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APPROXIMATION OF FRACTIONAL RESOLVENTS AND APPLICATIONS TO TIME OPTIMAL CONTROL PROBLEMS*

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Abstract We investigate the approximation of fractional resolvents, extending and improving some corresponding results on semigroups and resolvents. As applications, we utilize the approach of Meyer approximation to analyze the time optimal control problem of a Riemann-Liouville fractional system without Lipschitz continuity. A fractional diffusion model is also presented to confirm our theoretical findings.

Keywords Resolvent, Riemann-Liouville fractional derivatives, time optimal controls, Meyer approximation.

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1. Introduction

Since the notion of resolvent was firstly proposed and studied by Da Prato and Iannelli in [13,14], there has been considerable interest in introducing and analyzing the notions related to resolvent, such as solution operators, (a, k)-regularized families, α-order fractional resolvents and so on (refer to [6,8,15]). Recently, Li and Peng [6] investigated a Riemann-Liouville fractional evolution problem by introducing the notion of α-order fractional resolvent.

As we all know, the approximation of semigroups is of great importance in the study of optimal control problems (see [17, 18]). Many researchers thereby show tremendous interest in analyzing the approximation of semigroups and resolvents. For example, the approximation of solution operators was analyzed in [1]. The approximation of (a, k)-regularized families was studied in [9]. However, limited work has been done in the approximation of α-order fractional resolvents. Considering that the resolvent technique is a convenient and efficient approach in studying fractional evolution systems [6,22–24], we will investigate the approximation of fractional resolvents. The main difficulty in the study is that these resolvents possess singularity at zero. In this article, by introducing a new concept of exponential boundedness for $s \geq s_0$ and constructing resolvents with parameters, we explore the approximation problem.

On the other hand, time optimal control problems for evolution systems have recent years drawn tremendous attention based upon their broad applications in

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many fields, such as control theory, industrial application and space technology, etc. Many researchers analyzed them by setting up time optimal sequence pairs (see [4,5,7,10,19]). Recently, with the aid of the Lipschitz assumption on the nonlinear term f and the approximation of semigroups, the problems were tackled by Meyer approximation (refer to [16–18]). That is, the authors formulated a sequence of Meyer problems to approximate the time optimal control problems. Naturally, one may ask how to analyze the problems for Riemann-Liouville evolution systems by Meyer approximation if we remove the Lipschitz continuity. Therefore, the present paper is intended to conduct some investigations on time optimal control problems for a Riemann-Liouville fractional evolution system by Meyer approximation, when the Lipschitz condition is not satisfied. In this paper, we first transform the original fractional evolution system into an approximate system and propose a Meyer problem. Then, we deal with the Meyer problem by constructing minimizing sequences twice. Finally, we analyze the time optimal problem by Meyer approximation.

The following enumerates two aspects of novelties of this paper:

1) Considering that the fractional resolvent has singularity at zero, we introduce a new concept of exponential boundedness for $s \geq s_0$. Furthermore, we analyze the approximation of fractional resolvents by setting up resolvents with parameters.

2) We combine the approximation of fractional resolvents and the time control problem organically. In addition, the new method of constructing minimizing sequences twice is employed to compensate the lack of Lipschitz assumption, when addressing the Meyer problem.

The outline of this article is as follows. We present some preliminaries in Section 2. Section 3 deals with the approximation of fractional resolvents. As applications, in Section 4, we treat the time optimal control problem of a Riemann-Liouville fractional system by Meyer approximation and present an example on a Riemann-Liouville fractional partial differential system to confirm our theoretical findings.

2. Preliminaries

We compile here some preliminaries, including the notions and facts about fractional resolvents. Let V be a Banach space and Y a reflexive separable Banach space. From now on, unless otherwise stated, we assume that $0 < \beta < 1$. Set $J = [0, T]$, $J' = (0, T]$ and

$$C_{1-\beta}(J;V) = \{z \in C(J';V) | \widetilde{z}(\tau) = \tau^{1-\beta} z(\tau),\ \widetilde{z}(0) = \lim_{\tau \downarrow 0} \widetilde{z}(\tau),\ \widetilde{z} \in C(J;V)\}.$$

If the space $C_{1-\beta}(J;V)$ is normed by $\|z\|_{C_{1-\beta}} = \sup_{\tau \in J} \|\widetilde{z}(\tau)\|$, it is a Banach space. Let $\mathscr{L}(Y;V)$ represent the collection of all bounded linear operators from Y to V and $\mathscr{L}(V)$ denote $\mathscr{L}(V;V)$. Furthermore, we utilize the symbol $P_{fc}(Y)$ to stand for a class of nonempty closed convex subset of Y and the notation $*$ to mean the convolution, i.e., $(f * g)(s) = \int_0^s f(s-\tau)g(\tau)\mathrm{d}\tau$.

Definition 2.1 ([6]). By a β-order fractional resolvent, we understand a strongly continuous family $\{R_\beta(s)\}_{s>0} \subseteq \mathscr{L}(V)$ satisfying

(a) $\lim_{s \downarrow 0} \Gamma(\beta) s^{1-\beta} R_\beta(s) z = z$ for any $z \in V$;

(b) $R_\beta(\tau) R_\beta(s) = R_\beta(s) R_\beta(\tau)$ for $s,\ \tau > 0$;

(c) $R_\beta(\tau)J_s^\beta R_\beta(s) - J_\tau^\beta R_\beta(\tau)R_\beta(s) = g_\beta(\tau)J_s^\beta R_\beta(s) - g_\beta(s)J_\tau^\beta R_\beta(\tau)$ for $s,\ \tau > 0$,

where $g_\beta(\tau) = \frac{\tau^{\beta-1}}{\Gamma(\beta)}, \tau > 0$ and the notation J_τ^β stands for the β-order fractional integral operator, i.e., $J_\tau^\beta f(\tau) = (g_\beta * f)(\tau)$ (see [12]).

In addition, the generator $A : D(A) \subseteq V \to V$ of the resolvent $\{R_\beta(s)\}_{s>0}$ is

$$D(A) = \left\{ z \in V : \lim_{s\downarrow 0} \frac{s^{1-\beta}R_\beta(s)z - \frac{z}{\Gamma(\beta)}}{s^\beta} \text{ exists} \right\},$$

$$Az = \Gamma(2\beta) \lim_{s\downarrow 0} \frac{s^{1-\beta}R_\beta(s)z - \frac{z}{\Gamma(\beta)}}{s^\beta},\ z \in V.$$

Remark 2.1. Based on (a) of Definition 2.1, we see that the resolvent $\{R_\beta(s)\}_{s>0}$ has singularity at zero.

Remark 2.2. Let $\{R_\beta(s)\}_{s>0}$ be a resolvent. Then, by using Definition 2.1 and the uniform boundedness principle, we can easily show that $\sup_{s\in J} \|s^{1-\beta}R_\beta(s)\| < \infty$, where $s^{1-\beta}R_\beta(s)|_{s=0} = \lim_{s\downarrow 0} s^{1-\beta}R_\beta(s)$.

Lemma 2.1 ([6]). *Assume that A generates a resolvent $\{R_\beta(s)\}_{s>0}$. Then*

(a) *$R_\beta(s)D(A) \subseteq D(A)$ and $AR_\beta(s)z = R_\beta(s)Az$ for any $z \in D(A)$;*

(b) *$R_\beta(s)z = g_\beta(s)z + J_s^\beta R_\beta(s)Az$ for any $z \in D(A)$;*

(c) *$R_\beta(s)z = g_\beta(s)z + A(g_\beta * R_\beta)(s)z$ for any $z \in V$;*

(d) *$\overline{D(A)} = V$.*

Based on Lemmas 3.4 and 3.5 of [2], we can propose the following important properties of $\{R_\beta(s)\}_{s>0}$.

Lemma 2.2. *Let $\{t^{1-\beta}R_\beta(t)\}_{t>0}$ be compact and equicontinuous. Then, for $t \in J'$,*

(a) $\lim_{\tau\downarrow 0} \left\| (t+\tau)^{1-\beta}R_\beta(t+\tau) - \left(\Gamma(\beta)\tau^{1-\beta}R_\beta(\tau)\right)\left(t^{1-\beta}R_\beta(t)\right) \right\| = 0$;

(b) $\lim_{\tau\downarrow 0} \left\| t^{1-\beta}R_\beta(t) - \left(\Gamma(\beta)\tau^{1-\beta}R_\beta(\tau)\right)\left((t-\tau)^{1-\beta}R_\beta(t-\tau)\right) \right\| = 0$.

Lemma 2.3. *Let $p > \frac{1}{\beta}$ and $\{t^{1-\beta}R_\beta(t)\}_{t>0}$ be compact and equicontinuous. Assume that $g \in L^p(J;V)$ and $\Lambda : L^p(J;V) \to C_{1-\beta}(J;V)$ is a map defined by $(\Lambda g)(\cdot) = (R_\beta * g)(\cdot)$. Then $R_\beta * g \in C(J;V)$ and Λ is compact.*

Proof. With the help of Lemma 2.2, we can easily verify the statement of this lemma by following the verification of Lemmas 3.1 and 4.2 in [23]. □

3. Approximation of fractional resolvents

This section is intended to display some approximation theorems of the fractional resolvent $\{R_\beta(s)\}_{s>0}$.

As we all know, any C_0-semigroup is (M,ω) type. However, the resolvent $\{R_\beta(s)\}_{s>0}$ is not (M,ω) type since it has singularity at zero. Now, we introduce the following new definition:

Definition 3.1. Let $s_0 > 0$. The resolvent $\{R_\beta(s)\}_{s>0}$ is called exponentially bounded for $s \geq s_0$, if there exist two nonnegative constants ω and $\overline{M}$ such that

$$\|R_\beta(s)\| \leq \overline{M} e^{\omega s}, \ s \geq s_0. \tag{3.1}$$

For convenience, we employ the symbol $A \in C^\beta_{s_0}(\overline{M}, \omega)$ to mean that A generates a resolvent $\{R_\beta(s)\}_{s>0}$ satisfying (3.1).

Remark 3.1. Let $A \in C^\beta_{s_0}(\overline{M}, \omega)$. Then by using (3.1) and Remark 2.2, it is easy to show that $R_\beta(s)$ is Laplace transformable for $\lambda > \omega$. In fact, for $t > s_0$, we have

$$\begin{aligned}\int_0^t e^{-\lambda s}\|R_\beta(s)\|\mathrm{d}s &\leq \int_0^{s_0} e^{-\lambda s}\|R_\beta(s)\|\mathrm{d}s + \int_{s_0}^t e^{-\lambda s}\|R_\beta(s)\|\mathrm{d}s \\ &\leq M\int_0^{s_0} s^{\beta-1}\mathrm{d}s + \int_{s_0}^t e^{-(\lambda-\omega)s}\overline{M}\mathrm{d}s \\ &\leq \frac{M s_0^\beta}{\beta} + \frac{\overline{M}}{\lambda-\omega},\end{aligned}$$

where $M = \sup\limits_{s\in[0,s_0]} \|s^{1-\beta}R_\beta(s)\|$. Thus, due to the strong continuity of $\{R_\beta(s)\}_{s>0}$, $R_\beta(s)$ is Laplace transformable for $\lambda > \omega$.

Lemma 3.1. *Let $s_0 > 0$. Then $A \in C^\beta_{s_0}(\overline{M}, \omega)$ if and only if $(\omega^\beta, \infty) \subseteq \rho(A)$ and there is a family $\{R_\beta(s)\}_{s>0} \subseteq \mathscr{L}(V)$ satisfying*

- (a) *for any $z \in V$, $R_\beta(\cdot)z \in C(\mathbb{R}_+; V)$ and $\lim\limits_{s\downarrow 0} \Gamma(\beta)s^{1-\beta}R_\beta(s)z = z$;*
- (b) *for $s \geq s_0$, $\|R_\beta(s)\| \leq \overline{M}e^{\omega s}$;*
- (c) *$R_\beta(s)R_\beta(\tau) = R_\beta(\tau)R_\beta(s)$ for $s, \tau > 0$;*
- (d) *for $\lambda > \omega$ and $z \in V$,*

$$(\lambda^\beta I - A)^{-1}z = \int_0^\infty e^{-\lambda s}R_\beta(s)z\mathrm{d}s.$$

In such a case, $\{R_\beta(s)\}_{s>0}$ is a resolvent generated by A.

Proof. (Necessity) Let $A \in C^\beta_{s_0}(\overline{M}, \omega)$. Then (a), (b) and (c) hold. Based upon Remark 3.1, $R_\beta(s)$ is Laplace transformable for $\lambda > \omega$. Set

$$R(\lambda) = \int_0^\infty e^{-\lambda s}R_\beta(s)\mathrm{d}s, \ \lambda > \omega.$$

Due to (a) and (b) of Lemma 2.1, we can employ Laplace transform to get that for any $z \in D(A)$,

$$R(\lambda)z = \lambda^{-\beta}z + \lambda^{-\beta}R(\lambda)Az = \lambda^{-\beta}z + \lambda^{-\beta}AR(\lambda)z.$$

Thus, according to (d) of Lemma 2.1, we can deduce that for $\lambda > \omega$ and $z \in V$,

$$(\lambda^\beta I - A)^{-1}z = R(\lambda)z = \int_0^\infty e^{-\lambda s}R_\beta(s)z\mathrm{d}s.$$

(Sufficiency) We suppose that (a), (b), (c) and (d) hold. In view of the resolvent identity, we get

$$(\mu^{-\beta}-\lambda^{-\beta})R(\lambda)R(\mu)=\lambda^{-\beta}\mu^{-\beta}R(\mu)-\mu^{-\beta}\lambda^{-\beta}R(\lambda).$$

Hence, by utilizing inverse Laplace transform, we derive

$$R_\beta(\tau)J_s^\beta R_\beta(s)-J_\tau^\beta R_\beta(\tau)R_\beta(s)=g_\beta(\tau)J_s^\beta R_\beta(s)-g_\beta(s)J_\tau^\beta R_\beta(\tau),$$

which indicates that $\{R_\beta(s)\}_{s>0}$ is a resolvent, hence that $A\in C_{s_0}^\beta(\overline{M},\omega)$. □

Lemma 3.2. *Let both $\{R_\beta(s)\}_{s>0}$ and $\{T_\beta(s)\}_{s>0}$ be fractional resolvents generated by A, Then $R_\beta(s)=T_\beta(s)$ for $s>0$.*

Proof. Due to Lemma 2.1, we have

$$\begin{aligned}g_\beta * R_\beta &= (T_\beta - A(g_\beta * T_\beta)) * R_\beta\\ &= T_\beta * (R_\beta - A(g_\beta * R_\beta)) = T_\beta * g_\beta = g_\beta * T_\beta,\end{aligned}$$

which indicates that $R_\beta(s)z=T_\beta(s)z$ for every $z\in D(A)$. As such, based on $\overline{D(A)}=V$, we get $R_\beta(s)=T_\beta(s)$ for $s>0$. □

Lemma 3.3. *Let $A\in C_{s_0}^\beta(\overline{M},\omega)$ and $k\in[0,+\infty)$. Then kA generates a resolvent $\{R_\beta^k(s)\}_{s>0}$, where*

$$R_\beta^k(s)=\begin{cases}k^{\left(-1+\frac{1}{\beta}\right)}R_\beta\left(sk^{\frac{1}{\beta}}\right), & k>0,\\ \frac{s^{\beta-1}I}{\Gamma(\beta)}, & k=0.\end{cases}$$

Proof. For clarity, we verify this lemma by considering the following two cases. **Case 1** $k>0$. Because of $A\in C_{s_0}^\beta(\overline{M},\omega)$, we can easily derive the strong continuity of $\{R_\beta^k(s)\}_{s>0}$ and the commutativity of $R_\beta^k(s)$ and $R_\beta^k(\tau)$. Moreover, we can see that $\{R_\beta^k(s)\}_{s>0}\subseteq\mathscr{L}(V)$.

Additionally, due to $A\in C_{s_0}^\beta(\overline{M},\omega)$ and (a) of Lemma 3.1, we derive

$$\lim_{s\downarrow 0}\Gamma(\beta)s^{1-\beta}R_\beta^k(s)z=\lim_{s\downarrow 0}\Gamma(\beta)\left(sk^{\frac{1}{\beta}}\right)^{1-\beta}R_\beta\left(sk^{\frac{1}{\beta}}\right)z=z$$

for any $z\in V$.

Furthermore, for $s\geq s_0>0$, we get from (b) of Lemma 3.1 that

$$\|R_\beta^k(s)\|=\left\|k^{(-1+\frac{1}{\beta})}R_\beta(sk^{\frac{1}{\beta}})\right\|\leq k^{(-1+\frac{1}{\beta})}\overline{M}e^{\omega k^{\frac{1}{\beta}}s}.$$

As such, we can choose $\overline{M_1}>0$ and $\omega_1>0$ such that $\|R_\beta^k(s)\|\leq\overline{M_1}e^{\omega_1 s}$, $s\geq s_0$.

In addition, according to (d) of Lemma 3.1, we can conclude that for any $z\in V$,

$$\begin{aligned}\int_0^\infty e^{-\lambda s}R_\beta^k(s)z\mathrm{d}s &= \int_0^\infty e^{-\lambda s}k^{(-1+\frac{1}{\beta})}R_\beta\left(sk^{\frac{1}{\beta}}\right)z\mathrm{d}s\\ &= k^{-1}\int_0^\infty e^{-\lambda k^{-\frac{1}{\beta}}t}R_\beta(t)z\mathrm{d}t=(\lambda^\beta I-kA)^{-1}z,\ \lambda>\omega_1.\end{aligned}$$

Thus, based on Lemmas 3.1 and 3.2, $\{R_\beta^k(s)\}_{s>0}$ is a resolvent generated by kA. **Case 2** $k=0$. Firstly, we see at once that $\{R_\beta^k(s)\}_{s>0}$ is strongly continuous,

$\{R^k_\beta(s)\}_{s>0} \subseteq \mathscr{L}(V)$ and $R^k_\beta(s)R^k_\beta(\tau) = R^k_\beta(\tau)R^k_\beta(s)$. Furthermore, we can pick $\overline{M_2} > 0$ and $\omega_2 > 0$ to ensure that $\|R^k_\beta(s)\| \le \overline{M_2}e^{\omega_2 s}$, $s \ge s_0$. Moreover, it is clear that for $z \in V$, $\lim_{s\downarrow 0}\Gamma(\beta)s^{1-\beta}R^k_\beta(s)z = z$.

Additionally, we have

$$\begin{aligned}\int_0^\infty e^{-\lambda s}R^k_\beta(s)z\mathrm{d}s &= \int_0^\infty e^{-\lambda s}\frac{s^{\beta-1}}{\Gamma(\beta)}z\mathrm{d}s\\ &= \frac{\lambda^{-\beta}}{\Gamma(\beta)}\int_0^\infty e^{-t}t^{\beta-1}z\mathrm{d}t = (\lambda^\beta I - \mathbf{0})^{-1}z,\ \lambda > \omega_2,\end{aligned}$$

where the notation $\mathbf{0}$ stands for zero operator.

Consequently, $\{R^k_\beta(s)\}_{s>0}$ is a resolvent generated by $\mathbf{0}$. □

Theorem 3.1. *Let $A \in C^\beta_{s_0}(\overline{M},\omega)$ and $k_n \ge 0$. If $k_n \to k_\varepsilon$ as $n \to \infty$, then for any $z \in V$ and $t \ge 0$,*

$$t^{1-\beta}R^{k_n}_\beta(t)z \to t^{1-\beta}R^{k_\varepsilon}_\beta(t)z, \tag{3.2}$$

where $\left(t^{1-\beta}R^{k_n}_\beta(t)z\right)|_{t=0} = \lim_{t\downarrow 0}\left(t^{1-\beta}R^{k_n}_\beta(t)z\right)$. Moreover, we have

$$t^{1-\beta}R^{k_n}_\beta(t) \xrightarrow{s} t^{1-\beta}R^{k_\varepsilon}_\beta(t),$$

uniformly in $t \in [a,b] \subseteq [0,+\infty)$, as $n \to \infty$. Here the notation $\xrightarrow{s}$ means the strong operator topology.

Proof. We consider the following cases.
Case 1 $k_n = 0$. In view of Lemma 3.3, the proof is immediate.
Case 2 $k_n > 0$ and $k_\varepsilon = 0$. Let $z \in V$. If $t = 0$, according to Lemmas 3.1 and 3.3, we derive

$$t^{1-\beta}R^{k_n}_\beta(t)z|_{t=0} = \lim_{t\downarrow 0}\left(tk_n^{\frac{1}{\beta}}\right)^{1-\beta}R_\beta\left(tk_n^{\frac{1}{\beta}}\right)z = \frac{z}{\Gamma(\beta)}.$$

If $t > 0$, we get

$$\lim_{n\to\infty}t^{1-\beta}R^{k_n}_\beta(t)z = \lim_{n\to\infty}\left(tk_n^{\frac{1}{\beta}}\right)^{1-\beta}R_\beta\left(tk_n^{\frac{1}{\beta}}\right)z = \frac{z}{\Gamma(\beta)}.$$

On the other hand, due to $k_\varepsilon = 0$ and Lemma 3.3, we have $t^{1-\beta}R^{k_\varepsilon}_\beta(t)z = \frac{z}{\Gamma(\beta)}$. Hence, (3.2) holds.
Case 3 $k_n > 0$ and $k_\varepsilon > 0$. If $t = 0$, by means of Lemma 3.3, we derive $t^{1-\beta}R^{k_n}_\beta(t)z|_{t=0} = \frac{z}{\Gamma(\beta)}$ and $t^{1-\beta}R^{k_\varepsilon}_\beta(t)z|_{t=0} = \frac{z}{\Gamma(\beta)}$. Thus, the proof is straightforward. If $t > 0$, we have

$$\begin{aligned}&\left\|t^{1-\beta}R^{k_n}_\beta(t)z - t^{1-\beta}R^{k_\varepsilon}_\beta(t)z\right\|\\ &= \left\|\left(tk_n^{\frac{1}{\beta}}\right)^{1-\beta}R_\beta\left(tk_n^{\frac{1}{\beta}}\right)z - \left(tk_\varepsilon^{\frac{1}{\beta}}\right)^{1-\beta}R_\beta\left(tk_\varepsilon^{\frac{1}{\beta}}\right)z\right\|\\ &\le \left\|\left(tk_n^{\frac{1}{\beta}}\right)^{1-\beta}\right\|\left\|R_\beta\left(tk_n^{\frac{1}{\beta}}\right)z - R_\beta\left(tk_\varepsilon^{\frac{1}{\beta}}\right)z\right\|\end{aligned}$$

$$+\left\|k_n^{(-1+\frac{1}{\beta})}-k_\varepsilon^{(-1+\frac{1}{\beta})}\right\|t^{1-\beta}\left\|R_\beta\left(tk_\varepsilon^{\frac{1}{\beta}}\right)z\right\|.$$

Firstly, the fact that $k_n \to k_\varepsilon,\ n \to \infty$ implies the boundedness of $\{k_n\}$. In addition, if $s_0 \in (0, tk_\varepsilon^{\frac{1}{\beta}})$, by employing the strong continuity of $\{R_\beta(t)\}_{t>0}$, the exponential boundedness for $t \geq s_0$ and $k_n \to k_\varepsilon$, one can easily see that (3.2) holds. If $s_0 \geq tk_\varepsilon^{\frac{1}{\beta}}$, by utilizing the boundedness of $\left(tk_\varepsilon^{\frac{1}{\beta}}\right)^{1-\beta} R_\beta\left(tk_\varepsilon^{\frac{1}{\beta}}\right)$ on $\left[tk_\varepsilon^{\frac{1}{\beta}}, s_0\right]$ and the strong continuity of $\{R_\beta(t)\}_{t>0}$, we can easily conclude that (3.2) holds.

As for $t \in [a,b] \subseteq [0,+\infty)$, we derive

$$a\left|k_n^{\frac{1}{\beta}}-k_\varepsilon^{\frac{1}{\beta}}\right| \leq \left|tk_n^{\frac{1}{\beta}}-tk_\varepsilon^{\frac{1}{\beta}}\right| \leq b\left|k_n^{\frac{1}{\beta}}-k_\varepsilon^{\frac{1}{\beta}}\right|,$$

which indicates that $\left|tk_n^{\frac{1}{\beta}}-tk_\varepsilon^{\frac{1}{\beta}}\right| \to 0$, uniformly in t. Hence, it is easily seen that $t^{1-\beta}R_\beta^{k_n}(t) \xrightarrow{s} t^{1-\beta}R_\beta^{k_\varepsilon}(t)$, uniformly in t. □

Theorem 3.2. *Let* $\lim\limits_{n\to\infty} k_n = k_\varepsilon,\ k_n, k_\varepsilon \in (0,+\infty)$ *and* $\{t^{1-\beta}R_\beta(t)\}_{t>0}$ *be continuous in the uniform operator topology sense. Then*

$$t^{1-\beta}R_\beta^{k_n}(t) \xrightarrow{\tau_u} t^{1-\beta}R_\beta^{k_\varepsilon}(t),$$

uniformly in $t \in [a,b] \subseteq (0,+\infty)$*, where the notation* $\xrightarrow{\tau_u}$ *stands for the uniform operator topology.*

Proof. It follows from $\lim\limits_{n\to\infty} k_n = k_\varepsilon,\ k_n, k_\varepsilon \in (0,+\infty)$ and

$$a\left|k_n^{\frac{1}{\beta}}-k_\varepsilon^{\frac{1}{\beta}}\right| \leq \left|tk_n^{\frac{1}{\beta}}-tk_\varepsilon^{\frac{1}{\beta}}\right| \leq b\left|k_n^{\frac{1}{\beta}}-k_\varepsilon^{\frac{1}{\beta}}\right|$$

that $\left|tk_n^{\frac{1}{\beta}}-tk_\varepsilon^{\frac{1}{\beta}}\right| \to 0,\ n \to \infty$, uniformly in t. Thus, due to the assumption on $\{t^{1-\beta}R_\beta(t)\}_{t>0}$, we derive

$$\begin{aligned}
&\left\|t^{1-\beta}R_\beta^{k_n}(t)-t^{1-\beta}R_\beta^{k_\varepsilon}(t)\right\| \\
=&\left\|\left(tk_n^{\frac{1}{\beta}}\right)^{1-\beta}R_\beta\left(tk_n^{\frac{1}{\beta}}\right)-\left(tk_\varepsilon^{\frac{1}{\beta}}\right)^{1-\beta}R_\beta\left(tk_\varepsilon^{\frac{1}{\beta}}\right)\right\| \\
\to&\ 0,\ n \to \infty,
\end{aligned}$$

uniformly in t, which establishes the conclusion. □

Theorem 3.3. *Let* $A \in C_{s_0}^\beta(\overline{M},\omega)$ *and* $\{t^{1-\beta}R_\beta(t)\}_{t>0}$ *be compact and equicontinuous. Then* $\{t^{1-\beta}R_\beta^k(t)\}_{t>0}$ $(k>0)$ *is compact and equicontinuous.*

Proof. Since $k > 0$ and $t^{1-\beta}R_\beta^k(t) = \left(tk^{\frac{1}{\beta}}\right)^{1-\beta}R_\beta\left(tk^{\frac{1}{\beta}}\right)$, we can see that $\{t^{1-\beta}R_\beta^k(t)\}_{t>0}$ is compact and equicontinuous, by the compactness and equicontinuity of $\{t^{1-\beta}R_\beta(t)\}_{t>0}$. □

Remark 3.2. With the aid of the resolvent properties, we have explored the approximation of resolvents by introducing the notion of exponential boundedness for $s \geq s_0$ and constructing resolvents with parameters. Emphasis here is that our method differs from the approach in [11, 17]. Moreover, our technique can also be applied to the case of C_0-semigroups. However, the question whether the Trotter-Kato type approximation theorem holds for fractional resolvents is at present far from being solved, since these resolvents have singularity at zero.

4. Time optimal control problems

In this section, with the help of the approximation theory in Section 3, we deal with the time optimal control problem of a Riemann-Liouville fractional evolution system. We first propose the time optimal control problem (P) of a Riemann-Liouville fractional control system and the Meyer problem (P_ε) of a transformation system. Then, we tackle the Meyer problem (P_ε) by constructing minimizing sequences twice. Finally, we deal with the problem (P) by Meyer approximation.

Consider the following evolution control system with a β-order Riemann-Liouville fractional derivative:

$$\begin{cases} D_t^\beta z(t) = Az(t) + B(t)v(t) + f(t, z(t)),\ t \in J', \\ \lim\limits_{t\downarrow 0} \Gamma(\beta) t^{1-\beta} z(t) = z_0, \\ v \in V_{ad}. \end{cases} \tag{4.1}$$

Here A generates a β-order resolvent $\{R_\beta(t)\}_{t>0}$, $f : J \times V \to V$ is a continuous function. Moreover, V_{ad} is an appropriate admissible set.

From now on, we impose the following conditions:

(HA) $A \in C_{s_0}^\beta(\overline{M}, \omega)$ and $t^{1-\beta} R_\beta(t)$ is compact and equicontinuous on J'.

(Hf) $f : J \times V \to V$ is continuous and there exists a positive constant N to guarantee that for $(t, z) \in J \times V$, $\|f(t, z)\| \leq N$.

(HB) $B \in L^\infty(J, \mathscr{L}(Y, V))$.

Furthermore, we introduce an admissible control set

$$V_{ad} = \{v \in L^p(J; G) | v(t) \in U(t)\ a.e.\},$$

where $p > \frac{1}{\beta}$, $U : J \to P_{fc}(Y)$ is a measurable multi-valued mapping, $G \subseteq Y$ is a bounded set and $U(\cdot) \subseteq G$. Thanks to Proposition 2.1.7 in [3], we know that $V_{ad} \neq \emptyset$. In addition, condition (HB) indicates that for all $v \in V_{ad}$, $Bv \in L^p(J; V)$.

Due to Lemma 2.3 and our previous work (see Lemma 3.2 in [22]), we introduce the notion of mild solutions to system (4.1).

Definition 4.1. For fixed $v \in V_{ad}$, by a mild solution to system (4.1) associated with v, we mean the function $z \in C_{1-\beta}(J; V)$ satisfying

$$z(t) = R_\beta(t) z_0 + \int_0^t R_\beta(t - \tau)\big(B(\tau)v(\tau) + f(\tau, z(\tau))\big) \mathrm{d}\tau,\ t \in J'.$$

For convenience, put

$$S(v) = \big\{z \in C_{1-\beta}(J; V) : z \text{ is a mild solution to (4.1) depending on } v \in V_{ad}\big\}.$$

By the standard technique utilized for Riemann-Liouville fractional evolution systems (see Theorem 3.1 in [22]), we exhibit the existence result of (4.1).

Theorem 4.1. *Let conditions* (HA), (Hf) *and* (HB) *hold. Then* $S(v) \neq \emptyset$.

In what follows, we first introduce some notations and propose the time optimal control problem of system (4.1).

Let $z_0, z_1 \in V$ and $z_0 \neq z_1$. Set $\mathcal{A}_d = \{(z, v) : v \in V_{ad},\ z \in S(v)\}$,

$$\mathcal{A}_d^{z_1} = \{(z, v) \in \mathcal{A}_d : \text{there exists } t \geq 0 \text{ to ensure that } \Gamma(\beta)t^{1-\beta}z(t) = z_1\}$$

and

$$V_0 = \{v \in V_{ad} : \text{there exists } z \in S(v) \text{ to guarantee that } (z, v) \in \mathcal{A}_d^{z_1}\}.$$

Let $\mathcal{A}_d^{z_1} \neq \emptyset$. For any $(z, v) \in \mathcal{A}_d^{z_1}$, we denote by

$$t(z, v) = \min\{t \geq 0 : \Gamma(\beta)t^{1-\beta}z(t) = z_1\}$$

the transition time related to the state-control pair (z, v).

Set $t^* = \inf\{t(z, v) : (z, v) \in \mathcal{A}_d^{z_1}\}$. We propose the time optimal control problem (P) of system (4.1):

Seek a state-control pair $(z^*, v^*) \in \mathcal{A}_d^{z_1}$ to ensure that $t(z^*, v^*) = t^*$.

Remark 4.1. Note that the mild solution $z \in S(v)$ has singularity at zero, but that $\Gamma(\beta)(\cdot)^{1-\beta}z(\cdot) \in C(J; V)$. We thereby employ $\Gamma(\beta)t^{1-\beta}z(t) = z_1$ instead of $z(t) = z_1$, when we define the transition time $t(z, v)$. Below, we check that the definition of the transition time is well-defined. To simplify notation, set $\tau(z, v) = \{t \geq 0 : \Gamma(\beta)t^{1-\beta}z(t) = z_1\}$ and $\bar{t} = \inf \tau(z, v)$. If $\tau(z, v)$ contains only finite elements, the proof is trivial. Otherwise, we can take a sequence $\{t_n\}_{n\geq 1} \subseteq \tau(z, v)$ satisfying that $\lim_{n\to\infty} t_n = \bar{t}$. As such, $\Gamma(\beta)t_n^{1-\beta}z(t_n) = z_1$. Moreover, we conclude from $\Gamma(\beta)(\cdot)^{1-\beta}z(\cdot) \in C(J; V)$ that

$$\lim_{n\to\infty} \Gamma(\beta)t_n^{1-\beta}z(t_n) = \Gamma(\beta)\bar{t}^{1-\beta}z(\bar{t}),$$

hence that $\Gamma(\beta)\bar{t}^{1-\beta}z(\bar{t}) = z_1$, and finally that the definition is well-defined.

Next, by introducing the linear transformation

$$t = ks,\ 0 \leq s \leq 1 \text{ and } k \in [0, +\infty),$$

we transform the system (4.1) into the following system:

$$\begin{cases} D_s^\beta x(s) = k^\beta Ax(s) + k^\beta(B(ks)u(s) + f(ks, x(s))),\ s,\ k > 0, \\ \Gamma(\beta)(ks)^{1-\beta}x(s) = z_0,\ s = 0 \text{ or } k = 0, \\ (u, k) \in W, \end{cases} \tag{4.2}$$

where $x(s) = z(ks)$, $u(s) = v(ks)$, $\Gamma(\beta)(ks)^{1-\beta}x(s)|_{k=0} = \lim_{k\downarrow 0} \Gamma(\beta)(ks)^{1-\beta}x(s)$,

$$\Gamma(\beta)(ks)^{1-\beta}x(s)|_{s=0} = \lim_{s\downarrow 0} \Gamma(\beta)(ks)^{1-\beta}x(s),$$

$$W = \{(u,k) : u(s) = v(ks),\ s \in [0,1],\ v \in V_0,\ k \in [0,+\infty)\}.$$

Due to Lemma 3.3, $k^\beta A$ generates a resolvent (written $\{R^k_\beta(s)\}_{s>0}$, for short). From Theorem 3.3 and (HA), we see that $\{s^{1-\beta}R^k_\beta(s)\}_{s>0}$ is compact and equicontinuous for $k > 0$. Furthermore, for fixed $k \in [0,+\infty)$, it is easily seen that $\sup_{s\in[0,1]} \|s^{1-\beta}R^k_\beta(s)\| < \infty$. For convenience, we assume that $\sup_{s\in[0,1]} \|s^{1-\beta}R^k_\beta(s)\| < M$.

Write

$$C^k_\beta([0,1];V) = \left\{x | \widetilde{x}(\tau) = (k\tau)^{1-\beta}x(\tau),\ \widetilde{x} \in C([0,1];V)\right\},$$

where $\widetilde{x}(0) = \lim_{\tau\downarrow 0} \widetilde{x}(\tau)$ and $\widetilde{x}(\tau)|_{k=0} = \lim_{k\downarrow 0} \widetilde{x}(\tau)$.

Based on Definition 4.1, we can introduce the following notion:

Definition 4.2. For fixed $w = (u,k) \in W$, by a mild solution of (4.2) related to w, we understand the function $x \in C^k_\beta([0,1];V)$ satisfying that for $s \in (0,1]$ and $k > 0$,

$$x(s) = R^k_\beta(s)k^{-(1-\beta)}z_0 + \int_0^s R^k_\beta(s-\tau)k^\beta\big(B(k\tau)u(\tau) + f(k\tau, x(\tau))\big)\mathrm{d}\tau,$$

and for $s = 0$ or $k = 0$, $\Gamma(\beta)(ks)^{1-\beta}x(s) = z_0$.

For simplicity, set

$$S(w) = \big\{x \in C^k_\beta([0,1];V) : x \text{ is a mild solution of (4.2) related to } w \in W\big\}.$$

By means of Theorem 4.1, we can establish the following existence result:

Theorem 4.2. *Assume that (HA), (Hf) and (HB) hold. Then $S(w) \neq \emptyset$.*

Then, we analyze the following Meyer problem (P_ε) of system (4.2):

Seek a state-control pair $(x_\varepsilon, w_\varepsilon)$ to guarantee that

$$J_\varepsilon(x_\varepsilon, w_\varepsilon) = \inf_{(x,w)\in S(w)\times W} J_\varepsilon(x,w),$$

where $w = (u,k) \in W$ and $J_\varepsilon(x,w) = \frac{1}{2\varepsilon}\left\|\Gamma(\beta)k^{1-\beta}x(1) - z_1\right\|^2 + k$.

To treat the Meyer problem (P_ε), we need the following lemma:

Lemma 4.1. *Let $k > 0$ and assumptions (HA) and (HB) hold. Then the operator $\phi : L^p([0,1];V_{ad}) \to C^k_\beta([0,1];V)$, $p > \frac{1}{\beta}$, given by*

$$(\phi u)(s) = \int_0^s R^k_\beta(s-\tau)B(k\tau)u(\tau)\mathrm{d}\tau,$$

is compact.

Proof. Assume that $\{u_n\}_{n\geq 1}$ is a bounded sequence in $L^p([0,1];V_{ad})$. Then we infer from (HB) that $\{B(k\cdot)u_n(\cdot)\}_{n\geq 1} \subseteq L^p([0,1];V)$ is bounded. Thus, by employing Lemma 2.3, ϕ is compact. □

Since the uniqueness of the solutions cannot be acquired, the method of setting up a minimizing state-control pair sequence in [16–18] breaks down. We now utilize a new technique of establishing minimizing sequences twice to deal with the Meyer problem (P_ε).

Theorem 4.3. *Let (HA), (Hf) and (HB) hold. Then for fixed $\varepsilon > 0$,* Meyer *problem (P_ε) possesses at least one optimal trajectory-control pair $(x_\varepsilon, w_\varepsilon)$.*

Proof. For clarity, we split the proof into the following procedures.
Step 1 For fixed $w \in W$, we will seek $\overline{x} \in S(w)$ to ensure that $J_\varepsilon(\overline{x}, w) = J_\varepsilon(w)$, where $J_\varepsilon(w) = \inf\limits_{x\in S(w)} J_\varepsilon(x, w)$.

Below, we consider the following two cases.
Case 1 $k = 0$. From $\Gamma(\beta)k^{1-\beta}x(1) = z_0$ for any $x \in S(w)$, we deduce that $J_\varepsilon(x, w)$ is a constant, hence that the proof is obvious.
Case 2 $k > 0$. Since it is trivial for the two cases when $J_\varepsilon(w) = +\infty$ or the solution set $S(w)$ possesses only finite elements, we can suppose that $J_\varepsilon(w) < +\infty$. Thus, we can take $\{x_n\}_{n\geq 1} \in S(w)$ to guarantee that $\lim\limits_{n\to\infty} J_\varepsilon(x_n, w) = J_\varepsilon(w)$.

By $\{x_n\}_{n\geq 1} \in S(w)$, we get that for $s > 0$,

$$x_n(s) = R^k_\beta(s)k^{-(1-\beta)}z_0 + \int_0^s R^k_\beta(s-\tau)k^\beta\big(B(k\tau)u(\tau) + f(k\tau, x_n(\tau))\big)\mathrm{d}\tau. \quad (4.3)$$

Since $\{s^{1-\beta}R^k_\beta(s)\}_{s>0}$ is compact and equicontinuous, we can obtain the compactness of $\{x_n\}_{n\geq 1}$ in $C^k_\beta([0,1];V)$. This follows by the same argument as in Step 3 of Theorem 3.1 in [22]. As such, we can choose $\overline{x} \in C^k_\beta([0,1];V)$ and a subsequence extracted from $\{x_n\}_{n\geq 1}$, still written $\{x_n\}_{n\geq 1}$, such that $\lim\limits_{n\to\infty} x_n = \overline{x}$. Therefore, by letting $n \to \infty$ on both sides of (4.3), we conclude from the dominated convergence theorem that

$$\overline{x}(s) = R^k_\beta(s)k^{-(1-\beta)}z_0 + \int_0^s R^k_\beta(s-\tau)k^\beta\big(B(k\tau)u(\tau) + f(k\tau, \overline{x}(\tau))\big)\mathrm{d}\tau,$$

hence that $\overline{x} \in S(w)$, finally that

$$\begin{aligned} J_\varepsilon(w) &= \lim_{n\to\infty} J_\varepsilon(x_n, w) = \lim_{n\to\infty} \frac{1}{2\varepsilon}\left\|\Gamma(\beta)k^{(1-\beta)}x_n(1) - z_1\right\|^2 + k \\ &\geq \frac{1}{2\varepsilon}\left\|\Gamma(\beta)k^{(1-\beta)}\overline{x}(1) - z_1\right\|^2 + k = J_\varepsilon(\overline{x}, w) \\ &\geq J_\varepsilon(w). \end{aligned}$$

This indicates that $J_\varepsilon(\overline{x}, w) = J_\varepsilon(w)$.
Step 2 We shall look for w_ε satisfying $J_\varepsilon(w_\varepsilon) = \inf\limits_{w\in W} J_\varepsilon(w)$. For simplicity of notation, set $m_\varepsilon = \inf\limits_{w\in W} J_\varepsilon(w)$.

We only need to consider the case $m_\varepsilon < +\infty$, since the case $m_\varepsilon = +\infty$ is trivial.

Due to $m_\varepsilon < +\infty$, we can pick $\{w_n\}_{n\geq 1} \subseteq W$ to ensure that $\lim\limits_{n\to\infty} J_\varepsilon(w_n) = m_\varepsilon$, where $w_n = (u_n, k_n) \in V_{ad} \times [0, +\infty)$.

Based on the definitions of m_ε, $J_\varepsilon(w)$, $J_\varepsilon(x, w)$ and W, $\{k_n\}$ is bounded. Accordingly, one can extract a subsequence from $\{k_n\}$, relabeled by it again, to guarantee that $k_n \to k_\varepsilon,\ n \to \infty$, for some $k_\varepsilon \in [0, +\infty)$.

Thanks to $\{u_n\}_{n\geq 1} \in V_{ad}$, a subsequence of $\{u_n\}_{n\geq 1} \subseteq V_{ad}$ can be extracted, written $\{u_n\}_{n\geq 1}$ again, such that $u_n \overset{w}{\to} u_\varepsilon,\ n \to \infty$, for some $u_\varepsilon \in V_{ad}$. Since V_{ad} is close and convex, we infer from Mazur lemma that $u_\varepsilon \in V_{ad}$.

By virtue of Step 1, one can take $x_n \in S(w_n)$ to ensure that $J_\varepsilon(x_n, w_n) = J_\varepsilon(w_n)$. On account of $x_n \in S(w_n)$, it yields that for $k_n > 0$ and $s \in (0, 1]$,

$$x_n(s) = R^{k_n}_\beta(s)k_n^{-(1-\beta)}z_0 + \int_0^s R^{k_n}_\beta(s-\tau)k_n^\beta\big(B(k_n\tau)u_n(\tau) + f(k_n\tau, x_n(\tau))\big)\mathrm{d}\tau, \quad (4.4)$$

and for $k_n = 0$,

$$\Gamma(\beta+\gamma(1-\beta))(k_n s)^{(1-\beta)(1-\gamma)}x_n(s) = z_0.$$

Below, we consider the following three cases.

Case 1 $k_n > 0$ and $k_\varepsilon > 0$. We see from Theorem 3.3 that $\{t^{1-\beta}R_\beta^{k_n}(t)\}_{t>0}$ is compact and equicontinuous, hence that $\{x_n\}_{n\geq 1}$ is compact by the same method in Step 1, finally that we can suppose, without loss of generality, that $x_n \to x_\varepsilon$.

Firstly, according to $k_n \to k_\varepsilon$ and $k_\varepsilon > 0$, we can derive the boundedness of $\{k_n\}_{n\geq 1}$ and $\left\{k_n^{-(1-\beta)}\right\}_{n\geq 1}$. Thus, from $\sup_{s\in[0,1]}\|s^{1-\beta}R_\beta^k(s)\| < M$ for fixed $k \in [0,\infty)$ and Theorem 3.1, we can deduce that for fixed $s\in(0,1]$,

$$\begin{aligned}
&\left\|R_\beta^{k_n}(s)k_n^{-(1-\beta)}z_0 - R_\beta^{k_\varepsilon}(s)k_\varepsilon^{-(1-\beta)}z_0\right\| \\
\leq &\left\|s^{1-\beta}R_\beta^{k_n}(s)z_0 - s^{1-\beta}R_\beta^{k_\varepsilon}(s)z_0\right\|(sk_n)^{-(1-\beta)} \\
&+\left\|s^{1-\beta}R_\beta^{k_\varepsilon}(s)\right\|\left|(sk_n)^{-(1-\beta)} - (sk_\varepsilon)^{-(1-\beta)}\right|\|z_0\| \\
\to &\, 0,\ n\to\infty.
\end{aligned}$$

Secondly, we deduce from Lemma 4.1 and $u_n \xrightarrow{w} u_\varepsilon$ that for $s\in(0,1]$,

$$\int_0^s R_\beta^{k_\varepsilon}(s-\tau)k_\varepsilon^\beta B(k_\varepsilon\tau)(u_n(\tau)-u_\varepsilon(\tau))\mathrm{d}\tau \to 0,\ n\to\infty.$$

In addition, according to the definition of V_{ad}, we get $\{u_n(s) : n\geq 1, a.e.\ s\in[0,1]\}\subseteq G$. Since G is bounded, we can suppose that $\|u_n(s)\|\leq M_1$, uniformly for $s\in[0,1]$ and $n\geq 1$. As such,

$$\begin{aligned}
&\left\|\int_0^s R_\beta^{k_\varepsilon}(s-\tau)k_\varepsilon^\beta(B(k_n\tau)-B(k_\varepsilon\tau))u_n(\tau)\mathrm{d}\tau\right\| \\
\leq\ & MM_1k_\varepsilon^\beta\int_0^s(s-\tau)^{-(1-\beta)}\|B(k_n\tau)-B(k_\varepsilon\tau)\|\mathrm{d}\tau \\
\leq\ & MM_1k_\varepsilon^\beta\left(\frac{p-1}{\beta p-1}\right)^{1-\frac{1}{p}}\left(\int_0^1\|B(k_n\tau)-B(k_\varepsilon\tau)\|^p\mathrm{d}\tau\right)^{\frac{1}{p}}.
\end{aligned}$$

Hence, we have

$$\int_0^s R_\beta^{k_\varepsilon}(s-\tau)k_\varepsilon^\beta(B(k_n\tau)-B(k_\varepsilon\tau))u_n(\tau)\mathrm{d}\tau\to 0,\ n\to\infty,$$

which is due to the p-mean continuity (see problem 23.9 on page 445 in [20]).

Furthermore, on account of $\sup_{s\in[0,1]}\|s^{1-\beta}R_\beta^k(s)\| < M$ for $k\in[0,+\infty)$ and $\|u_n(s)\|\leq M_1$, uniformly in $s\in[0,1]$ and $n\geq 1$, we derive that

$$\left\|\int_0^s R_\beta^{k_\varepsilon}(s-\tau)\left(k_n^\beta-k_\varepsilon^\beta\right)B(k_n\tau)u_n(\tau)\mathrm{d}\tau\right\|\leq\frac{MM_1\|B\|_\infty}{\beta}\left|k_n^\beta-k_\varepsilon^\beta\right|\to 0.$$

Additionally, we can infer from Theorem 3.2 and the dominated convergence theorem that

$$\left\|\int_0^s\left(R_\beta^{k_n}(s-\tau)-R_\beta^{k_\varepsilon}(s-\tau)\right)k_n^\beta B(k_n\tau)u_n(\tau)\mathrm{d}\tau\right\|$$

$$
\begin{aligned}
&\le k_n^{\beta}\|B\|_{\infty}M_1\int_0^s \|g_s(n,\tau)\|\,(s-\tau)^{-(1-\beta)}\mathrm{d}\tau\\
&\le k_n^{\beta}\|B\|_{\infty}M_1\left(\frac{p-1}{\beta p-1}\right)^{1-\frac{1}{p}}\left(\int_0^s \|g_s(n,\tau)\|^p\,\mathrm{d}\tau\right)^{\frac{1}{p}}\\
&\to 0, n\to\infty,
\end{aligned}
$$

where $g_s(n,\tau)=(s-\tau)^{1-\beta}R_\beta^{k_n}(s-\tau)-(s-\tau)^{1-\beta}R_\beta^{k_\varepsilon}(s-\tau)$.

Hence, we get

$$
\begin{aligned}
&\left\|\int_0^s R_\beta^{k_n}(s-\tau)k_n^\beta B(k_n\tau)u_n(\tau)-R_\beta^{k_\varepsilon}(s-\tau)k_\varepsilon^\beta B(k_\varepsilon\tau)u_\varepsilon(\tau)\mathrm{d}\tau\right\|\\
\le&\left\|\int_0^s \left(R_\beta^{k_n}(s-\tau)-R_\beta^{k_\varepsilon}(s-\tau)\right)k_n^\beta B(k_n\tau)u_n(\tau)\mathrm{d}\tau\right\|\\
&+\left\|\int_0^s R_\beta^{k_\varepsilon}(s-\tau)\left(k_n^\beta-k_\varepsilon^\beta\right)B(k_n\tau)u_n(\tau)\mathrm{d}\tau\right\|\\
&+\left\|\int_0^s R_\beta^{k_\varepsilon}(s-\tau)k_\varepsilon^\beta(B(k_n\tau)-B(k_\varepsilon\tau))u_n(\tau)\mathrm{d}\tau\right\|\\
&+\left\|\int_0^s R_\beta^{k_\varepsilon}(s-\tau)k_\varepsilon^\beta B(k_\varepsilon\tau)(u_n(\tau)-u_\varepsilon(\tau))\mathrm{d}\tau\right\|\\
\to&\ 0,\ n\to\infty.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\left\|\int_0^s R_\beta^{k_n}(s-\tau)k_n^\beta f(k_n(\tau),x_n(\tau))-R_\beta^{k_\varepsilon}(s-\tau)k_\varepsilon^\beta f(k_\varepsilon\tau,x_\varepsilon(\tau))\mathrm{d}\tau\right\|\\
\le&\left\|\int_0^s \left(R_\beta^{k_n}(s-\tau)-R_\beta^{k_\varepsilon}(s-\tau)\right)k_n^\beta f(k_n\tau,x_n(\tau))\mathrm{d}\tau\right\|\\
&+\left\|\int_0^s R_\beta^{k_\varepsilon}(s-\tau)\left(k_n^\beta-k_\varepsilon^\beta\right)f(k_n\tau,x_n(\tau))\mathrm{d}\tau\right\|\\
&+\left\|\int_0^s R_\beta^{k_\varepsilon}(s-\tau)k_\varepsilon^\beta(f(k_n\tau,x_n(\tau))-f(k_\varepsilon\tau,x_\varepsilon(\tau)))\mathrm{d}\tau\right\|\\
\to&\ 0,\ n\to\infty.
\end{aligned}
$$

Therefore, letting $n\to\infty$ on both sides of (4.4), we get

$$
x_\varepsilon(s)=R_\beta^{k_\varepsilon}(s)k_\varepsilon^{-(1-\beta)}z_0+\int_0^s R_\beta^{k_\varepsilon}(s-\tau)k_\varepsilon^\beta\big(B(k_\varepsilon\tau)u_\varepsilon(\tau)+f(k_\varepsilon\tau,x_\varepsilon(\tau))\big)\mathrm{d}\tau.
$$

Thus, $x_\varepsilon\in S(w_\varepsilon)$, where $w_\varepsilon=(u_\varepsilon,k_\varepsilon)$. In addition, according to the boundedness of $\{k_n\}$, $x_n\to x_\varepsilon$ and $k_n\to k_\varepsilon$, we get

$$
\begin{aligned}
&\left\|\Gamma(\beta)k_n^{1-\beta}x_n(1)-\Gamma(\beta)k_\varepsilon^{1-\beta}x_\varepsilon(1)\right\|\\
\le&\ \Gamma(\beta)k_n^{1-\beta}\|x_n(1)-x_\varepsilon(1)\|+\Gamma(\beta)\|x_\varepsilon(1)\|\left|k_n^{1-\beta}-k_\varepsilon^{1-\beta}\right|\\
\to&\ 0,\ n\to\infty.
\end{aligned}
$$

As such,

$$
\lim_{n\to\infty}\Gamma(\beta)k_n^{1-\beta}x_n(1)=\Gamma(\beta)k_\varepsilon^{1-\beta}x_\varepsilon(1).
$$

Case 2 $k_n > 0$ and $k_\varepsilon = 0$. By means of Lemma 3.3 and (4.4), we get

$$\lim_{n\to\infty} \Gamma(\beta)(k_n s)^{1-\beta} x_n(s) = \lim_{n\to\infty} \Gamma(\beta) R_\beta^{k_n}(s) s^{1-\beta} z_0$$
$$= \lim_{n\to\infty} \Gamma(\beta)(s k_n^{\frac{1}{\beta}})^{1-\beta} R_\beta(s k_n^{\frac{1}{\beta}}) z_0 = z_0.$$

In addition, Definition 4.2 shows that

$$\Gamma(\beta) k_\varepsilon^{(1-\beta)(1-\gamma)} x_\varepsilon(1) = z_0,$$

where $x_\varepsilon \in S(w_\varepsilon)$ and $w_\varepsilon = (u_\varepsilon, 0)$. Hence,

$$\lim_{n\to\infty} \Gamma(\beta) k_n^{1-\beta} x_n(1) = \Gamma(\beta) k_\varepsilon^{1-\beta} x_\varepsilon(1).$$

Case 3 $k_n = 0$. From $k_n \to k_\varepsilon,\ n \to \infty$, we get $k_\varepsilon = 0$. Moreover, due to Definition 4.2, we have

$$\Gamma(\beta) k_n^{1-\beta} x_n(1) = z_0$$

and

$$\Gamma(\beta) k_\varepsilon^{1-\beta} x_\varepsilon(1) = z_0,$$

where $x_\varepsilon \in S(w_\varepsilon)$ and $w_\varepsilon = (u_\varepsilon, 0)$. As such,

$$\lim_{n\to\infty} \Gamma(\beta) k_n^{1-\beta} x_n(1) = \Gamma(\beta) k_\varepsilon^{1-\beta} x_\varepsilon(1).$$

Thus, combining above cases, we can assert that

$$\begin{aligned} m_\varepsilon &= \lim_{n\to\infty} J_\varepsilon(w_n) = \lim_{n\to\infty} J_\varepsilon(x_n, w_n) \\ &= \lim_{n\to\infty} \left(\frac{1}{2\varepsilon} \left\| \Gamma(\beta) k_n^{1-\beta} x_n(1) - z_1 \right\|^2 + k_n \right) \\ &\geq \frac{1}{2\varepsilon} \| \Gamma(\beta) k_\varepsilon^{1-\beta} x_\varepsilon(1) - z_1 \|^2 + k_\varepsilon \\ &= J_\varepsilon(x_\varepsilon, w_\varepsilon) \geq J_\varepsilon(w_\varepsilon) \geq m_\varepsilon. \end{aligned}$$

Therefore,

$$J_\varepsilon(x_\varepsilon, w_\varepsilon) = m_\varepsilon = \inf_{w\in W} J_\varepsilon(w),$$

which indicates that $(x_\varepsilon, w_\varepsilon)$ is an optimal trajectory-control pair. □

We are now in a position to analyze the time optimal control problem (P) by Meyer approximation.

Theorem 4.4. *Suppose that conditions (HA), (Hf) and (HB) hold. Then problem (P) possesses at least one optimal trajectory-control pair.*

Proof. The proof will be divided into the following steps.
Step 1 According to $\mathcal{A}_d^{z_1} \neq \emptyset$, we can take $(\overline{z}, \overline{v}) \in \mathcal{A}_d^{z_1}$. Let $t(\overline{z}, \overline{v}) = \overline{t} < \infty$. Then one can assert from the definition of $t(\overline{z}, \overline{v})$ that $\Gamma(\beta)\overline{t}^{1-\beta}\overline{z}(\overline{t}) = z_1$. This gives $\overline{t} > 0$, which is based on the fact that $\Gamma(\beta) t^{1-\beta} \overline{z}(t)|_{t=0} = z_0$ and $z_0 \neq z_1$. Set $\overline{u}(s) = \overline{v}(\overline{t}s),\ 0 \leq s \leq 1$ and $\overline{w} = (\overline{u}, \overline{t}) \in W$. Then $\Gamma(\beta)\overline{t}^{1-\beta}\overline{x}(1) = z_1$ and $\overline{x} \in S(\overline{w})$, where $\overline{x}(\cdot) = \overline{z}(\overline{t}\cdot)$.

Due to Theorem 4.3, for any $\varepsilon > 0$, there exists a pair $(x_\varepsilon, w_\varepsilon)$ to ensure that

$$J_\varepsilon(x_\varepsilon, w_\varepsilon) = \frac{1}{2\varepsilon}\|\Gamma(\beta)k_\varepsilon^{1-\beta}x_\varepsilon(1) - z_1\|^2 + k_\varepsilon = \inf_{(x,w)\in S(w)\times W} J_\varepsilon(x, w).$$

Thus, we get

$$J_\varepsilon(\overline{x}, \overline{w}) = \overline{t} \geq J_\varepsilon(x_\varepsilon, w_\varepsilon) = \frac{1}{2\varepsilon}\|\Gamma(\beta)k_\varepsilon^{1-\beta}x_\varepsilon(1) - z_1\|^2 + k_\varepsilon,$$

which yields that for any $\varepsilon > 0$,

$$0 \leq k_\varepsilon \leq \overline{t} \text{ and } \|\Gamma(\beta)k_\varepsilon^{1-\beta}x_\varepsilon(1) - z_1\|^2 \leq 2\varepsilon\overline{t}.$$

As such, we can find a sequence $\{\varepsilon_n\}_{n\geq 1}$ to ensure that $\varepsilon_n \to 0$ as $n \to \infty$ and

$$k_{\varepsilon_n} \to k_0 \text{ for some } k_0 \in [0, \overline{t}],$$
$$\Gamma(\beta)k_{\varepsilon_n}^{1-\beta}x_{\varepsilon_n}(1) \to z_1,$$
$$u_{\varepsilon_n} \xrightarrow{w} u_0 \text{ for some } u_0 \text{ in } V_{ad}.$$

Thanks to $z_0 \neq z_1$, we have $k_{\varepsilon_n} > 0$ and $k_0 > 0$. In fact, if $k_{\varepsilon_n} = 0$, we can conclude from Definition 4.2 that $\Gamma(\beta)k_{\varepsilon_n}^{1-\beta}x_{\varepsilon_n}(1) = z_0$, which is a contradiction. If $k_0 = 0$, we see from step 2 in Theorem 4.3 that $\lim_{n\to\infty}\Gamma(\beta)k_{\varepsilon_n}^{1-\beta}x_{\varepsilon_n}(1) = z_0$, which is also a contradiction.

Step 2 On account of $x_{\varepsilon_n} \in S(w_{\varepsilon_n})$ and $k_{\varepsilon_n} > 0$, we get that for $s \in (0, 1]$,

$$x_{\varepsilon_n}(s)=R_\beta^{k_{\varepsilon_n}}(s)k_{\varepsilon_n}^{-(1-\beta)}z_0+\int_0^s R_\beta^{k_{\varepsilon_n}}(s-\tau)k_{\varepsilon_n}^\beta\big(B(k_{\varepsilon_n}\tau)u_{\varepsilon_n}(\tau)+f(k_{\varepsilon_n}\tau, x_{\varepsilon_n}(\tau))\big)\mathrm{d}\tau.$$

As in the proof of Theorem 4.3, we can extract a subsequence from $\{x_{\varepsilon_n}\}$, still written $\{x_{\varepsilon_n}\}$ again. Moreover, there is no loss of generality in assuming that $\lim_{n\to\infty} x_{\varepsilon_n} = x^0$. By the same reasoning as in Step 2 of Theorem 4.3, we obtain

$$x^0(s)=R_\beta^{k_0}(s)k_0^{-(1-\beta)}z_0+\int_0^s R_\beta^{k_0}(s-\tau)k_0{}^\beta\big(B(k_0\tau)u_0(\tau)+f(k_0\tau, x^0(\tau))\big)\mathrm{d}\tau,$$

which indicates that $x^0 \in S(w_0)$, where $w_0 = (u_0, k_0)$.

Since $x_{\varepsilon_n} \to x^0$ and $k_{\varepsilon_n} \to k_0$ as $n \to \infty$, we can conclude from the boundedness of $\{k_{\varepsilon_n}\}$ that

$$\begin{aligned}&\left\|\Gamma(\beta)k_{\varepsilon_n}^{1-\beta}x_{\varepsilon_n}(1) - \Gamma(\beta)k_0^{1-\beta}x^0(1)\right\|\\ \leq\ &\Gamma(\beta)k_{\varepsilon_n}^{1-\beta}\|x_{\varepsilon_n}(1) - x^0(1)\| + \Gamma(\beta)\|x^0(1)\|\left|k_{\varepsilon_n}^{1-\beta} - k_0^{1-\beta}\right|\\ \to\ &0,\ n \to \infty.\end{aligned}$$

On the other hand, due to Step 1, we derive $\left\|\Gamma(\beta)k_{\varepsilon_n}^{1-\beta}x_{\varepsilon_n}(1) - z_1\right\|^2 \leq 2\varepsilon_n\overline{t} \to 0$. We thereby get

$$\begin{aligned}&\left\|\Gamma(\beta)k_0^{1-\beta}x^0(1) - z_1\right\|\\ \leq\ &\left\|\Gamma(\beta)k_{\varepsilon_n}^{1-\beta}x_{\varepsilon_n}(1) - z_1\right\| + \left\|\Gamma(\beta)k_{\varepsilon_n}^{1-\beta}x_{\varepsilon_n}(1) - \Gamma(\beta)k_0^{1-\beta}x^0(1)\right\|\end{aligned}$$

$$\to 0,\ n \to \infty,$$

which indicates that $\Gamma(\beta)k_0^{1-\beta}x^0(1) = z_1$.

Step 3 Since $k_0 > 0$ (see Step 1), we can set $v_0(\cdot) = u_0(\cdot/k_0)$. Thus, we can infer that $z^0(\cdot) = x^0(\cdot/k_0)$ is a mild solution of (4.1) related to $v_0 \in V_0$ and

$$\Gamma(\beta)k_0^{1-\beta}z^0(k_0) = \Gamma(\beta)k_0^{1-\beta}x^0(1) = z_1.$$

This gives $(z^0, v_0) \in \mathcal{A}_d^{z_1}$. Let $t(z^0, v_0) = t_0$. According to the definition of $t(z^0, v_0)$, we get $t_0 \le k_0$. In addition, similar argument in step 1 shows that for any $\varepsilon_n > 0$, $t_0 \ge k_{\varepsilon_n}$. Letting $n \to \infty$, we get $t_0 \ge k_0$. As such, $t_0 = k_0$. Accordingly, due to $t^* = \inf\{t(z, v) : (z, v) \in \mathcal{A}_d^{z_1}\}$, we have $k_0 \ge t^*$.

For any $(z, v) \in \mathcal{A}_d^{z_1}$, the same reasoning as in step 1 tells us that for any $\varepsilon_n > 0$, $t(z, v) \ge k_{\varepsilon_n}$. Letting $n \to \infty$, for any $(z, v) \in \mathcal{A}_d^{z_1}$, we have $t(z, v) \ge k_0$. Therefore, we can conclude that $t^* \ge k_0$. As such, $t^* = k_0$. Hence, (z^0, v_0) is an optimal trajectory-control pair of problem (P). □

Remark 4.2. In most of the existing results on the time optimal control problems of evolution systems, many researchers explored them by setting up time optimal sequences (see [4, 5, 10, 19]). With the aid of the Lipschitz assumption on f, the authors in [16–18] coped with them by Meyer approximation. In the present paper, we have investigated the time control problem by Meyer approximation, when the Lipschitz condition is not satisfied. Thus, our results extend and generalize some recent results about time optimal controls of all evolution systems.

Finally, we address a fractional diffusion model by employing our theoretical findings.

Example 4.1. Consider the time optimal control problem of the following Riemann-Liouville fractional partial differential system:

$$\begin{cases} D_t^\beta z(t,y) = \frac{\partial^2}{\partial y^2} z(t,y) + f(t, z(t,y)) + v(t,y),\ t,\ y \in (0,1], \\ z(t,0) = z(t,1) = 0, \\ \lim\limits_{t \downarrow 0} \Gamma(\beta)t^{1-\beta}z(t,y) = g(y) = \sum\limits_{n=1}^{\infty} c_n \sin n\pi y. \end{cases} \tag{4.5}$$

Let $V = Y = L^2(0,1)$, $e_n(y) = \sqrt{2}\sin(n\pi y)$, $n = 1, 2, \cdots$ and $A = \frac{\partial^2}{\partial y^2}$ with domain

$$D(A) = \{\xi \in V : \xi', \xi'' \in V,\ \xi(0) = \xi(1) = 0\}.$$

Then A generates a β-order resolvent $R_\beta(t)$ (see [6]):

$$R_\beta(t)g(y) = \sum_{n=1}^{\infty} t^{\beta-1}E_{\beta,\beta}(-n^2\pi^2t^\beta)\langle g, e_n\rangle e_n(y),\ t > 0,\ g \in V.$$

Moreover, A also generates an analytic and compact semigroup $\{T(t)\}_{t>0}$ (see [11]):

$$T(t)g(y) = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t}\langle g, e_n\rangle e_n(y),\ t > 0,\ g \in V.$$

By utilizing probability density functions and Laplace transformations, we get

$$t^{1-\beta}R_\beta(t)=\beta\int_0^\infty \tau\xi_\beta(\tau)T(t^\beta\tau)\mathrm{d}\tau,$$

where

$$\xi_\beta(\tau)=\frac{1}{\beta}\tau^{-1-\frac{1}{\beta}}\varpi_\beta(\tau^{-\frac{1}{\beta}}),$$

$$\varpi_\beta(\tau)=\frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\tau^{-(n+1)\beta-1}\frac{\Gamma((n+1)\beta+1)}{(n+1)!}\sin\left((n+1)\pi\beta\right),\ \tau\in\mathbb{R}_+.$$

Based on the compactness of $\{T(t)\}_{t>0}$, we can acquire the compactness and equcontinuity of $\{t^{1-\alpha}R_\alpha(t)\}_{t>0}$ (see [21]). Since $\|T(t)\|\le 1$, we can choose $M>0$ and $\omega>0$ to ensure that $\|R_\alpha(t)\|\le Me^{\omega t}$ for $t\ge s_0$, which indicates that (HA) holds. In addition, due to Lemma 3.3 and Theorem 3.3, kA generates a compact and equicontinuous resolvent $\{R_\alpha^k(s)\}_{s>0}$.

Let $z(t)(y)=z(t,y)$, $B(t)=1$ and $v(t)(y)=v(t,y)$. We assume that $f:[0,1]\times V\to V$, defined by $f(t,z(t))(y)=f(t,z(t,y))$, is continuous. Furthermore, we introduce the following admissible control set

$$V_{ad}=\{v(t)(\cdot)\in Y:\ \text{there exists constant } N_1>0 \text{ such that } \|v(t)(\cdot)\|_Y\le N_1\}.$$

Then by means of Theorem 4.4, the time optimal control problem of system (4.5) possesses optimal trajectory-control pairs.

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References

[1] E. Bajlekova, *Fractional evolution equations in Banach spaces (Ph.D. thesis)*, University Press Facilities, Eindhoven University of Technology, 2001.

[2] Z. Fan, *Characterization of compactness for resolvents and its applications*, Appl. Math. Comput., 2014, 232, 60–67.

[3] S. Hu and N. S. Papageorgious, *Handbook of Multivalued Analysis (Theory)*, Kluwer Academic Publishers, Dordrecht Boston, London, 1997.

[4] Y. Jiang and N. Huang, *Solvability and optimal controls of fractional delay evolution inclusions with Clarke subdifferential*, Math. Meth. Appl. Sci., 2017, 40, 3026–3039.

[5] S. Kumar, *Mild solution and fractional optimal control of semilinear system with fixed delay*, J. Optim. Theory Appl., 2017, 174, 108–121.

[6] K. Li and J. Peng, *Fractional resolvents and fractional evolution equations*, Appl. Math. Lett., 2012, 25, 808–812.

[7] T. Lian, Z. Fan and G. Li, *Time optimal controls for fractional differential systems with Riemann-Liouville derivatives,* Fract. Calc. Appl. Anal., 2018, 21(6), 1524–1541.

[8] C. Lizama, *Regularized solutions for abstract Volterra equations*, J. Math. Anal. Appl., 2000, 243, 278–292.

[9] C. Lizama, *On approximation and representation of k-regularized resolvent families*, Integr. equ. oper. theory, 2001, 41, 223–229.

[10] F. Z. Mokkedem and X. L. Fu, *Optimal control problems for a semilinear evolution system with infinite delay*, Appl. Math. Optim., 2017. Doi:10.1007/s00245-017-9420-6.

[11] Z. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[12] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[13] G. Da Prato and M. Iannelli, *Linear abstract integrodifferential equations of hyperbolic type in Hilbert spaces*, Rend. Sem. Mat. Padova, 1980, 62, 191–206.

[14] G. Da Prato and M. Iannelli, *Linear integrodifferential equations in Banach spaces*, Rend. Sem. Mat. Padova, 1980, 62, 207–219.

[15] J. Prüss, *Evolutionary Integral Equations and Applications*, Birkhäuser, Basel, Berlin, 1993.

[16] J. R. Wang, M. Fečkan and A.Debbouche, *Time optimal control of a system governed by non-instantaneous impulsive differential equations*, J. Optim. Theory Appl., 2019, 182(2), 573–587.

[17] J. R. Wang, X. Xiang and W. Wei, *The constructive approach on existence of time optimal controls of system governed by nonlinear equations on banach spaces*, Electron. J. Qual. Theory Differ. Equ., 2009, 45, 1–10.

[18] J. R. Wang and Y. Zhou, *Study of an approximation process of time optimal control for fractional evolution systems in Banach spaces*, Adv. Differ. Equ., 2011. Doi:10.1155/2011/385324.

[19] J. Wang and Y. Zhou, *Time optimal control problem of a class of fractional distributed systems*, Int. J. Dyn. Syst. Differ. Equ., 2011, 3, 363–382.

[20] E. Zeidler, *Nonlinear Functional Analysis and Its Application II/A*, Springer-Verlag, New York, 1990.

[21] Y. Zhou and F. Jiao, *Existence of mild solutions for fractional neutral evolution equations*, Comput. Math. Appl., 2010, 59, 1063–1077.

[22] S. Zhu, Z. Fan and G. Li, *Optimal controls for Riemann-Liouville fractional evolution systems without Lipschitz assumption*, J. Optim. Theory Appl., 2017, 174, 47–64.

[23] S. Zhu, Z. Fan and G. Li, *Approximate controllability of Riemann-Liouville fractional evolution equations with integral contractor assumption*, J. Appl. Anal. Comput., 2018, 8, 532–548.

[24] S. Zhu, Z. Fan and G. Li, *Topological characteristics of solution sets for fractional evolution equations and applications to control systems*, Topol. Methods Nonlinear Anal., 2019. Doi: 10. 12775/TMNA. 2019. 033.