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EXISTENCE AND UNIQUENESS OF POSITIVE SOLUTION FOR NONLINEAR FRACTIONAL Q-DIFFERENCE EQUATION WITH INTEGRAL BOUNDARY CONDITIONS*

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Abstract This paper studies a class of nonlinear fractional q-difference equations with integral boundary conditions. By exploiting the properties of Green's function and two fixed point theorems for a sum operator, the existence and uniqueness of positive solutions for the boundary value problem are established. Iterative schemes for approximating the solutions are also obtained. Explicit examples are given to illustrate main results.

Keywords Fractional q-difference, integral boundary value problem, fixed point theorem, positive solution.

MSC(2010) 34A08, 34K30.

1. Introduction

In this paper, we consider an integral boundary value problem of fractional q-difference equations given by

$$\begin{cases} (D_q^\alpha u)(t) = -[f(t, u(t)) + g(t, u(t))], \quad 0 < t < 1, \\ u(0) = (D_q u)(0) = 0, \ u(1) = \int_0^1 p(s)u(s)d_q s, \end{cases} \tag{1.1}$$

where $2 < \alpha \leqslant 3$, $f, g : [0, 1] \times [0, +\infty) \to [0, +\infty)$ are continuous functions and the $p(s)$ satisfies the following condition:

(A) $p : [0, 1] \to [0, +\infty)$ with $p \in L^1[0, 1]$ and $\sigma_1 = \int_0^1 \tau^{\alpha-1}(1-\tau)p(\tau)d_q\tau > 0$, $\sigma_2 = \int_0^1 \tau^{\alpha-1}p(\tau)d_q\tau < 1$.

Nonlinear fractional q-difference equations appear in the mathematical modeling of many phenomena in engineering and science and have attracted much attention during the past decades (see, for example, [6–8, 12, 13, 19]). Boundary value problems involving fractional q-difference equations have been studied by applying different methods (for instance, see [2–5, 9–11, 14, 16, 18, 22, 23] and the references

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therein). Ahmad etc [3–5] investigated existence of solutions for nonlinear fractional q-difference equation with different boundary conditions by some classical fixed point theorems. Etemad etc [10] studied the existence of solutions for a new class of fractional q-integro-difference equation involving Riemann-Liouville q-derivatives and a q-integral of different orders, supplemented with boundary conditions containing q-integrals of different orders. Li and Yang [16] investigated the existence of positive solutions and two iterative schemes approximating the solutions for a class of nonlinear fractional q-difference equations with integral boundary conditions by applying monotone iterative method. Zhao and Yang [22] obtained sufficient conditions for the existence and uniqueness of solutions for a singular coupled integral boundary value problem of nonlinear higher-order fractional q-difference equations by using a mixed monotone method and Guo-Krasnoselskii fixed point theorem. Motivated by aforementioned works, we obtain the existence and uniqueness of positive solutions for the problem (1.1) by using the method of [11]. We also present sequences approximating a unique solution to the given problem.

2. Preliminarie

For the convenience, we collect here the necessary definitions from the theory of fractional q-calculus. Let $q \in (0,1)$ and define

$$[a]_q = \frac{1-q^a}{1-q}, a \in \mathbb{R}.$$

The q-analogue of the power function $(a-b)^n$ with $n \in \mathbb{N}$, $a, b \in \mathbb{R}$ is

$$(a-b)^0 = 1, (a-b)^{(n)} = \prod_{k=0}^{n-1}(a-bq^k).$$

More generally, if $\alpha \in \mathbb{R}$, then

$$(a-b)^{(\alpha)} = a^\alpha \prod_{n=0}^{\infty} \frac{a-bq^n}{a-bq^{\alpha+n}}.$$

If $b=0$, then $(a-b)^{(\alpha)} = a^{(\alpha)} = a^\alpha$. It is easy to see that $[a(t-s)]^{(\alpha)} = a^\alpha (t-s)^{(\alpha)}$ and $(a-b)^{(\alpha)} = (a-bq^{\alpha-1})(a-b)^{(\alpha-1)}$.

The q-gamma function is defined by

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, \quad x \in \mathbb{R} \setminus \{0, -1, -2, \cdots\},$$

and satisfies $\Gamma_q(x+1) = [x]_q \Gamma_q(x)$.

The following expression

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}$$

is called the q-derivative of the function $f(x)$. D_q has the following properties and formulas:

$$D_q(af(x) + bg(x)) = aD_q f(x) + bD_q g(x),$$

$$D_q(f(x)g(x)) = f(x)D_q g(x) + g(qx)D_q f(x),$$
$$_tD_q(t-s)^{(\alpha)} = [\alpha]_q(t-s)^{(\alpha-1)},$$
$$({}_xD_q \int_0^x f(x,t)d_qt)(x) = \int_0^x {}_xD_q f(x,t)d_qt + f(qx,x).$$

The q-integral of a function f defined on the interval $[0,b]$ is given by

$$I_q f(x) = \int_0^x f(t)d_qt = x(1-q)\sum_{n=0}^{\infty} f(xq^n)q^n, x \in [0,b].$$

If $a \in [0,b]$ and f is defined on $[0,b]$, its integral from a to b is defined by

$$\int_a^b f(t)d_qt = \int_0^b f(t)d_qt - \int_0^a f(t)d_qt.$$

Definition 2.1 (Agarwal [1])**.** Let $\alpha \geqslant 0$ and f be a function defined on $[0,1]$. The fractional q-integral of the Riemann-Liouville type is $(I_q^0 f)(x) = f(x)$ and

$$(I_q^\alpha f)(x) = \frac{1}{\Gamma_q(\alpha)}\int_0^x (x-qt)^{(\alpha-1)} f(t)d_qt, \alpha > 0, x \in [0,1].$$

Definition 2.2 (Rajković etc [17])**.** The fractional q-derivative of the Riemann-Liouville type of order $\alpha \geqslant 0$ is defined by

$$(D_q^\alpha f)(x) = (D_q^m I_q^{m-\alpha} f)(x),$$

where m is the smallest integer greater than or equal to α.

Lemma 2.1 (Agarwal [1])**.** *Let $\alpha, \beta \geqslant 0$ and f be a function defined on $[0,1]$. Then the following formulas hold:*

$$(I_q^\beta I_q^\alpha f)(x) = (I_q^{\alpha+\beta} f)(x),$$
$$(D_q^\alpha I_q^\alpha f)(x) = f(x).$$

Definition 2.3 (Ferreira [12])**.** Let $\alpha > 0$ and p be a positive integer. Then, the following equality holds:

$$(I_q^\alpha D_q^p f)(x) = (D_q^p I_q^\alpha f)(x) - \sum_{k=0}^{p-1} \frac{x^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)}(D_q^k f)(0).$$

Lemma 2.2. *The unique solution of the q-analogue of the fractional differential problem (1.1) is given by*

$$u(t) = \int_0^1 G(t,qs)[f(s,u(s)) + g(s,u(s))]d_qs,$$

where $2 < \alpha \leqslant 3, G(t,qs)$ is Green's function for the problem (1.1), which is given by

$$G(t,qs) = G_1(t,qs) + G_2(t,qs), \tag{2.1}$$

where

$$G_1(t,qs)=\frac{1}{\Gamma_q(\alpha)}\begin{cases}t^{\alpha-1}(1-qs)^{(\alpha-1)}-(t-qs)^{(\alpha-1)}, 0\leqslant qs\leqslant t\leqslant 1,\\ t^{\alpha-1}(1-qs)^{(\alpha-1)}, 0\leqslant t\leqslant qs\leqslant 1,\end{cases}$$

$$G_2(t,qs)=\frac{t^{\alpha-1}}{1-\sigma_2}\int_0^1 G_1(\tau,qs)p(\tau)d_q\tau,$$

where σ_2 *is defined in condition (A).*

Proof. By integrating the two sides on $(D_q^\alpha u)(t)=-[f(t,u(t))+g(t,u(t))]$, we can get

$$(I_q^\alpha D_q^\alpha u)(t)=-I_q^\alpha[f(t,u(t))+g(t,u(t))]. \tag{2.2}$$

In view of Definition 2.2, we deduce

$$(I_q^\alpha D_q^\alpha u)(t)=(I_q^\alpha D_q^3 I_q^{3-\alpha}u)(t). \tag{2.3}$$

By applying Definition 2.3 and Lemma 2.1, we have

$$\begin{aligned}(I_q^\alpha D_q^3 I_q^{3-\alpha}u)(t)=&(D_q^3 I_q^\alpha I_q^{3-\alpha}u)(t)-\sum_{k=0}^{2}\frac{t^{\alpha-3+k}}{\Gamma_q(\alpha+k-3+1)}(D_q^k f)(0)\\ =&u(t)-\sum_{k=0}^{2}\frac{t^{\alpha-3+k}}{\Gamma_q(\alpha+k-2)}(D_q^k f)(0).\end{aligned} \tag{2.4}$$

It follows from (2.2)-(2.4) and Definition 2.1, we have

$$\begin{aligned}u(t)=&c_1t^{\alpha-1}+c_2t^{\alpha-2}+c_3t^{\alpha-3}\\ &-\frac{1}{\Gamma_q(\alpha)}\int_0^t(t-qs)^{(\alpha-1)}[f(s,u(s))+g(s,u(s))]d_qs,\end{aligned} \tag{2.5}$$

where $c_1=\frac{(D_q^0 f)(0)}{\Gamma_q(\alpha-2)}, c_2=\frac{(D_q^1 f)(0)}{\Gamma_q(\alpha-1)}, c_3=\frac{(D_q^2 f)(0)}{\Gamma_q(\alpha)}$ are constants to be determined. Using the boundary conditions given by (1.1) in (2.5), we find that $c_3=0, c_2=0$ and $c_1=\frac{1}{\Gamma_q(\alpha)}\int_0^1(1-qs)^{(\alpha-1)}[f(s,u(s))+g(s,u(s))]d_qs+\int_0^1 p(s)u(s)d_qs$. Furthermore, we have

$$\begin{aligned}u(t)=&\frac{1}{\Gamma_q(\alpha)}\int_0^t(t^{\alpha-1}(1-qs)^{(\alpha-1)}-(t-qs)^{(\alpha-1)})[f(s,u(s))+g(s,u(s))]d_qs\\ &+\frac{1}{\Gamma_q(\alpha)}\int_t^1 t^{\alpha-1}(1-qs)^{(\alpha-1)}[f(s,u(s))+g(s,u(s))]d_qs+t^{\alpha-1}\int_0^1 p(s)u(s)d_qs.\end{aligned}$$

Set

$$G_1(t,qs)=\frac{1}{\Gamma_q(\alpha)}\begin{cases}t^{\alpha-1}(1-qs)^{(\alpha-1)}-(t-qs)^{(\alpha-1)},\quad 0\leqslant qs\leqslant t\leqslant 1,\\ t^{\alpha-1}(1-qs)^{(\alpha-1)}, 0\leqslant t\leqslant qs\leqslant 1,\end{cases}$$

then, we have

$$\int_0^1 p(s)u(s)d_qs=\int_0^1\int_0^1 p(s)G_1(s,\tau)d_qs[f(\tau,u(\tau))+g(\tau,u(\tau))]d_q\tau$$

$$+\int_0^1 s^{\alpha-1}p(s)d_qs\int_0^1 p(\tau)u(\tau)d_q\tau.$$

Therefore, it follows that

$$\int_0^1 p(s)u(s)d_qs = \frac{1}{1-\sigma_2}\int_0^1\int_0^1 G_1(s,\tau)p(s)d_qs[f(\tau,u(\tau))+g(\tau,u(\tau))]d_q\tau.$$

Set

$$G_2(t,qs) = \frac{t^{\alpha-1}}{1-\sigma_2}\int_0^1 G_1(\tau,qs)p(\tau)d_q\tau.$$

Finally, in order to solve the problem (1.1), it is sufficient to find the solution of the following integral equation:

$$u(t) = \int_0^1 G(t,qs)[f(s,u(s))+g(s,u(s))]d_qs,$$

where $G(t,qs) = G_1(t,qs) + G_2(t,qs)$. □

Lemma 2.3 (Lemma 2.4, [22]). *The Green's function $G_1(t,qs)$ defined in Lemma 2.2 satisfies the following properties:*

(1) $G_1(t,qs)$ is continuous on $(t,s)\in[0,1]\times[0,1]$ and $G_1(t,qs)>0$ for each $t,s\in(0,1)$;

(2) $q^{\alpha-1}t^{\alpha-1}(1-t)s(1-qs)^{(\alpha-1)} \leqslant \Gamma_q(\alpha)G_1(t,qs) \leqslant t^{\alpha-1}(1-qs)^{(\alpha-1)}$.

Lemma 2.4. *The Green's function $G(t,qs)$ defined in Lemma 2.2 satisfies the following inequality*

$$\frac{\sigma_1 q^{\alpha-1}s(1-qs)^{(\alpha-1)}t^{\alpha-1}}{(1-\sigma_2)\Gamma_q(\alpha)} \leqslant G(t,qs) \leqslant \frac{t^{\alpha-1}(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)(1-\sigma_2)}.$$

Proof. By Lemma 2.3, we have

$$\begin{aligned}
G(t,qs) &= G_1(t,qs)+G_2(t,qs) \geqslant G_2(t,qs) = \frac{t^{\alpha-1}}{1-\sigma_2}\int_0^1 G_1(\tau,qs)p(\tau)d_q\tau \\
&\geqslant \frac{t^{\alpha-1}}{1-\sigma_2}\int_0^1 \frac{q^{\alpha-1}\tau^{\alpha-1}(1-\tau)s(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}p(\tau)d_q\tau \\
&= \frac{\sigma_1 q^{\alpha-1}s(1-qs)^{(\alpha-1)}t^{\alpha-1}}{(1-\sigma_2)\Gamma_q(\alpha)}.
\end{aligned}$$

On the other hand, from the expression of $G_1(t,qs)$, it is obvious that

$$\begin{aligned}
G(t,qs) &= G_1(t,qs)+G_2(t,qs) \\
&\leqslant \frac{t^{\alpha-1}(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} + \frac{t^{\alpha-1}}{1-\sigma_2}\int_0^1 \frac{\tau^{\alpha-1}(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}p(\tau)d_q\tau \\
&= \frac{t^{\alpha-1}(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)(1-\sigma_2)}.
\end{aligned}$$

□

Definition 2.4 (Guo [15]). Let E be a real Banach space. A nonempty convex closed set P is called a cone provided that: (1) $au \in P$, for all $u \in P; a \geqslant 0$; (2) $u, -u \in P$ implies $u = 0$.

For all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leqslant y \leqslant \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h > \theta$(i.e., $h \geqslant \theta$ and $h \neq \theta$), we denote by P_h the set $P_h = \{x \in E : x \sim h\}$. It is easy to see that $P_h \subset P$.

Definition 2.5 (Guo [15]). Let γ be a real number with $0 < \gamma < 1$. An operator $A : P \to P$ is said to be γ-concave if it satisfies $A(tx) > t^\gamma Ax$ for all $t \in (0, 1), x \in P$. An operator $A : E \to E$ is said to be homogeneous if it satisfies $A(tx) = tAx$ for all $t > 0, x \in E$. An operator $A : P \to P$ is said to be sub-homogeneous if it satisfies $A(tx) \geqslant tAx$ for all $t > 0, x \in P$.

Theorem 2.1 (Theorem 2.2, [21]). *Let P be a normal cone in a real Banach space $E, A : P \to P$ be an increasing γ-concave operator, and $B : P \to P$ be an increasing sub-homogeneous operator. Assume that*

(i) there is $h > \theta$ such that $Ah \in P_h$ and $Bh \in P_h$;

(ii) there exists a constant $\delta_0 > 0$ such that $Ax \geqslant \delta_0 Bx$ for all $x \in P$. Then the operator equation $Ax + Bx = x$ has a unique solution x^* in P_h. Moreover, constructing successively the sequence $y_n = Ay_{n-1} + By_{n-1}, n = 1, 2, \cdots,$ for any initial value $y_0 \in P_h$, we have $y_n \to x^*$ as $n \to \infty$.

Theorem 2.2 (Theorem 2.1, [20]). *Let P be a normal cone a real Banach space $E, A : P \to P$ be an increasing operator, and $B : P \to P$ be a decreasing operator. Assume that:*

(i) for any $x \in P$ and $t \in (0, 1)$, there exist $\varphi_i(t) \in (t, 1)(i = 1, 2)$ such that

$$A(tx) \geqslant \varphi_1(t)Ax, B(tx) \leqslant \frac{1}{\varphi_2(t)}Bx;$$

(ii) there exists $h_0 \in P_h$ such that $Ah_0 + Bh_0 \in P_h$. Then the operator equation $Ax+Bx = x$ has a unique solution x^* in P_h. Moreover, for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences

$$x_n = Ax_{n-1} + By_{n-1}, y_n = Ay_{n-1} + Bx_{n-1}; n = 1, 2, \cdots ,$$

we have $x_n \to x^*, y_n \to x^*$ as $n \to \infty$.

Remark 2.1. When B is a null operator, Theorems 2.1, 2.2 also hold.

3. Main results and proofs

Consider the Banach space $E = C[0, 1]$ with the norm $\|x\| = \sup\{|x(t)| : t \in [0, 1]\}$. Define the cone on E: $P = \{x \in E : x(t) \geqslant 0, t \in [0, 1]\}$, then P is a normal cone in E and the normality constant is 1. E can be equipped with a partial order given by

$$x \leqslant y, \quad x, y \in E \Leftrightarrow x(t) \leqslant y(t), t \in [0, 1].$$

In order to solve the q-analogue of the fractional differential problem (1.1), it is sufficient to find positive solutions of the following integral equation:

$$u(t) := \int_0^1 G(t,qs)[f(s,u(s)) + g(s,u(s))]d_qs.$$

Define operators $A : P \to E$ and $B : P \to E$:

$$Au(t) := \int_0^1 G(t,qs)f(s,u(s))d_qs, \quad Bu(t) := \int_0^1 G(t,qs)g(s,u(s))d_qs,$$

then it is easy to see that u is the solution of problem (1.1) if and only if $u = Au+Bu$.

Theorem 3.1. *Suppose (A) and*

(H1) $f, g : [0,1] \times [0,+\infty) \to [0,+\infty)$ *are continuous and increasing with respect to the second argument,* $g(t,0) \neq 0$;

(H2) $g(t,\lambda x) \geqslant \lambda g(t,x)$ *for* $\lambda \in (0,1), t \in [0,1], x \in [0,+\infty)$, *and there exists a constant* $\gamma \in (0,1)$ *such that* $f(t,\lambda x) \geqslant \lambda^\gamma f(t,x)$ *for all* $t \in [0,1], \lambda \in (0,1), x \in [0,+\infty)$*;*

(H3) there exists a constant $\delta_0 > 0$ *such that* $f(t,x) \geqslant \delta_0 g(t,x), t \in [0,1], x \geqslant 0$.

Then problem (1.1) has a unique positive solution u^* *in* P_h, *where* $h(t) = t^{\alpha-1}, t \in [0,1]$. *Meanwhile, for any initial value* $u_0 \in P_h$, *constructing successively the sequence*

$$u_{n+1}(t) := \int_0^1 G(t,qs)[f(s,u_n(s)) + g(s,u_n(s))]d_qs,$$

we can get $u_n(t) \to u^*(t)$ *as* $n \to \infty$, *where* $G(t,qs)$ *is given in (2.1).*

Proof. First, from (H1) and Lemma 2.2, for $u, v \in P$, with $u \geqslant v$

$$Au(t) = \int_0^1 G(t,qs)f(s,u(s))d_qs \geqslant \int_0^1 G(t,qs)f(s,v(s))d_qs = Av(t),$$

then we have $Au \geqslant Av$. Similarly, $Bu \geqslant Bv$.

Second, from (H2), for $\gamma \in (0,1)$ and $u \in P$, we have

$$A(\lambda u)(t) = \int_0^1 G(t,qs)f(s,\lambda u(s))d_qs \geqslant \int_0^1 G(t,qs)\lambda^\gamma f(s,u(s))d_qs = \lambda^\gamma Au(t),$$
$$B(\lambda u)(t) = \int_0^1 G(t,qs)g(s,\lambda u(s))d_qs \geqslant \int_0^1 G(t,qs)\lambda g(s,u(s))d_qs = \lambda Bu(t),$$

then we have $A(\lambda u) \geqslant \lambda^\gamma Au, B(\lambda u) \geqslant \lambda Bu$. So the operator A is a γ-concave operator and the operator B is sub-homogeneous.

Third, from (H1) and Lemma 2.4, for $h(t) = t^{\alpha-1}, t \in [0,1]$,

$$Ah(t) = \int_0^1 G(t,qs)f(s,s^{\alpha-1})d_qs \leqslant \frac{t^{\alpha-1}}{\Gamma_q(\alpha)(1-\sigma_2)} \int_0^1 (1-qs)^{(\alpha-1)} f(s,1)d_qs,$$
$$Ah(t) = \int_0^1 G(t,qs)f(s,s^{\alpha-1})d_qs \geqslant \frac{\sigma_1 q^{\alpha-1} t^{\alpha-1}}{(1-\sigma_2)\Gamma_q(\alpha)} \int_0^1 s(1-qs)^{(\alpha-1)} f(s,0)d_qs.$$

In view of (H1) and (H3), it is clear that $f(s,1) \geqslant f(s,0) \geqslant \delta_0 g(t,0)$. Then we have

$$\begin{aligned}\int_0^1 (1-qs)^{(\alpha-1)} f(s,1) d_q s &\geqslant \int_0^1 s(1-qs)^{(\alpha-1)} f(s,0) d_q s\\ &\geqslant \delta_0 \int_0^1 s(1-qs)^{(\alpha-1)} g(s,0) d_q s > 0.\end{aligned}$$

Set $l_1 = \frac{1}{\Gamma_q(\alpha)(1-\sigma_2)} \int_0^1 (1-qs)^{(\alpha-1)} f(s,1) d_q s, l_2 = \frac{\sigma_1 q^{\alpha-1}}{(1-\sigma_2)\Gamma_q(\alpha)} \int_0^1 s(1-qs)^{(\alpha-1)} f(s,0) d_q s$, then $l_2 h(t) \leqslant Ah(t) \leqslant l_1 h(t), t \in [0,1]$. Therefore, $Ah \in P_h$. Similarly, $Bh \in P_h$.

Next, from (H3), it is easy to see that $Au \geqslant \delta_0 Bu, u \in P$. Obviously, all the conditions of Theorem 2.1 are satisfied. Consequently, the problem (1.1) has a unique positive solution u^* in P_h by Theorem 2.1, where $h(t) = t^{\alpha-1}, t \in [0,1]$. And, for any initial value $u_0 \in P_h$, constructing successively the sequence

$$u_{n+1}(t) := \int_0^1 G(t,qs)[f(s,u_n(s)) + g(s,u_n(s))] d_q s,$$

we can obtain $u_n(t) \to u^*(t)$ as $n \to \infty$. □

Corollary 3.1. *Suppose (A) and*

(H1′) $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ *is continuous and increasing with respect to the second argument,* $f(t,0) \neq 0$;

(H2′) there exists a constant $\gamma \in (0,1)$ *such that* $f(t,\lambda x) \geqslant \lambda^\gamma f(t,x)$ *for all* $t \in [0,1], \lambda \in (0,1), x \in [0,+\infty)$.

Then, the following problem

$$\begin{cases} (D_q^\alpha u)(t) = -f(t,u(t)), 0 < t < 1, \\ u(0) = (D_q u)(0) = 0, u(1) = \int_0^1 p(s)u(s) d_q s \end{cases} \tag{3.1}$$

has a unique positive solution u^* *in* P_h*, where* $h(t) = t^{\alpha-1}, t \in [0,1]$*. Meanwhile, for any initial value* $u_0 \in P_h$*, constructing successively the sequence*

$$u_{n+1}(t) := \int_0^1 G(t,qs) f(s,u_n(s)) d_q s,$$

we can get $u_n(t) \to u^*(t)$ *as* $n \to \infty$*, where* $G(t,qs)$ *is given in (2.1).*

Proof. From Remark 1 and Theorem 3.1 the conclusions hold. □

Theorem 3.2. *Suppose (A) and*

(H4) $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ *are continuous and increasing with respect to the second argument, and* $f(t,0) \neq 0$;

(H5) $g : [0,1] \times [0,+\infty) \to [0,+\infty)$ *are continuous and decreasing with respect to the second argument, and* $g(t,1) \neq 0$;

(H6) there exists $\varphi_i(\lambda) \in (\lambda,1)$ *such that*

$$f(t,\lambda x) \geqslant \varphi_1(\lambda) f(t,x), \quad g(t,\lambda x) \leqslant \frac{1}{\varphi_2(\lambda)} g(t,x)$$

for all $t \in [0,1], \lambda \in (0,1), x \in [0,+\infty)$.

Then problem (1.1) has a unique positive solution u^* in P_h, where $h(t) = t^{\alpha-1}, t \in [0,1]$. Meanwhile, for any initial value $x_0, y_0 \in P_h$, constructing successively the sequence

$$x_{n+1}(t) := \int_0^1 G(t,qs)[f(s,x_n(s)) + g(s,y_n(s))]d_qs,$$
$$y_{n+1}(t) := \int_0^1 G(t,qs)[f(s,y_n(s)) + g(s,x_n(s))]d_qs,$$

we can get $x_n(t) \to u^*(t), y_n(t) \to u^*(t)$ as $n \to \infty$, where $G(t,qs)$ is given in (2.1).

Proof. First, from (H4) and (H5), $A: P \to P$ be an increasing operator, and $B: P \to P$ be a decreasing operator. Second, from (H6), (ii) in Theorem 3.1 hold. Finally, for $h(t) = t^{\alpha-1}, t \in [0,1]$,

$$Ah(t) + Bh(t) \leqslant \frac{t^{\alpha-1}}{\Gamma_q(\alpha)(1-\sigma_2)} \int_0^1 (1-qs)^{(\alpha-1)}[f(s,1) + g(s,0)]d_qs,$$
$$Ah(t) + Bh(t) \geqslant \frac{\sigma_1 q^{\alpha-1} t^{\alpha-1}}{(1-\sigma_2)\Gamma_q(\alpha)} \int_0^1 s(1-qs)^{(\alpha-1)}[f(s,0) + g(s,1)]d_qs.$$

In view of (H4) and (H5), it is clear that $f(s,1) + g(s,0) \geqslant f(s,0) + g(s,1) \geqslant 0$. Then we have

$$\int_0^1 (1-qs)^{(\alpha-1)}[f(s,1) + g(s,0)]d_qs \geqslant \int_0^1 s(1-qs)^{(\alpha-1)}[f(s,0) + g(s,1)]d_qs \geqslant 0.$$

Since $f(s,0) + g(s,1) \neq 0$, then

$$\int_0^1 s(1-qs)^{(\alpha-1)}[f(s,0) + g(s,1)]d_qs > 0.$$

Set $l_3 = \frac{1}{\Gamma_q(\alpha)(1-\sigma_2)} \int_0^1 (1-qs)^{(\alpha-1)}[f(s,1) + g(s,0)]d_qs$, $\quad l_4 = \frac{\sigma_1 q^{\alpha-1}}{(1-\sigma_2)\Gamma_q(\alpha)} \int_0^1 s(1-qs)^{(\alpha-1)}[f(s,0) + g(s,1)]d_qs$, then $l_4 h(t) \leqslant Ah(t) + Bh(t) \leqslant l_3 h(t), t \in [0,1]$. Therefore, it is clear that $Ah + Bh \in P_h$. Hence, by Theorem 2.2, we obtain that problem (1.1) has a unique positive solution u^* in P_h, where $h(t) = t^{\alpha-1}, t \in [0,1]$. Meanwhile, for any initial value $x_0, y_0 \in P_h$, constructing successively the sequence

$$x_{n+1}(t) := \int_0^1 G(t,qs)[f(s,x_n(s)) + g(s,y_n(s))]d_qs,$$
$$y_{n+1}(t) := \int_0^1 G(t,qs)[f(s,y_n(s)) + g(s,x_n(s))]d_qs,$$

we can get $x_n(t) \to u^*(t), y_n(t) \to u^*(t)$ as $n \to \infty$. □

Corollary 3.2. *Suppose (A), (H1′) and*

(H6′) there exists $\varphi(\lambda) \in (\lambda, 1)$ such that $f(t, \lambda x) \geqslant \varphi(\lambda) f(t,x)$, for all $t \in [0,1], \lambda \in (0,1), x \in [0, +\infty)$.

Then problem (3.1) has a unique positive solution u^* in P_h, where $h(t) = t^{\alpha-1}, t \in [0,1]$. Meanwhile, for any initial value $u_0 \in P_h$, constructing successively the sequence

$$u_{n+1}(t) := \int_0^1 G(t,qs) f(s,u_n(s))d_qs,$$

we can get $u_n(t) \to u^*(t)$ *as* $n \to \infty$*, where* $G(t, qs)$ *is given in (2.1).*

Proof. From Remark 1 and Theorem 3.2 the conclusions hold. □

4. Example

Example 4.1. Consider the following boundary value problem

$$\begin{cases} -(D_{\frac{1}{2}}^{\frac{5}{2}}u)(t) = 2u^{\frac{1}{2}} + ue^t + 2t + 1, \quad 0 < t < 1, \\ u(0) = u'(0) = 0, \ u(1) = \int_0^1 \sqrt{s}u(s)d_{\frac{1}{2}}s. \end{cases} \tag{4.1}$$

Let $p(t) = \sqrt{t}$, $f(t, u) = 2u^{\frac{1}{2}} + t + 1$, $g(t, u) = ue^t + t$ and $\gamma = \frac{1}{2}$, then $p : [0, 1] \to [0, +\infty)$ with $p \in L^1[0, 1]$ and $\sigma_1 = \int_0^1 \tau^{\frac{3}{2}}(1 - \tau)\sqrt{\tau}d_{\frac{1}{2}}\tau = \frac{4}{105} > 0$, $\sigma_2 = \int_0^1 \tau^{\frac{3}{2}}\sqrt{\tau}d_{\frac{1}{2}}\tau = \frac{4}{7} < 1$. Obviously, $f, g : [0, 1] \times [0, +\infty) \to [0, +\infty)$ are continuous and increasing with respect to the second argument, $g(t, 0) \neq 0$. On the one hand, for $\lambda \in (0, 1), t \in [0, 1], u \in [0, +\infty)$,

$$g(t, \lambda u) = \lambda ue^t + t \geqslant \lambda g(t, u)$$

and

$$f(t, \lambda u) = 2\lambda^{\frac{1}{2}}u^{\frac{1}{2}} + t + 1 \geqslant \lambda^{\frac{1}{2}}f(t, u).$$

On the other hand, there exists a constant $0 < \delta_0 < \frac{1}{2}$ such that $f(t, u) \geqslant \delta_0 g(t, u), t \in [0, 1], u \geqslant 0$. By Theorem 3.1, it is easy to know that problem (4.1) has a unique positive solution in P_h, where $h(t) = t^{\frac{3}{2}}, t \in [0, 1]$.

Example 4.2. Consider the following boundary value problem

$$\begin{cases} -(D_{\frac{1}{2}}^{\frac{5}{2}}u)(t) = 2u^{\frac{1}{2}} + \frac{e^t}{u} + 2t + 1, \quad 0 < t < 1, \\ u(0) = u'(0) = 0, \ u(1) = \int_0^1 \sqrt{s}u(s)d_{\frac{1}{2}}s. \end{cases} \tag{4.2}$$

Let $p(t) = \sqrt{t}$, $f(t, u) = 2u^{\frac{1}{2}} + t + 1$, $g(t, u) = \frac{e^t}{u^{\frac{1}{2}}} + t$ and $\gamma = \frac{1}{2}$. Similar to Example 4.1, the condition (A) is satisfied. Obviously, $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ are continuous and increasing with respect to the second argument, $f(t, 0) = t + 1 \neq 0$, $g : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is continuous and decreasing with respect to the second argument, $g(t, 1) = e^t + t \neq 0$.

Let $\varphi_1(\lambda) = \lambda^{\frac{1}{2}}, \varphi_2(\lambda) = \lambda^{\frac{1}{2}}$, then $\varphi_1(\lambda), \varphi_2(\lambda) \in (\lambda, 1)$ For $\lambda \in (0, 1), t \in [0, 1], u \in [0, +\infty)$,

$$g(t, \lambda u) = \frac{e^t}{\lambda^{\frac{1}{2}}u^{\frac{1}{2}}} + t \leqslant \lambda^{\frac{1}{2}}g(t, u) = \varphi_2(\lambda)g(t, u),$$

and

$$f(t, \lambda u) = 2\lambda^{\frac{1}{2}}u^{\frac{1}{2}} + t + 1 \geqslant \frac{1}{\lambda^{\frac{1}{2}}}f(t, u) = \frac{1}{\varphi_1(\lambda)}f(t, u).$$

Then by Theorem 3.2, problem (4.2) has a unique positive solution in P_h, where $h(t) = t^{\frac{3}{2}}$, $t \in [0, 1]$.

5. Conclusions

We have discussed the existence and uniqueness of positive solutions for a class of nonlinear fractional q-difference equations with integral boundary conditions by two fixed point theorems of a sum operator in partial ordering Banach space. We also present sequences approximating a unique solution to the given problem. In particular, we skillfully use a sum operator to solve the inconsistency of monotonicity of nonlinear terms. In other words, our results do not require super-linearity, sublinearity or boundness of nonlinear terms. We yield several new results to boundary value problems of fractional q-difference equations.

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