Journal of Applied Analysis and Computation Website:http://jaac.ijournal.cn
Volume 9, Number 5, October 2019, 1973–1986 DOI:10.11948/20190053

PARAMETRIC MULHOLLAND-TYPE INEQUALITIES*

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Abstract By means of the weight functions and the idea of introducing parameters, a discrete Mulholland-type inequality with the general homogeneous kernel and the equivalent form are given. The equivalent statements of the best possible constant factor related to some parameters, the operator expressions and some particular examples are considered.

Keywords Weight function, Mulholland-type inequality, equivalent statement, parameter, operator expression.

MSC(2010) 26D15.

1. Introduction

Assuming that $0 < \sum_{m=1}^{\infty} a_m^2 < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, we have the following discrete Hilbert's inequality with the best possible constant factor π (cf [3], Theorem 315):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \pi\left(\sum_{m=1}^{\infty} a_m^2 \sum_{n=1}^{\infty} b_n^2\right)^{\frac{1}{2}}. \tag{1.1}$$

We still have the following Mulholland's inequality with the same best possible constant factor π (cf. [3], Theorem 343):

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{a_m b_n}{\ln mn} < \pi\left(\sum_{m=2}^{\infty} m a_m^2 \sum_{n=2}^{\infty} n b_n^2\right)^{\frac{1}{2}}, \tag{1.2}$$

If $0 < \int_0^{\infty} f^2(x)dx < \infty$ and $0 < \int_0^{\infty} g^2(y)dy < \infty$, then we have the following Hilbert's integral inequality:

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}dxdy < \pi\left(\int_0^{\infty} f^2(x)dx \int_0^{\infty} g^2(y)dy\right)^{\frac{1}{2}}, \tag{1.3}$$

with the best possible constant factor π (cf. [3], Theorem 316).

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* The authors were supported by National Natural Science Foundation of China (No. 61772140), National Science Foundation of Jishou University (No. Jd16012) and Science and Technology Planning Project Item of Guangzhou City (No. 201707010229).

Inequalities (1.1), (1.2), (1.3) and their extensions by introducing independent parameters and conjugate exponents $(p,q)(p > 1, \frac{1}{p} + \frac{1}{q} = 1)$ are important in analysis and its applications (cf. [1,2,15–17,19–22,26]).

The following half-discrete Hilbert-type inequality was provided (cf. [3], Theorem 351): If $K(x)(x>0)$ is decreasing, $p>1, \frac{1}{p}+\frac{1}{q}=1, 0<\phi(s)=\int_0^\infty K(x)x^{s-1}dx<\infty$, then

$$\int_0^\infty x^{p-2}(\sum_{n=1}^\infty K(nx)a_n)^p dx < \phi^p(\frac{1}{p})\sum_{n=1}^\infty a_n^p. \tag{1.4}$$

In the last ten years, some new extensions of (1.4) with their applications were provided by [4,12–14,23] and [24].

In 2016, by the use of the technique of real analysis, Hong [5] considered some equivalent statements of the extensions of (1.1) with the best possible constant factor related to a few parameters. The other similar works on the extensions of (1.3) and (1.4) were given by [6–8,18] and [9].

In this paper, following the way of [5], by the use of the weight functions and the idea of introducing parameters, a discrete Mulholland-type inequality with the general homogeneous kernel and the equivalent form are given, which are extension of (1.2). The equivalent statements of the best possible constant factor related to a few parameters, the operator expressions and some particular examples are also considered.

2. Some lemmas

In what follows, we suppose that $p>1, \frac{1}{p}+\frac{1}{q}=1, \alpha,\beta\in(0,1], \lambda\in\mathbf{R}$,$\lambda_1, \lambda-\lambda_2 \leq \frac{1}{\alpha}, \lambda_2\ \lambda-\lambda_1 \leq \frac{1}{\beta}, k_\lambda(x,y)$ is a positive homogeneous function of degree $-\lambda$, satisfying for any $u,x,y>0$,

$$k_\lambda(ux,uy)=u^{-\lambda}k_\lambda(x,y).$$

Also, $k_\lambda(x,y)$ is decreasing with respect to $x,y>0$ (or

$$\frac{\partial}{\partial x}k_\lambda(x,y)\leq 0, \frac{\partial}{\partial y}k_\lambda(x,y)\leq 0(x,y>0)),$$

such that for any $\gamma=\lambda_1,\lambda-\lambda_2$,

$$k_\lambda(\gamma):=\int_0^\infty k_\lambda(u,1)u^{\gamma-1}du\in\mathbf{R}_+=(0,\infty). \tag{2.1}$$

In this paper, we still assume that $a_m, b_n \geq 0$ $(m,n\in\mathbf{N}\backslash\{1\}=\{2,3,\cdots\})$, such that

$$0<\sum_{m=2}^\infty \frac{\ln^{p[1-\alpha(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}m}{m^{1-p}}a_m^p<\infty, \text{ and}$$
$$0<\sum_{n=2}^\infty \frac{\ln^{q[1-\beta(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})]-1}n}{n^{1-q}}b_n^q<\infty.$$

Definition 2.1. We define the following weight functions:

$$\omega_\lambda(\lambda_2, m) := \ln^{\alpha(\lambda-\lambda_2)} m \sum_{n=2}^{\infty} k_\lambda(\ln^\alpha m, \ln^\beta n) \frac{\ln^{\beta\lambda_2 - 1} n}{n} \ (m \in \mathbf{N}\backslash\{1\}), \quad (2.2)$$

$$\varpi_\lambda(\lambda_1, n) := \ln^{\beta(\lambda-\lambda_1)} n \sum_{m=2}^{\infty} k_\lambda(\ln^\alpha m, \ln^\beta n) \frac{\ln^{\alpha\lambda_1 - 1} m}{m} \ (n \in \mathbf{N}\backslash\{1\}). \quad (2.3)$$

Lemma 2.1. *We have the following inequalities:*

$$\omega_\lambda(\lambda_2, m) < \frac{1}{\beta} k_\lambda(\lambda - \lambda_2) \ (m \in \mathbf{N}\backslash\{1\}), \quad (2.4)$$

$$\varpi_\lambda(\lambda_1, n) < \frac{1}{\alpha} k_\lambda(\lambda_1) \ (n \in \mathbf{N}\backslash\{1\}). \quad (2.5)$$

Proof. For $\beta\lambda_2 - 1 \leq 0$, it is evident that $k_\lambda(\ln^\alpha m, \ln^\beta t)\frac{\ln^{\beta\lambda_2-1} t}{t}$ is a strictly decreasing function with respect to $t > 0$. By the decreasing property, setting $u = \frac{\ln^\alpha m}{\ln^\beta t}$, we find that

$$\begin{aligned}\omega_\lambda(\lambda_2, m) &< \ln^{\alpha(\lambda-\lambda_2)} m \int_1^\infty k_\lambda(\ln^\alpha m, \ln^\beta t) \frac{\ln^{\beta\lambda_2-1} t}{t} dt \\ &= \frac{1}{\beta} \int_0^\infty k_\lambda(u, 1) u^{(\lambda-\lambda_2)-1} du = \frac{1}{\beta} k_\lambda(\lambda - \lambda_2).\end{aligned}$$

Hence, we have (2.4).

For $\alpha\lambda_1 - 1 \leq 0$, it is evident that $k_\lambda(\ln^\alpha t, \ln^\beta n)\frac{\ln^{\alpha\lambda_1-1} t}{t}$ is a strictly decreasing function with respect to $t > 0$. By the decreasing property, setting $u = \frac{\ln^\alpha t}{\ln^\beta n}$, we find that

$$\begin{aligned}\varpi_\lambda(\lambda_1, n) &< \ln^{\beta(\lambda-\lambda_1)} n \int_1^\infty k_\lambda(\ln^\alpha t, \ln^\beta n) \frac{\ln^{\alpha\lambda_1-1} t}{t} dt \\ &= \frac{1}{\alpha} \int_0^\infty k_\lambda(u, 1) u^{\lambda_1-1} du = \frac{1}{\alpha} k_\lambda(\lambda_1).\end{aligned}$$

Hence, we have (2.5). □

Lemma 2.2. *We have the following inequality:*

$$\begin{aligned} I &:= \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} k_\lambda(\ln^\alpha m, \ln^\beta n) a_m b_n \\ &< \frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{p}}(\lambda - \lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1) \left\{ \sum_{m=2}^{\infty} \frac{\ln^{p[1-\alpha(\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q})]-1} m}{m^{1-p}} a_m^p \right\}^{\frac{1}{p}} \\ &\times \left\{ \sum_{n=2}^{\infty} \frac{\ln^{q[1-\beta(\frac{\lambda_2}{p} + \frac{\lambda-\lambda_1}{q})]-1} n}{n^{1-q}} b_n^q \right\}^{\frac{1}{q}}. \end{aligned} \quad (2.6)$$

Proof. By Hölder's inequality with weight (cf. [10]), we obtain

$$I = \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} k_\lambda(\ln^\alpha m, \ln^\beta n) \left[\frac{\ln^{(\beta\lambda_2-1)/p} n}{n^{1/p}} \frac{\ln^{(1-\alpha\lambda_1)/q} m}{m^{-1/q}} a_m \right]$$

$$\times \left[\frac{\ln^{(\alpha\lambda_1-1)/q} m}{m^{1/q}} \frac{\ln^{(1-\beta\lambda_2)/p} n}{n^{-1/p}} b_n \right]$$

$$\leq \left\{ \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} k_\lambda(\ln^\alpha m, \ln^\beta n) \frac{\ln^{\beta\lambda_2-1} n}{n} \frac{\ln^{(p-1)(1-\alpha\lambda_1)} m}{m^{1-p}} a_m^p \right\}^{\frac{1}{p}}$$

$$\times \left\{ \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} k_\lambda(\ln^\alpha m, \ln^\beta n) \frac{\ln^{\alpha\lambda_1-1} m}{m} \frac{\ln^{(q-1)(1-\beta\lambda_2)} n}{n^{1-q}} b_n^q \right\}^{\frac{1}{q}}$$

$$= \left\{ \sum_{m=2}^{\infty} \omega_\lambda(\lambda_2, m) \frac{\ln^{p[1-\alpha(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})} m}{m^{1-p}} a_m^p \right\}^{\frac{1}{p}}$$

$$\times \left\{ \sum_{n=2}^{\infty} \varpi_\lambda(\lambda_1, n) \frac{\ln^{q[1-\beta(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})} n}{n^{1-q}} b_n^q \right\}^{\frac{1}{q}}.$$

Then by (2.4) and (2.5), we have (2.6). □

Remark 2.1. By (10), for $\lambda_1 + \lambda_2 = \lambda$, we find

$$0 < \sum_{m=2}^{\infty} \frac{\ln^{p(1-\alpha\lambda_1)-1} m}{m^{1-p}} a_m^p < \infty \text{ and } 0 < \sum_{n=2}^{\infty} \frac{\ln^{q(1-\beta\lambda_2)-1} n}{n^{1-q}} b_n^q < \infty,$$

and the following inequality:

$$\sum_{n=2}^{\infty} \sum_{m=2}^{\infty} k_\lambda(\ln^\alpha m, \ln^\beta n) a_m b_n$$
$$< \frac{k_\lambda(\lambda_1)}{\beta^{1/p}\alpha^{1/q}} \left[\sum_{m=2}^{\infty} \frac{\ln^{p(1-\alpha\lambda_1)-1} m}{m^{1-p}} a_m^p \right]^{\frac{1}{p}} \left[\sum_{n=2}^{\infty} \frac{\ln^{q(1-\beta\lambda_2)-1} n}{n^{1-q}} b_n^q \right]^{\frac{1}{q}}. \tag{2.7}$$

In particular, for $\alpha = \beta = 1$, we have

$$\sum_{n=2}^{\infty} \sum_{m=2}^{\infty} k_\lambda(\ln m, \ln n) a_m b_n$$
$$< k_\lambda(\lambda_1) \left[\sum_{m=2}^{\infty} \frac{\ln^{p(1-\lambda_1)-1} m}{m^{1-p}} a_m^p \right]^{\frac{1}{p}} \left[\sum_{n=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} n}{n^{1-q}} b_n^q \right]^{\frac{1}{q}}. \tag{2.8}$$

For $p = q = 2, \lambda = 1, k_\lambda(x, y) = \frac{1}{x+y}, \lambda_1 = \lambda_2 = \frac{1}{2}$, (2.8) reduces to (1.2). Hence, (2.7) is an extension of (2.8) and (1.2).

Lemma 2.3. *The constant factor $\frac{k_\lambda(\lambda_1)}{\beta^{1/p}\alpha^{1/q}}$ in (2.7) is the best possible.*

Proof. For any $\varepsilon > 0$, we set

$$\widetilde{a}_m := \frac{\ln^{\alpha(\lambda_1-\frac{\varepsilon}{p})-1} m}{m}, \widetilde{b}_n := \frac{\ln^{\beta(\lambda_2-\frac{\varepsilon}{q})-1} n}{n} \ (m, n \in \mathbf{N}\backslash\{1\}).$$

If there exists a constant $M \leq \frac{k_\lambda(\lambda_1)}{\beta^{1/p}\alpha^{1/q}}$, such that (2.7) is valid when replacing $\frac{k_\lambda(\lambda_1)}{\beta^{1/p}\alpha^{1/q}}$ by M, then in particular, we have

$$\widetilde{I} \ : \ = \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} k_\lambda(\ln^\alpha m, \ln^\beta n) \widetilde{a}_m \widetilde{b}_n$$

$$< M\left[\sum_{m=2}^{\infty}\frac{\ln^{p(1-\alpha\lambda_1)-1}m}{m^{1-p}}\widetilde{a}_m^p\right]^{\frac{1}{p}}\left[\sum_{n=2}^{\infty}\frac{\ln^{q(1-\beta\lambda_2)-1}n}{n^{1-q}}\widetilde{b}_n^q\right]^{\frac{1}{q}}.$$

We obtain

$$\begin{aligned}
\widetilde{I} &< M\left[\sum_{m=2}^{\infty}\frac{\ln^{p(1-\alpha\lambda_1)-1}m}{m^{1-p}}\frac{\ln^{p\alpha(\lambda_1-\frac{\varepsilon}{p})-p}m}{m^p}\right]^{\frac{1}{p}}\\
&\quad\times\left[\sum_{n=2}^{\infty}\frac{\ln^{q(1-\beta\lambda_2)-1}n}{n^{1-q}}\frac{\ln^{q\beta(\lambda_2-\frac{\varepsilon}{q})-q}n}{n^q}\right]^{\frac{1}{q}}\\
&= M\left(\frac{\ln^{-\alpha\varepsilon-1}2}{2}+\sum_{m=3}^{\infty}\frac{\ln^{-\alpha\varepsilon-1}m}{m}\right)^{\frac{1}{p}}\left(\frac{\ln^{-\beta\varepsilon-1}2}{2}+\sum_{n=3}^{\infty}\frac{\ln^{-\beta\varepsilon-1}n}{n}\right)^{\frac{1}{q}}\\
&< M\left(\frac{\ln^{-\alpha\varepsilon-1}2}{2}+\int_2^{\infty}\frac{\ln^{-\alpha\varepsilon-1}t}{t}dt\right)^{\frac{1}{p}}\left(\frac{\ln^{-\beta\varepsilon-1}2}{2}+\int_2^{\infty}\frac{\ln^{-\beta\varepsilon-1}t}{t}dt\right)^{\frac{1}{q}}\\
&= \frac{M}{\varepsilon}\left(\frac{\varepsilon\ln^{-\alpha\varepsilon-1}2}{2}+\frac{\ln^{-\alpha\varepsilon}2}{\alpha}\right)^{\frac{1}{p}}\left(\frac{\ln^{-\beta\varepsilon-1}2}{2}+\frac{\ln^{-\beta\varepsilon}2}{\beta}\right)^{\frac{1}{q}}.
\end{aligned}$$

By the decreasing property and the Fubini theorem (cf. [11]), we find

$$\begin{aligned}
\widetilde{I} &= \sum_{n=2}^{\infty}\sum_{m=2}^{\infty}k_\lambda(\ln^\alpha m,\ln^\beta n)\frac{\ln^{\alpha\lambda_1-1}m}{m\ln^{\frac{\alpha\varepsilon}{p}}m}\cdot\frac{\ln^{\beta\lambda_2-1}n}{n\ln^{\frac{\beta\varepsilon}{q}}n}\\
&\geq \int_2^{\infty}\left[k_\lambda(\ln^\alpha x,\ln^\beta y)\frac{\ln^{\alpha\lambda_1-1}x}{x\ln^{\frac{\alpha\varepsilon}{p}}x}\cdot\frac{\ln^{\beta\lambda_2-1}y}{y\ln^{\frac{\beta\varepsilon}{q}}y}dx\right]dy\ \left(u=\frac{\ln^\alpha x}{\ln^\beta y}\right)\\
&= \frac{1}{\alpha}\int_2^{\infty}\frac{\ln^{-\beta\varepsilon-1}y}{y}\left(\int_{\frac{\ln^\alpha 2}{\ln^\beta y}}^{\infty}k_\lambda(u,1)u^{\lambda_1-\frac{\varepsilon}{p}-1}du\right)dy\\
&= \frac{1}{\alpha}\int_2^{\infty}\frac{\ln^{-\beta\varepsilon-1}y}{y}\left(\int_{\frac{\ln^\alpha 2}{\ln^\beta y}}^{1}k_\lambda(u,1)u^{\lambda_1-\frac{\varepsilon}{p}-1}du\right)dy\\
&\quad+\frac{1}{\alpha}\int_2^{\infty}\frac{\ln^{-\beta\varepsilon-1}y}{y}\left(\int_1^{\infty}k_\lambda(u,1)u^{\lambda_1-\frac{\varepsilon}{p}-1}du\right)dy\\
&= \frac{1}{\alpha}\int_0^1\left[\int_{\eta+e^{(u^{-1}\ln^\alpha 2)^{1/\beta}}}^{\infty}\frac{\ln^{-\beta\varepsilon-1}y}{y}dy\right]k_\lambda(u,1)u^{\lambda_1-\frac{\varepsilon}{p}-1}du\\
&\quad+\frac{1}{\alpha\beta\varepsilon\ln^{\beta\varepsilon}2}\int_1^{\infty}k_\lambda(u,1)u^{\lambda_1-\frac{\varepsilon}{p}-1}du\\
&= \frac{1}{\alpha\beta\varepsilon}\left(\frac{1}{\ln^{\alpha\varepsilon}2}\int_0^1 k_\lambda(u,1)u^{\lambda_1+\frac{\varepsilon}{q}-1}du+\frac{1}{\ln^{\beta\varepsilon}2}\int_1^{\infty}k_\lambda(u,1)u^{\lambda_1-\frac{\varepsilon}{p}-1}du\right).
\end{aligned}$$

Then we have

$$\frac{1}{\alpha\beta}\left(\frac{1}{\ln^{\alpha\varepsilon}2}\int_0^1 k_\lambda(u,1)u^{\lambda_1+\frac{\varepsilon}{q}-1}du+\frac{1}{\ln^{\beta\varepsilon}2}\int_1^{\infty}k_\lambda(u,1)u^{\lambda_1-\frac{\varepsilon}{p}-1}du\right)$$

$$\leq \varepsilon \widetilde{I} < M \left(\frac{\varepsilon \ln^{-\alpha\varepsilon-1} 2}{2} + \frac{\ln^{-\alpha\varepsilon} 2}{\alpha} \right)^{\frac{1}{p}} \left(\frac{\ln^{-\beta\varepsilon-1} 2}{2} + \frac{\ln^{-\beta\varepsilon} 2}{\beta} \right)^{\frac{1}{q}}.$$

For $\varepsilon \to 0^+$, by Fatou lemma (cf. [11]), we find

$$\begin{aligned}
\frac{1}{\alpha\beta} k_\lambda(\lambda_1) &= \frac{1}{\alpha\beta} \left(\varliminf_{\varepsilon\to 0^+} \frac{1}{\ln^{\alpha\varepsilon} 2} \int_0^1 k_\lambda(u,1) u^{\lambda_1 + \frac{\varepsilon}{q} - 1} du \right. \\
&\left. + \varliminf_{\varepsilon\to 0^+} \frac{1}{\ln^{\beta\varepsilon} 2} \int_1^\infty k_\lambda(u,1) u^{\lambda_1 - \frac{\varepsilon}{p} - 1} du \right) \\
&\leq \frac{1}{\alpha\beta} \varliminf_{\varepsilon\to 0^+} \left(\frac{1}{\ln^{\alpha\varepsilon} 2} \int_0^1 k_\lambda(u,1) u^{\lambda_1 + \frac{\varepsilon}{q} - 1} du \right. \\
&\left. + \frac{1}{\ln^{\beta\varepsilon} 2} \int_1^\infty k_\lambda(u,1) u^{\lambda_1 - \frac{\varepsilon}{p} - 1} du \right) \\
&\leq M \varliminf_{\varepsilon\to 0^+} \left(\frac{\varepsilon \ln^{-\alpha\varepsilon-1} 2}{2} + \frac{\ln^{-\alpha\varepsilon} 2}{\alpha} \right)^{\frac{1}{p}} \left(\frac{\ln^{-\beta\varepsilon-1} 2}{2} + \frac{\ln^{-\beta\varepsilon} 2}{\beta} \right)^{\frac{1}{q}} \\
&= M \left(\frac{1}{\alpha} \right)^{\frac{1}{p}} \left(\frac{1}{\beta} \right)^{\frac{1}{q}},
\end{aligned}$$

namely, $\frac{k_\lambda(\lambda_1)}{\beta^{1/p}\alpha^{1/q}} \leq M$.

Hence, $M = \frac{k_\lambda(\lambda_1)}{\beta^{1/p}\alpha^{1/q}}$ is the best possible constant factor of (2.7). □

Remark 2.2. Setting $\widehat{\lambda}_1 := \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}$, $\widehat{\lambda}_2 := \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p}$, we find

$$\begin{aligned}
\widehat{\lambda}_1 + \widehat{\lambda}_2 &= \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q} + \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p} \\
&= \frac{\lambda}{p} + \frac{\lambda}{q} = \lambda, \\
\widehat{\lambda}_1 &\leq \frac{1}{p\alpha} + \frac{1}{q\alpha} = \frac{1}{\alpha}, \widehat{\lambda}_2 \leq \frac{1}{q\beta} + \frac{1}{p\beta},
\end{aligned}$$

and by Hölder's inequality (cf. [10]), we obtain

$$\begin{aligned}
k_\lambda(\lambda - \widehat{\lambda}_2) &= k_\lambda(\widehat{\lambda}_1) = k_\lambda(\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}) \\
&= \int_0^\infty k_\lambda(u,1) u^{\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q} - 1} du \\
&= \int_0^\infty k_\lambda(u,1) (u^{\frac{\lambda-\lambda_2-1}{p}})(u^{\frac{\lambda_1-1}{q}}) du \\
&\leq \left(\int_0^\infty k_\lambda(u,1) u^{\lambda-\lambda_2-1} du \right)^{\frac{1}{p}} \left(\int_0^\infty k_\lambda(u,1) u^{\lambda_1-1} du \right)^{\frac{1}{q}} \\
&= k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1) < \infty. \qquad (2.9)
\end{aligned}$$

We can rewrite (2.6) as follows:

$$I < \frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1) \left\{ \sum_{m=2}^\infty \frac{\ln^{p(1-\alpha\widehat{\lambda}_1)-1} m}{m^{1-p}} a_m^p \right\}^{\frac{1}{p}}$$

$$\times \left\{ \sum_{n=2}^{\infty} \frac{\ln^{q(1-\beta\widehat{\lambda}_2)-1} n}{n^{1-q}} b_n^q \right\}^{\frac{1}{q}}. \tag{2.10}$$

Lemma 2.4. *If the constant factor* $\frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1)$ *in (2.6) is the best possible, then* $\lambda_1+\lambda_2=\lambda$.

Proof. If the constant factor $\frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1)$ in (2.6) is the best possible, then by (2.10) and (2.7), the unique best possible constant factor must be $\frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda(\widehat{\lambda}_1)$ ($\in \mathbf{R}_+$), namely,

$$k_\lambda(\widehat{\lambda}_1) = k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1).$$

We find that (2.9) keeps the form of equality if and only if there exist constants A and B, such that they are not all zero and (cf. [10])

$$Au^{\lambda-\lambda_2-1} = Bu^{\lambda_1-1} \text{ a.e. in } \mathbf{R}_+.$$

Assuming that $A \neq 0$ (otherwise, $B = A = 0$), it follows that $u^{\lambda-\lambda_2-\lambda_1} = \frac{B}{A}$ a.e. in $\mathbf{R}_+$, and then $\lambda-\lambda_2-\lambda_1 = 0$, namely, $\lambda_1+\lambda_2=\lambda$. □

3. Main results

Theorem 3.1. *Inequality (2.6) is equivalent to*

$$J := \left[\sum_{n=2}^{\infty} \frac{\ln^{p\beta(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})-1} n}{n} \left(\sum_{m=2}^{\infty} k_\lambda(\ln^\alpha m, \ln^\beta n) a_m \right)^p \right]^{\frac{1}{p}}$$
$$< \frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1) \left\{ \sum_{m=2}^{\infty} \frac{\ln^{p[1-\alpha(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1} m}{m^{1-p}} a_m^p \right\}^{\frac{1}{p}}. \tag{3.1}$$

If the constant factor in (2.6) is the best possible, then, so is the constant factor in (3.1).

Proof. Suppose that (3.1) is valid. By Hölder's inequality (cf. [10]), we find

$$I = \sum_{n=2}^{\infty} \left[\frac{\ln^{\frac{-1}{p}+\beta(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})} n}{n^{1/p}} \sum_{m=2}^{\infty} k_\lambda(\ln^\alpha m, \ln^\beta n) a_m \right]$$
$$\times \left[\frac{\ln^{\frac{1}{p}-\beta(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})} n}{n^{-1/p}} b_n \right]$$
$$\leq J \left\{ \sum_{n=2}^{\infty} \frac{\ln^{q[1-\beta(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})]-1} n}{n^{1-q}} b_n^q \right\}^{\frac{1}{q}}. \tag{3.2}$$

Then by (3.1), we obtain (2.6).

On the other hand, assuming that (2.6) is valid, we set

$$b_n := \frac{\ln^{p\beta(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})-1} n}{n} \left(\sum_{m=2}^{\infty} k_\lambda(\ln^\alpha m, \ln^\beta n) a_m \right)^{p-1}, n \in \mathbf{N}\backslash\{1\}.$$

If $J = 0$, then (3.1) is naturally valid; if $J = \infty$, then it is impossible that makes (3.1) valid, namely, $J < \infty$. Suppose that $0 < J < \infty$. By (2.6), it follows that

$$\begin{aligned}
&\sum_{n=2}^{\infty} \frac{\ln^{q[1-\beta(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})]-1} n}{n^{1-q}} b_n^q = J^p = I \\
&< \frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1) \left\{ \sum_{m=2}^{\infty} \frac{\ln^{p[1-\alpha(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1} m}{m^{1-p}} a_m^p \right\}^{\frac{1}{p}} \\
&\times \left\{ \sum_{n=2}^{\infty} \frac{\ln^{q[1-\beta(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})]-1} n}{n^{1-q}} b_n^q \right\}^{\frac{1}{q}}, \\
J &= \left\{ \sum_{n=2}^{\infty} \frac{\ln^{q[1-\beta(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})]-1} n}{n^{1-q}} b_n^q \right\}^{\frac{1}{p}} \\
&< \frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1) \left\{ \sum_{m=2}^{\infty} \frac{\ln^{p[1-\alpha(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1} m}{m^{1-p}} a_m^p \right\}^{\frac{1}{p}},
\end{aligned}$$

namely, (3.1) follows, which is equivalent to (2.6).

If the constant factor in (2.6) is the best possible, then so is constant factor in (3.1). Otherwise, by (3.2), we would reach a contradiction that the constant factor in (2.6) is not the best possible. □

Theorem 3.2. *The following statements (i), (ii), (iii) and (iv) are equivalent:*

(i) $k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1)$ is independent of p, q;
(ii) $k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1)$ is expressed by a single integral;
(iii) $\frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1)$ is the best possible constant factor of (2.6);
(iv) $\lambda_1 + \lambda_2 = \lambda$.

If the statement (iv) follows, namely, $\lambda_1 + \lambda_2 = \lambda$, then we have (2.7) and the following equivalent inequality with the best possible constant factor $\frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda(\lambda_1)$:

$$\begin{aligned}
&\left[\sum_{n=2}^{\infty} \frac{\ln^{p\beta\lambda_2 - 1} n}{n} \left(\sum_{m=2}^{\infty} k_\lambda(\ln^\alpha m, \ln^\beta n) a_m \right)^p \right]^{\frac{1}{p}} \\
&< \frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda(\lambda_1) \left[\sum_{m=2}^{\infty} \frac{\ln^{p(1-\alpha\lambda_1)-1} m}{m^{1-p}} a_m^p \right]^{\frac{1}{p}}. \qquad (3.3)
\end{aligned}$$

Proof. $(i) => (ii)$. Since $k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1)$ is independent of p, q,we find

$$k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1) = \lim_{p\to\infty} \lim_{q\to 1^+} k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1) = k_\lambda(\lambda_1),$$

namely, $k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1)$ is expressed by a single integral

$$k_\lambda(\lambda_1) = \int_0^\infty k_\lambda(u, 1) u^{\lambda_1 - 1} du.$$

$(ii) => (iv)$. In (2.9), if $k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)$ is expressed by a single integral, then (2.9) keeps the form of equality, which follows that $\lambda_1+\lambda_2=\lambda$.

$(iv) => (i)$. If $\lambda_1+\lambda_2=\lambda$, then $k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)=k_\lambda(\lambda_1)$, which is independent of p,q. Hence, we have $(i) <=> (ii) <=> (iv)$.

$(iii) => (iv)$. By Lemma 2.4, we have $\lambda_1+\lambda_2=\lambda$.

$(iv) => (iii)$. By Lemma 2.3, for $\lambda_1+\lambda_2=\lambda$, $\frac{1}{\beta^{1/p}\alpha^{1/q}}k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)(=\frac{1}{\beta^{1/p}\alpha^{1/q}}k_\lambda(\lambda_1))$ is the best possible constant factor of (2.6). Therefore, we have $(iii) <=> (iv)$.

Hence, the statements (i), (ii), (iii) and (iv) are equivalent. □

Remark 3.1. (i) For $\alpha=\beta=\lambda=1,\lambda_1=\frac{1}{q},\lambda_2=\frac{1}{p}$ in (2.7) and (3.3), we have the following equivalent inequalities with the best possible constant factor $k_1(\frac{1}{q})$:

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}k_1(\ln m,\ln n)a_m b_n < k_1(\frac{1}{q})\left(\sum_{m=2}^{\infty}\frac{1}{m^{1-p}}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=2}^{\infty}\frac{1}{n^{1-q}}b_n^q\right)^{\frac{1}{q}}, \tag{3.4}$$

$$\left[\sum_{n=2}^{\infty}\frac{1}{n}\left(\sum_{m=2}^{\infty}k_\lambda(\ln m,\ln n)a_m\right)^p\right]^{\frac{1}{p}} < k_\lambda(\frac{1}{q})\left(\sum_{m=2}^{\infty}\frac{1}{m^{1-p}}a_m^p\right)^{\frac{1}{p}}. \tag{3.5}$$

(ii) For $\alpha=\beta=\lambda=1,\lambda_1=\frac{1}{p},\lambda_2=\frac{1}{q}$ in (2.7) and (3.3), we have the following equivalent inequalities with the best possible constant factor $k_1(\frac{1}{p})$:

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}k_1(\ln m,\ln n)a_m b_n < k_1(\frac{1}{p})\left(\sum_{m=2}^{\infty}\frac{\ln^{p-2}m}{m^{1-p}}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=2}^{\infty}\frac{\ln^{q-2}n}{n^{1-q}}b_n^q\right)^{\frac{1}{q}}, \tag{3.6}$$

$$\left[\sum_{n=2}^{\infty}\frac{\ln^{p-2}n}{n}\left(\sum_{m=2}^{\infty}k_\lambda(\ln m,\ln n)a_m\right)^p\right]^{\frac{1}{p}} < k_1(\frac{1}{p})\left(\sum_{m=2}^{\infty}\frac{\ln^{p-2}m}{m^{1-p}}a_m^p\right)^{\frac{1}{p}}. \tag{3.7}$$

(iii) For $p=q=2$, both (3.4) and (3.6) reduce to

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}k_1(\ln m,\ln n)a_m b_n < k_1(\frac{1}{2})\left(\sum_{m=2}^{\infty}ma_m^2\sum_{n=2}^{\infty}nb_n^2\right)^{\frac{1}{2}}, \tag{3.8}$$

and both (3.5) and (3.7) reduce to the equivalent form of (3.8) as follows:

$$\left[\sum_{n=2}^{\infty}\frac{1}{n}\left(\sum_{m=2}^{\infty}k_{\lambda}(\ln m,\ln n)a_m\right)^2\right]^{\frac{1}{2}} < k_1(\frac{1}{2})\left(\sum_{m=2}^{\infty}ma_m^2\right)^{\frac{1}{2}}. \tag{3.9}$$

4. Operator expressions and some particular examples

We set functions

$$\varphi(m) := \frac{\ln^{p[1-\alpha(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1} m}{m^{1-p}}, \psi(n) := \frac{\ln^{q[1-\beta(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})]-1} n}{n^{1-q}},$$

wherefrom,

$$\psi^{1-p}(n) = \frac{\ln^{p\beta(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})-1} n}{n}\ (m,n\in \mathbf{N}\backslash\{1\}).$$

Define the following real normed spaces:

$$l_{p,\varphi} : = \left\{a = \{a_m\}_{m=2}^{\infty}; ||a||_{p,\varphi} := (\sum_{m=2}^{\infty}\varphi(m)|a_m|^p)^{\frac{1}{p}} < \infty\right\},$$

$$l_{q,\psi} : = \left\{b = \{b_n\}_{n=2}^{\infty}; ||b||_{q,\psi} := (\sum_{n=2}^{\infty}\psi(n)|b_n|^q)^{\frac{1}{q}} < \infty\right\},$$

$$l_{p,\psi^{1-p}} : = \left\{c = \{c_n\}_{n=2}^{\infty}; ||c||_{p,\psi^{1-p}} := (\sum_{n=2}^{\infty}\psi^{1-p}(n)|c_n|^p)^{\frac{1}{p}} < \infty\right\}.$$

Assuming that $a\in l_{p,\varphi}$, setting

$$c = \{c_n\}_{n=2}^{\infty}, c_n := \sum_{m=2}^{\infty}k_{\lambda}(\ln^{\alpha} m, \ln^{\beta} n)a_m, n\in\mathbf{N}\backslash\{1\},$$

we can rewrite (3.1) as follows:

$$||c||_{p,\psi^{1-p}} < \frac{1}{\beta^{1/p}\alpha^{1/q}}k_{\lambda}^{\frac{1}{p}}(\lambda-\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)||a||_{p,\varphi} < \infty,$$

namely, $c\in l_{p,\psi^{1-p}}$.

Definition 4.1. Define a Mulholland-type operator $T : l_{p,\varphi}\to l_{p,\psi^{1-p}}$ as follows: For any $a\in l_{p,\varphi}$, there exists a unique representation $c\in l_{p,\psi^{1-p}}$. Define the formal inner product of Ta and $b\in l_{q,\psi}$, and the norm of T as follows:

$$(Ta,b) : = \sum_{n=2}^{\infty}\left(\sum_{m=2}^{\infty}k_{\lambda}(\ln^{\alpha} m,\ln^{\beta} n)a_m\right)b_n = I,$$

$$||T|| : = \sup_{a(\neq\theta)\in l_{p,\varphi}}\frac{||Ta||_{p,\psi^{1-p}}}{||a||_{p,\varphi}}.$$

By Theorem 3.1 and Theorem 3.2, we have

Theorem 4.1. *If $a \in l_{p,\varphi}, b \in l_{q,\psi}, ||a||_{p,\varphi}, ||b||_{q,\psi} > 0$, then we have the following equivalent inequalities:*

$$(Ta, b) < \frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{p}}(\lambda - \lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1) ||a||_{p,\varphi} ||b||_{q,\psi}, \tag{4.1}$$

$$||Ta||_{p,\psi^{1-p}} < \frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{p}}(\lambda - \lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1) ||a||_{p,\varphi}. \tag{4.2}$$

Moreover, $\lambda_1 + \lambda_2 = \lambda$ if and only if the constant factor $\frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{p}}(\lambda - \lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1)$ in (4.1) and (4.2) is the best possible, namely,

$$||T|| = \frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda(\lambda_1). \tag{4.3}$$

Example 4.1. We set $k_\lambda(x, y) = \frac{1}{(cx+y)^\lambda}$ $(c, \lambda > 0; x, y > 0)$. Then we find

$$k_\lambda(\ln^\alpha m, \ln^\beta n) = \frac{1}{(c \ln^\alpha m + \ln^\beta n)^\lambda}.$$

For $0 < \lambda_1, \lambda - \lambda_2 \leq \frac{1}{\alpha}, 0 < \lambda_2, \lambda - \lambda_1 \leq \frac{1}{\beta}, k_\lambda(x, y)$ is a positive homogeneous function of degree $-\lambda$, satisfying that $k_\lambda(x, y)$ is decreasing with respect to $x, y > 0$, and for $\gamma = \lambda_1, \lambda - \lambda_2$,

$$\begin{aligned} k_\lambda(\gamma) &= \int_0^\infty \frac{u^{\gamma-1}}{(cu+1)^\lambda} du \\ &= \frac{1}{c^\gamma} \int_0^\infty \frac{v^{\gamma-1}}{(v+1)^\lambda} dv = \frac{1}{c^\gamma} B(\gamma, \lambda - \gamma) \in \mathbf{R}_+. \end{aligned}$$

In view of Theorem 4.1, it follows that $\lambda_1 + \lambda_2 = \lambda$ if and only if

$$||T|| = \frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda(\lambda_1) = \frac{1}{\beta^{1/p}\alpha^{1/q}} \frac{1}{c^{\lambda_1}} B(\lambda_1, \lambda_2).$$

Example 4.2. We set $k_\lambda(x, y) = \frac{\ln(cx/y)}{(cx)^\lambda - y^\lambda}$ $(c, \lambda > 0; x, y > 0)$. Then we find

$$k_\lambda(\ln^\alpha m, \ln^\beta n) = \frac{\ln(c \ln^\alpha m / \ln^\beta n)}{c^\lambda \ln^{\alpha\lambda} m - \ln^{\beta\lambda} n}.$$

For $0 < \lambda_1, \lambda - \lambda_2 \leq \frac{1}{\alpha}, 0 < \lambda_2, \lambda - \lambda_1 \leq \frac{1}{\beta}, k_\lambda(x, y)$ is a positive homogeneous function of degree $-\lambda$, satisfying that $k_\lambda(x, y)$ is decreasing with respect to $x, y > 0$ (cf. [19], Example 2.2.1), and for $\gamma = \lambda_1, \lambda - \lambda_2$,

$$\begin{aligned} k_\lambda(\gamma) &= \int_0^\infty \frac{u^{\gamma-1} \ln(cu)}{(cu)^\lambda - 1} du \\ &= \frac{1}{c^\gamma \lambda^2} \int_0^\infty \frac{v^{(\gamma/\lambda)-1} \ln v}{v - 1} dv = \frac{1}{c^\gamma} [\frac{\pi}{\lambda \sin(\pi\gamma/\lambda)}]^2 \in \mathbf{R}_+. \end{aligned}$$

In view of Theorem 4.1, it follows that $\lambda_1 + \lambda_2 = \lambda$ if and only if

$$||T|| = \frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda(\lambda_1) = \frac{1}{\beta^{1/p}\alpha^{1/q}} \frac{1}{c^{\lambda_1}} [\frac{\pi}{\lambda \sin(\pi\lambda_1/\lambda)}]^2.$$

Example 4.3. We set $k_\lambda(x,y)=\prod_{k=1}^{s}\frac{1j\neq}{(x^{\lambda/s}+c_k y^{\lambda/s})}$ $(0<c_1\leq\cdots c_s,\lambda>0;x,y>0)$. Then we find

$$k_\lambda(\ln^\alpha m,\ln^\beta n)=\prod_{k=1}^{s}\frac{1}{(\ln^{\alpha\lambda/s}m+c_k\ln^{\beta\lambda/s})}.$$

For $0<\lambda_1,\lambda-\lambda_2\leq\frac{1}{\alpha},0<\lambda_2,\ \lambda-\lambda_1\leq\frac{1}{\beta},k_\lambda(x,y)$ is a positive homogeneous function of degree $-\lambda$, satisfying that $k_\lambda(x,y)$ is decreasing with respect to $x,y>0$, and for $\gamma=\lambda_1,\lambda-\lambda_2$, by Example 1 of [25], it follows that

$$\begin{aligned}k_\lambda^{(s)}(\gamma)&=\int_0^\infty\prod_{k=1}^{s}\frac{1}{u^{\lambda/s}+c_k}u^{\gamma-1}du\\&=\frac{\pi s}{\lambda\sin(\frac{\pi s\gamma}{\lambda})}\sum_{k=1}^{s}c_k^{\frac{s\gamma}{\lambda}}\prod_{j=1(j\neq k)}^{s}\frac{1}{c_j-c_k}\in\mathbf{R}_+.\end{aligned}$$

In view of Theorem 4.1, it follows that $\lambda_1+\lambda_2=\lambda$ if and only if

$$||T||=\frac{k_\lambda^{(s)}(\lambda_1)}{\beta^{1/p}\alpha^{1/q}}=\frac{1}{\beta^{1/p}\alpha^{1/q}}\frac{\pi s}{\lambda\sin(\frac{\pi s\lambda_1}{\lambda})}\sum_{k=1}^{s}c_k^{\frac{s\lambda_1}{\lambda}}\prod_{j=1(j\neq k)}^{s}\frac{1}{c_j-c_k}.$$

In particular, for $c_1=\cdots=c_s=c$, we have $k_\lambda(x,y)=\frac{1}{(x^{\lambda/s}+cy^{\lambda/s})^s}$ and

$$\begin{aligned}\widetilde{k}_\lambda^{(s)}(\lambda_1)\ :&=\int_0^\infty\frac{u^{\lambda_1-1}du}{(u^{\lambda/s}+c)^s}=\frac{s}{\lambda c^{\frac{\lambda_2 s}{\lambda}}}\int_0^\infty\frac{v^{(s\lambda_1/\lambda)-1}}{(v+1)^s}dv\\&=\frac{s}{\lambda c^{\frac{\lambda_2 s}{\lambda}}}B(\frac{s\lambda_1}{\lambda},\frac{s\lambda_2}{\lambda})\in\mathbf{R}_+.\end{aligned}$$

If $s=1$, then we have $k_\lambda(x,y)=\frac{1}{x^\lambda+cy^\lambda}$and

$$||T||=\frac{\widetilde{k}_\lambda^{(1)}(\lambda_1)}{\beta^{1/p}\alpha^{1/q}}=\frac{1}{\beta^{1/p}\alpha^{1/q}}\frac{s}{\lambda c^{\frac{\lambda_2}{\lambda}}}\frac{\pi}{\sin(\frac{\pi\lambda_1}{\lambda})}.$$

5. Conclusions

In this paper, by the use of the weight functions and the idea of introducing parameters, a discrete Mulholland-type inequality with the general homogeneous kernel and the equivalent form are given in Lemma 2.2 and Theorem 3.1. The equivalent statements of the best possible constant factor related to some parameters are considered in Theorem 3.2. The operator expressions and some particular examples are given in Theorem 4.1 and Examples 4.1–4.3. The lemmas and theorems provide an extensive account of this type of inequalities.

Acknowledgements. The authors would like to thank the referee for his useful propose in reforming the paper.

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