau) \\
\leq& m_1\left\langle \frac{s}{a_1+s}\right\rangle - 1 - \frac{\sigma_1^2}{2}\left\langle \left(\frac{s}{a_1+s}\right)^2\right\rangle \\
&+ \frac{\sigma_1}{t}\int_0^t \frac{s}{a_1+s} dB_1(\tau) - \frac{\sigma_2}{t}\int_0^t \frac{y}{a_2+x} dB_2(\tau) \\
\leq& m_1\left\langle \frac{s}{a_1+s}\right\rangle - 1 - \frac{\sigma_1^2}{2}\left\langle \frac{s}{a_1+s}\right\rangle^2 \\
&+ \frac{\sigma_1}{t}\int_0^t \frac{s}{a_1+s} dB_1(\tau) - \frac{\sigma_2}{t}\int_0^t \frac{y}{a_2+x} dB_2(\tau) \\
:=& f(z) + \frac{\sigma_1}{t}\int_0^t \frac{s}{a_1+s} dB_1(\tau) - \frac{\sigma_2}{t}\int_0^t \frac{y}{a_2+x} dB_2(\tau), \qquad (5.1)
\end{aligned}$$

where $f : \left(0, \frac{1}{a_1+1}\right) \to \mathbb{R}$ is defined by

$$f(z) = -\frac{\sigma_1^2}{2} z^2 + m_1 z - 1$$

$$= -\frac{\sigma_1^2}{2}\left(z - \frac{m_1}{\sigma_1^2}\right)^2 + \frac{m_1^2}{2\sigma_1^2} - 1, \quad z = \left\langle \frac{s}{a_1+s} \right\rangle \in \left(0, \frac{1}{a_1+1}\right). \tag{5.2}$$

Case 1. When $\sigma_1^2 > \frac{m_1^2}{2}$, by (5.2), we get

$$f(z) \le f\left(\frac{m_1}{\sigma_1^2}\right) = \frac{m_1^2}{2\sigma_1^2} - 1.$$

Then by (5.1), one can see that

$$\begin{aligned}\frac{\log x(t)}{t} \le & \frac{\log x(0)}{t} + \frac{m_1^2}{2\sigma_1^2} - 1 \\ & + \frac{\sigma_1}{t}\int_0^t \frac{s}{a_1+s} dB_1(\tau) - \frac{\sigma_2}{t}\int_0^t \frac{y}{a_2+x} dB_2(\tau).\end{aligned} \tag{5.3}$$

On the other hand, from the strong law of large number for local martingales [10], it follows that

$$\lim_{t\to\infty} \frac{\sigma_1}{t}\int_0^t \frac{s}{a_1+s} dB_1(\tau) = 0, \tag{5.4}$$

$$\lim_{t\to\infty} \frac{\sigma_2}{t}\int_0^t \frac{y}{a_2+x} dB_2(\tau) = 0. \tag{5.5}$$

So taking the superior limit on both sides of (5.3) and combining (5.4) and (5.5), we have

$$\limsup_{t\to\infty} \frac{\log x(t)}{t} \le \frac{m_1^2}{2\sigma_1^2} - 1 < 0 \; a.s.$$

Case 2. When $\sigma_1^2 \le m_1(a_1+1)$ and $\frac{m_1}{a_1+1} - \frac{\sigma_1^2}{2(a_1+1)^2} - 1 < 0$, by (5.2), we obtain

$$f(z) \le f\left(\frac{1}{a_1+1}\right) = \frac{m_1}{a_1+1} - \frac{\sigma_1^2}{2(a_1+1)^2} - 1.$$

In view of (5.1), one can derive that

$$\begin{aligned}\frac{\log x(t)}{t} \le & \frac{\log x(0)}{t} + \frac{m_1}{a_1+1} - \frac{\sigma_1^2}{2(a_1+1)^2} - 1 \\ & + \frac{\sigma_1}{t}\int_0^t \frac{s}{a_1+s} dB_1(\tau) - \frac{\sigma_2}{t}\int_0^t \frac{y}{a_2+x} dB_2(\tau).\end{aligned} \tag{5.6}$$

Then taking the superior limit on both sides of (5.6) and combining (5.4) and (5.5), we get

$$\limsup_{t\to\infty} \frac{\log x(t)}{t} \le \frac{m_1}{a_1+1} - \frac{\sigma_1^2}{2(a_1+1)^2} - 1 < 0 \; a.s.,$$

which implies $\lim_{t\to\infty} x(t) = 0$ a.s. This completes the proof. □

Remark 5.1. Theorem 5.1 indicates that if the noise intensity σ_1^2 is sufficiently large such that $\sigma_1^2 > \frac{m_1^2}{2}$ or it satisfies adequate conditions $\sigma_1^2 \le m_1(a_1+1)$ and $\frac{m_1}{a_1+1} - \frac{\sigma_1^2}{2(a_1+1)^2} - 1 < 0$, then the prey x will go extinct exponentially with probability one. In the circumstances, predator y will also be extinct.

Theorem 5.2. *Let $(x(t), y(t))$ be solution of model (2.1) with any given initial value $(x(0), y(0)) \in \Gamma_*$. If one of the following conditions holds*

(c) $\sigma_2^2 > \frac{m_2^2}{2}$, *or*

(d) $\sigma_2^2 \leq m_2(a_2+1)$ *and* $\frac{m_2}{a_2+1} - \frac{\sigma_2^2}{2(a_2+1)^2} - 1 < 0$, *then*

$$\limsup_{t\to\infty} \frac{\log y(t)}{t} \leq \frac{m_2^2}{2\sigma_2^2} - 1 < 0 \ a.s. \quad \textit{If (c) holds};$$

$$\limsup_{t\to\infty} \frac{\log y(t)}{t} \leq \frac{m_2}{a_2+1} - \frac{\sigma_2^2}{2(a_2+1)^2} - 1 < 0 \ a.s. \quad \textit{If (d) holds.}$$

That is to say, the predator will go extinct exponentially with probability one.

Proof. Making use of Itô formula to $\log y$ yields

$$d\log y = \left[\frac{m_2 x}{a_2+x} - 1 - \frac{\sigma_2^2}{2}\left(\frac{x}{a_2+x}\right)^2\right] dt + \frac{\sigma_2 x}{a_2+x} dB_2(t).$$

Integrating this equation from 0 to t and dividing by t on both sides, we have

$$\begin{aligned}
&\frac{\log y(t) - \log y(0)}{t} \\
=& m_2\left\langle \frac{x}{a_2+x}\right\rangle - 1 - \frac{\sigma_2^2}{2}\left\langle\left(\frac{x}{a_2+x}\right)^2\right\rangle + \frac{\sigma_2}{t}\int_0^t \frac{x}{a_2+x} dB_2(\tau) \\
\leq& m_2\left\langle \frac{x}{a_2+x}\right\rangle - 1 - \frac{\sigma_2^2}{2}\left\langle \frac{x}{a_2+x}\right\rangle^2 + \frac{\sigma_2}{t}\int_0^t \frac{x}{a_2+x} dB_2(\tau) \\
:=& g(w) + \frac{\sigma_2}{t}\int_0^t \frac{x}{a_2+x} dB_2(\tau),
\end{aligned}$$

where $g : \left(0, \frac{1}{a_2+1}\right) \to \mathbb{R}$ is defined by

$$\begin{aligned}
g(w) =& -\frac{\sigma_2^2}{2} w^2 + m_2 w - 1 \\
=& -\frac{\sigma_2^2}{2}\left(w - \frac{m_2}{\sigma_2^2}\right)^2 + \frac{m_2^2}{2\sigma_2^2} - 1, \quad w = \left\langle \frac{x}{a_2+x}\right\rangle \in \left(0, \frac{1}{a_2+1}\right).
\end{aligned}$$

The reminder of the proof is similar to Theorem 5.1, so we omit it. □

Remark 5.2. Theorem 5.2 reveals that if the noise intensity σ_2^2 is sufficiently large such that $\sigma_2^2 > \frac{m_2^2}{2}$ or it satisfies adequate conditions $\sigma_2^2 \leq m_2(a_2+1)$ and $\frac{m_2}{a_2+1} - \frac{\sigma_2^2}{2(a_2+1)^2} - 1 < 0$, then the predator y will become extinct.

6. Numerical simulations and discussion

In this section, we will illustrate our main theoretical results by numerical simulations with the help of Milstein's higher order method developed in [5]. Since $s(t) = 1 - x(t) - y(t)$ in model (2.1), numerical simulations for $s(t)$ are also given. We take initial value $(s(0), x(0), y(0)) = (0.2, 0.4, 0.4)$.

Example 6.1. Choose $m_1 = 4$, $m_2 = 3$, $a_1 = 1$, $a_2 = 1$, $\sigma_1 = 0.3$, $\sigma_2 = 0.2$. Simple computation results $c_1 = 0.7200$. Then there exists a constant $c_2 = 6.0400$ and

$$\lambda := \frac{m_2\bar{x}}{a_2+\bar{x}} - 1 - \frac{c_2\bar{x}\sigma_1^2}{2}\left(\frac{1}{a_1+1}\right)^2 - \frac{\sigma_2^2}{2}\left(\frac{1}{a_2+1}\right)^2 = 0.1497 > 0.$$

That is to say, the conditions of Theorem 4.1 are satisfied. It follows that system (2.1) has a unique ergodic stationary distribution. Simulation in Figure 1 can confirm this conclusion.

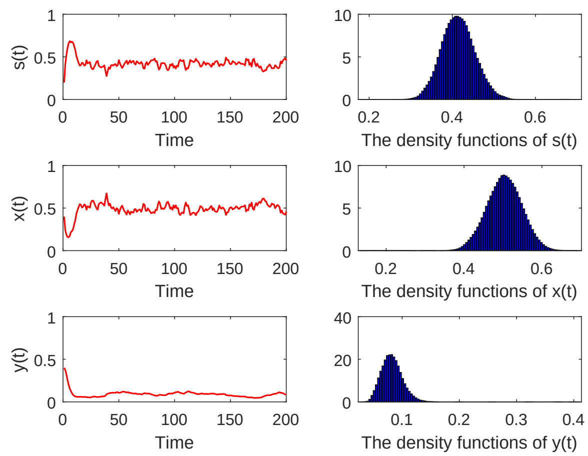

Figure 1. Numerical simulations of the solution trajectories (the left pictures) and the density functions (the right pictures) of $s(t)$, $x(t)$ and $y(t)$ for system (2.1) with initial value $(s(0), x(0), y(0)) = (0.2, 0.4, 0.4)$, $m_1 = 4$, $m_2 = 3$, $a_1 = 1$, $a_2 = 1$ and the noise intensities $\sigma_1 = 0.3$, $\sigma_2 = 0.2$.(Color figure online)

Example 6.2. In order to get the extinction of prey and predator in stochastic system (2.1), in view of Theorem 5.1, we choose constant parameter values $m_1 = 0.4$, $m_2 = 3$, $a_1 = 0.5$, $a_2 = 1$. Consider the following two situations.

Case (1). Let the white noise $\sigma_1 = 0.8$, $\sigma_2 = 0.4$. Then

$$\sigma_1^2 = 0.64 > 0.08 = \frac{m_1^2}{2}.$$

Hence, the condition (a) in Theorem 5.1 is satisfied, and the prey and predator species go extinct. Result of this simulation is presented in Figure 2.

Case (2). Set the white noise $\sigma_1 = 0.3$, $\sigma_2 = 0.4$. Then

$$\sigma_1^2 = 0.09 < 0.6 = m_1(a_1+1),$$

and

$$\frac{m_1}{a_1+1} - \frac{\sigma_1^2}{2(a_1+1)^2} - 1 = -0.7533 < 0.$$

In view of condition (b) in Theorem 5.1, we can obtain that the prey will become extinct exponentially with probability one and the predator will also go extinct. This result is supported by Figure 3.

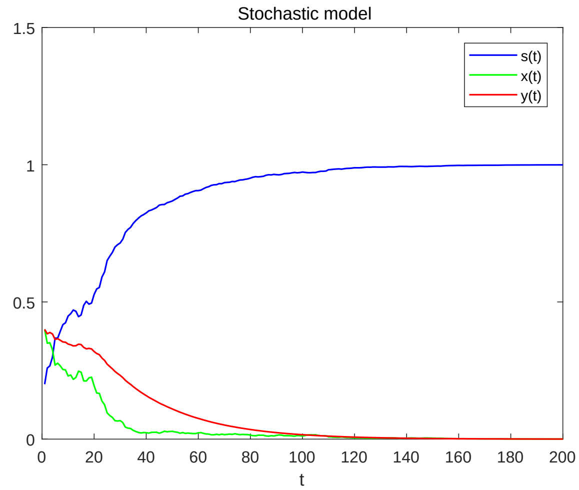

Figure 2. Solutions of system (2.1) with initial value $(s(0), x(0), y(0)) = (0.2, 0.4, 0.4)$, $m_1 = 0.4$, $m_2 = 3$, $a_1 = 0.5$, $a_2 = 1$ and the noise intensities $\sigma_1 = 0.8$, $\sigma_2 = 0.4$. (Color figure online)

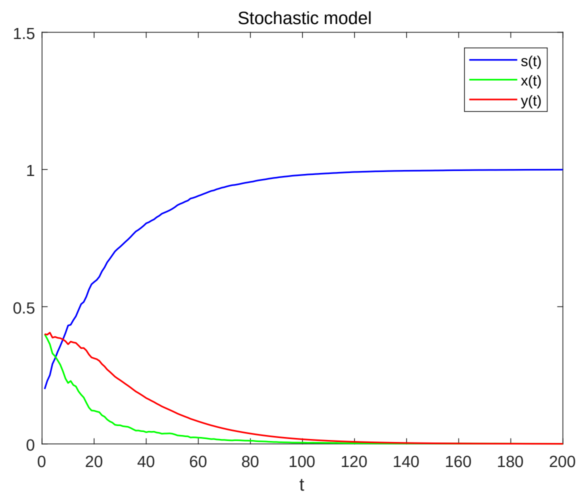

Figure 3. Solutions of system (2.1) with initial value $(s(0), x(0), y(0)) = (0.2, 0.4, 0.4)$, $m_1 = 0.4$, $m_2 = 3$, $a_1 = 0.5$, $a_2 = 1$ and the noise intensities $\sigma_1 = 0.3$, $\sigma_2 = 0.4$. (Color figure online)

Example 6.3. In order to obtain the extinction of predator in system (2.1), we select constant parameter values $m_1 = 3$, $m_2 = 0.4$, $a_1 = 1$, $a_2 = 0.5$. Consider the following two cases.

Case (3). Let the white noise $\sigma_1 = 0.5$, $\sigma_2 = 0.8$. Then

$$\sigma_2^2 = 0.64 > 0.08 = \frac{m_2^2}{2}.$$

It follows from condition (c) in Theorem 5.2 that the predator will be extinct. Figure 4 confirms this.

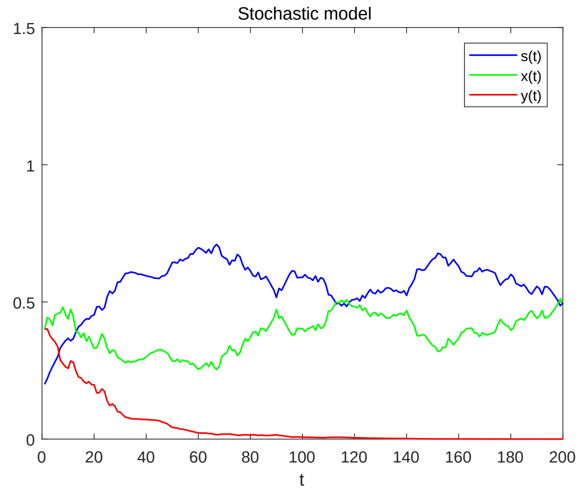

Figure 4. Solutions of system (2.1) with initial value $(s(0), x(0), y(0)) = (0.2, 0.4, 0.4)$, $m_1 = 3$, $m_2 = 0.4$, $a_1 = 1$, $a_2 = 0.5$ and the noise intensities $\sigma_1 = 0.5$, $\sigma_2 = 0.8$. (Color figure online)

Case (4). Choose the white noise $\sigma_1 = 0.5$, $\sigma_2 = 0.4$. Then

$$\sigma_2^2 = 0.16 < 0.6 = m_2(a_2 + 1),$$

and

$$\frac{m_2}{a_2 + 1} - \frac{\sigma_2^2}{2(a_2 + 1)^2} - 1 = -0.7689 < 0.$$

According to condition (d) in Theorem 5.2, we can see that the predator species goes extinct exponentially with probability one. Simulation in Figure 5 can confirm this conclusion.

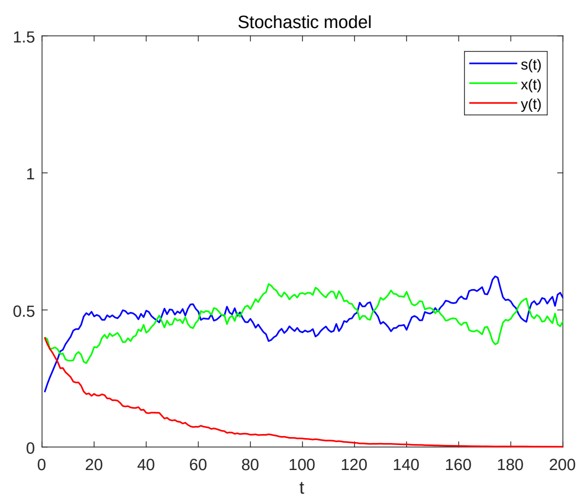

Figure 5. Solutions of system (2.1) with initial value $(s(0), x(0), y(0)) = (0.2, 0.4, 0.4)$, $m_1 = 3$, $m_2 = 0.4$, $a_1 = 1$, $a_2 = 0.5$ and the noise intensities $\sigma_1 = 0.5$, $\sigma_2 = 0.4$. (Color figure online)

Notice that some scholars [19, 22, 28] have studied certain stochastic chemostat models in which the environmental noise is proportional to the variables. Next, we will investigate the dynamics of the food chain chemostat model in which the noise is proportional to the variables. In addition, one can propose some more realistic models by considering the effects of telegraph noise on system (2.1) and it is worthy of further study.

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