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POSITIVE SOLUTIONS FOR A NONLINEAR DISCRETE FRACTIONAL BOUNDARY VALUE PROBLEM WITH A *P*-LAPLACIAN OPERATOR*

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Abstract In this paper using the monotone iterative technique we establish the existence and uniqueness of positive solutions for a nonlinear discrete fractional boundary value problem with a p-Laplacian operator. Also we discuss an iterative sequence which yields the approximate solution for this problem.

Keywords Fractional difference equations, p-Laplacian, positive solutions, Iteration.

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1. Introduction

For $a, b \in \mathbb{R}$, let $[a, b]_E = [a, b] \cap E$ for some set E with $a < b$. In this paper we investigate the existence and uniqueness of positive solutions for the following nonlinear discrete fractional boundary value problem with a p-Laplacian operator:

$$\begin{cases} \Delta^{\nu}_{\nu-1}(\phi_p(\Delta^{\nu}_{\nu-1}y(t))) = f(y(t+\nu-1)), t \in [0, T]_{\mathbb{Z}}, \\ y(\nu-1) = y(\nu+T), \Delta^{\nu}_{\nu-1}y(\nu-1) = \Delta^{\nu}_{\nu-1}y(\nu+T), \end{cases} \tag{1.1}$$

where $\nu \in (0, 1)$ is a real number, $\Delta^{\nu}_{\nu-1}$ is a discrete fractional operator, and $\phi_p(s) = |s|^{p-2}s$ is the p-Laplacian with $s \in \mathbb{R}$, $p > 1$. For the nonlinearity f, we assume that

(H1) $f \in C(\mathbb{R}^+, \mathbb{R}^+)$, and $f(y) > 0$ if $y > 0$,

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(H2) $f(y)$ is nondecreasing about y, and for $l \in (0,1)$, there exists $\alpha(l) \in (l,1)$ such that

$$f(ly) \geq (\alpha(l))^{p-1} f(y), \text{ for } y \in \mathbb{R}^+.$$

(H3) $f(y)$ is nonincreasing in y, and there exist two positive-valued functions $\varphi(\tau), \omega(\tau)$ on $[\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}$ such that $\varphi : [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}} \to (0,1)$ is a surjection and $\omega(\tau) > \varphi(\tau)$, $\forall \tau \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}$ with $f\left(\frac{y}{\varphi(\tau)}\right) \geq (\omega(\tau))^{p-1} f(y)$, $\forall \tau \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}, y \geq 0$.

Fractional-order models are used in in physics, chemistry, polymer rheology, economics, control theory, biophysics and blood flow phenomena. For example, in [32] the authors studied the abstract evolution of the system for HIV-1 population dynamics, which takes the fractional form:

$$\begin{cases} D_t^\alpha u(t) + \lambda f(t, u(t), D_t^\beta u(t), v(t)) = 0, \\ D_t^\gamma v(t) + \lambda g(t, u(t)) = 0, 0 < t < 1, \\ D_t^\beta u(0) = D_t^{\beta+1} u(0) = 0, D_t^\beta u(1) = \int_0^1 D_t^\beta u(s) dA(s), \\ v(0) = v'(0) = 0, \ v(1) = \int_0^1 v(s) dB(s); \end{cases} \tag{1.2}$$

here $f : (0,1) \times [0,+\infty)^3 \to (-\infty,+\infty)$, and $g : (0,1) \times [0,+\infty) \to (-\infty,+\infty)$ are two semipositone functions (for other related models see [8, 9, 15, 18, 30, 41, 43–48, 53, 56] and the references therein).

However there are only a small number of papers in the literature on discrete fractional equations (see for example [1, 6, 7, 10–12, 16, 21, 26, 29, 37, 42, 54]). In [10] the author used the Guo-Krasnosel'skii fixed point theorem to establish a positive solution for the discrete fractional boundary value problem

$$\begin{cases} \Delta^\nu y(t) = \lambda f(t+\nu-1, y(t+\nu-1)), t \in [0,T]_{\mathbb{Z}}, \\ y(\nu-1) = y(\nu+T) + \sum\limits_{i=1}^{N} F(t_i, y(t_i)), \end{cases} \tag{1.3}$$

where f is a semipositone nonlinearity and satisfies the sublinear growth condition:

$$\lim_{y \to +\infty} \frac{f(t,y)}{y} = 0, \text{ uniformly for } t \in [\nu-1, \nu+T]_{\mathbb{Z}_{\nu-1}}.$$

In [1, 37], the authors extended (1.3) to systems of discrete equations and used the fixed point index to establish the existence of positive solutions for their systems. In [54] for p-Laplacian systems the authors used the contraction mapping theorem and the Brouwer fixed point theorem to study the existence and uniqueness of solutions for the discrete fractional boundary value problem:

$$\begin{cases} \Delta^\beta(\phi_p(\Delta^\alpha y(t))) + f(\alpha+\beta+t-1, y(\alpha+\beta+t-1)) = 0, t \in [0,b]_{\mathbb{N}_0}, \\ \Delta^\alpha y(\beta-2) = \Delta^\alpha y(\beta+b) = 0, \\ y(\alpha+\beta-4) = y(\alpha+\beta+b) = 0, \end{cases} \tag{1.4}$$

where $f : [\alpha+\beta-4, \alpha+\beta+b]_{\mathbb{N}_{\alpha+\beta-4}} \times \mathbb{R} \to \mathbb{R}$ is a Lipschitz function.

The monotone iterative technique combined with the method of lower and upper solutions can be used in studying the existence of solutions for nonlinear problems (see [2–5, 14, 17, 19, 20, 22–25, 27, 28, 31, 33–36, 38–40, 49–52, 55] and the references therein). In [2] the authors studied the boundary value problems for the nonlinear fractional differential equation:

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, t \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1.5}$$

where $\alpha \in (1, 2]$ is a real number, and D_{0+}^{α} is the Riemann-Liouville fractional derivative. When f satisfies an appropriate Lipschitz condition the authors used the Banach's contraction mapping principle and the theory of linear operators to establish the uniqueness of solutions for (1.5) and they presented an iterative sequence (for other similar papers see [3, 4, 33, 39]). In [36], the authors studied the fractional differential equation with a p-Laplacian operator:

$$\begin{cases} -D_{0+}^{\alpha}(\phi_p(-D_{0+}^{\gamma} z(x))) = f(x, z(x)), x \in (0,1), \\ z(0) = 0, D_{0+}^{\gamma} z(0) = D_{0+}^{\gamma} z(1) = 0, z(1) = \int_0^1 z(x) d\chi(x), \end{cases} \tag{1.6}$$

where $D_{0+}^{\alpha}, D_{0+}^{\gamma}$ are the Riemann-Liouville fractional derivative, $\int_0^1 z(x) d\chi(x)$ is a Riemann-Stieltjes integral and χ is a function of bounded variation. The authors used the condition

(H)$_{\text{Zhang}}$ $f(x, z)$ is decreasing in z and for any $r \in (0, 1)$, there exists $\mu \in (0, \frac{1}{p-1})$ with $p > 1$ such that

$$f(x, rz) \le r^{-\mu} f(x, z), \forall (x, z) \in (0, 1) \times (0, +\infty),$$

to establish a unique solution for (1.6) and using an iterative technique the authors presented appropriate sequences converging uniformly to the unique positive solution (in addition they derived estimates of the approximation error and the convergence rate).

Motivated by the above in this paper we investigate the existence and uniqueness of positive solutions for the discrete fractional p-Laplacian problem (1.1) and we present iterative sequences which uniformly converge to the unique solution.

2. Preliminaries

In this section we give some necessary definitions from discrete fractional calculus.

Definition 2.1 (see [11]). We define $t^{\underline{\nu}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$ for any $t, \nu \in \mathbb{R}$ for which the right-hand side is well-defined. We use the convention that if $t + 1 - \nu$ is a pole of the Gamma function and $t + 1$ is not a pole, then $t^{\underline{\nu}} = 0$.

Definition 2.2 (see [11]). For $\nu > 0$, the $\nu-$th fractional sum of a function f is

$$\Delta_a^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t - s - 1)^{\underline{\nu-1}} f(s), \text{ for } t \in \mathbb{N}_{a+N-\nu}.$$

We also define the $\nu-$th fractional difference for $\nu > 0$ by

$$\Delta_a^{\nu} f(t) = \Delta^N \Delta_a^{\nu-N} f(t), \text{ for } t \in \mathbb{N}_{a+N-\nu},$$

where $N \in \mathbb{N}$ with $0 \le N - 1 < \nu \le N$.

Let $\phi_q = \phi_p^{-1}$ with $1/q + 1/p=1$. Then we have the following lemma.

Lemma 2.1. *The discrete fractional boundary value problem* (1.1) *can be transformed into its equivalent sum equation, which takes the form*

$$y(t) = \sum_{s=0}^{T} \frac{G(t,s)}{\Gamma(\nu)} \phi_q \left(\sum_{r=0}^{T} \frac{G(s+\nu-1,r)}{\Gamma(\nu)} f(y(r+\nu-1)) \right), t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}, \tag{2.1}$$

where

$$G(t,s) = \begin{cases} \frac{(\nu+T-s-1)^{\underline{\nu-1}} t^{\underline{\nu-1}}}{\Gamma(\nu)-(\nu+T)^{\underline{\nu-1}}} + (t-s-1)^{\underline{\nu-1}}, 0 \le s \le t-\nu \le T, \\ \frac{(\nu+T-s-1)^{\underline{\nu-1}} t^{\underline{\nu-1}}}{\Gamma(\nu)-(\nu+T)^{\underline{\nu-1}}}, t-\nu < s \le T. \end{cases} \tag{2.2}$$

Proof. Let $\phi_p(\Delta_{\nu-1}^{\nu} y(t)) = x(t)$. Then from (1.1) we have

$$\begin{cases} \Delta_{\nu-1}^{\nu} x(t) = f(y(t+\nu-1)), t \in [0,T]_{\mathbb{Z}}, \\ x(\nu-1) = x(\nu+T). \end{cases} \tag{2.3}$$

Using [10] and [37], we obtain

$$x(t) = \sum_{s=0}^{T} \frac{G(t,s)}{\Gamma(\nu)} f(y(s+\nu-1)),\ t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}. \tag{2.4}$$

Note that $\phi_p(\Delta_{\nu-1}^{\nu} y(t)) = x(t)$, and thus $\Delta_{\nu-1}^{\nu} y(t) = \phi_q(x(t))$. Hence, from (1.1) we have

$$\begin{cases} \Delta_{\nu-1}^{\nu} y(t) = \phi_q(x(t+\nu-1)), t \in [0,T]_{\mathbb{Z}}, \\ y(\nu-1) = y(\nu+T). \end{cases} \tag{2.5}$$

Using [10] and [37] again, we get

$$y(t) = \sum_{s=0}^{T} \frac{G(t,s)}{\Gamma(\nu)} \phi_q \left(\sum_{r=0}^{T} \frac{G(s+\nu-1,r)}{\Gamma(\nu)} f(y(r+\nu-1)) \right),\ t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}. \tag{2.6}$$

This completes the proof. □

Lemma 2.2 (see [10, Lemma 2.5])**.** *Let* $C^* = 1 + \frac{\Gamma(\nu)-(\nu+T)^{\underline{\nu-1}}}{(\nu+T-1)^{\underline{\nu-1}}}$ *for* $(t,s) \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}} \times [0,T]_{\mathbb{Z}}$*. Then the Green's function* G *satisfies:*

$$0 < \frac{(\nu+T)^{\underline{\nu-1}}}{\Gamma(\nu)-(\nu+T)^{\underline{\nu-1}}} (\nu+T-s-1)^{\underline{\nu-1}} \le G(t,s) \le \frac{C^*\Gamma(\nu)}{\Gamma(\nu)-(\nu+T)^{\underline{\nu-1}}} (\nu+T-s-1)^{\underline{\nu-1}}. \tag{2.7}$$

Let $\mathcal{E}$ be the collection of all maps from $[\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}$ to $\mathbb{R}$ with the norm

$$\|y\| = \max_{t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}} |y(t)|.$$

Then $(\mathcal{E}, \|\cdot\|)$ is a Banach space, Then we define two sets on $\mathcal{E}$ as follows:

$$P = \{y \in \mathcal{E} : y(t) \ge 0, \forall t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}\},$$
$$P_0 = \left\{ y \in \mathcal{E} : y(t) \ge \frac{(\nu+T)^{\underline{\nu-1}}}{C^*\Gamma(\nu)} \|y\|, \forall t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}} \right\}.$$

Now P, P_0 are cones on $\mathcal{E}$. From Lemma 2.1 we can define an operator S on $\mathcal{E}$ as follows:

$$(Sy)(t)=\sum_{s=0}^{T}\frac{G(t,s)}{\Gamma(\nu)}\phi_q\left(\sum_{r=0}^{T}\frac{G(s+\nu-1,r)}{\Gamma(\nu)}f(y(r+\nu-1))\right),\ t\in[\nu-1,\nu+T-1]_{\mathbb{Z}_{\nu-1}}.$$

From the Arzelà-Ascoli theorem a standard argument guarantees that $S:\mathcal{E}\to\mathcal{E}$ is a completely continuous operator and the existence of solutions for (1.1) follows from the existence of fixed points for S. Moreover, by Lemma 2.2 we easily obtain that $S(P)\subset P_0$(see [10]).

For $x, y\in\mathcal{E}$, $x\sim y$ is defined by: there exist $\delta,\gamma>0$ such that $\delta x\le y\le\gamma x$. Let $P_h=\{x\in\mathcal{E}: x\sim h\}$, where $h\in P\backslash\{0\}$.

Lemma 2.3 (see [38, Theorem 2.1])**.** *Let $h>0$ and P be a normal cone. Assume that:*

(D1) $S:P\to P$ is nondecreasing, and there exist $\delta,\gamma>0$ such that $\delta h\le Sh\le\gamma h$, i.e., $Sh\in P_h$,

(D2) for any $y\in P$ and $l\in(0,1)$, there exists $\alpha(l)\in(l,1)$ such that $S(ly)\ge\alpha(l)Sy$.

Then the following two conclusions hold:

(i) there are $u_0, v_0\in P_h$ and $l\in(0,1)$ such that $lv_0\le u_0<v_0$, $u_0\le Su_0\le Sv_0\le v_0$,

(ii) the operator equation $y=Sy$ has a unique positive solution in P_h.

Lemma 2.4 (see [13])**.** *Let $\mathcal{E}$ be a partially ordered Banach space, and $x_0, y_0\in\mathcal{E}$ with $x_0\le y_0$, $D=[x_0,y_0]$. Suppose that $S:D\to\mathcal{E}$ satisfies the following conditions:*

(i) S is an increasing operator,

(ii) $x_0\le Sx_0, y_0\ge Sy_0$, i.e., x_0 and y_0 is a subsolution and a supersolution of S,

(iii) S is a continuous compact operator.

Then S has the smallest fixed point y^* and the largest fixed point y_* in $[x_0,y_0]$, respectively. Moreover, $y^*=\lim_{n\to\infty}S^nx_0$, and $y_*=\lim_{n\to\infty}S^ny_0$.

3. Main Results

We give our main results in this paper.

Theorem 3.1. *Suppose that (H1)-(H2) hold and $f(0)\neq 0$. Then* (1.1) *has a unique positive solution in P_h. Moreover, for any $y_0\in P\setminus\{0\}$, constructing successively the sequence (n=0,1,2,...)*

$$y_{n+1}(t)=\sum_{s=0}^{T}\frac{G(t,s)}{\Gamma(\nu)}\phi_q\left(\sum_{r=0}^{T}\frac{G(s+\nu-1,r)}{\Gamma(\nu)}f(y_n(r+\nu-1))\right),\ t\in[\nu-1,\nu+T-1]_{\mathbb{Z}_{\nu-1}},$$

we have that $y_n(t)$ converges uniformly to $y^*(t)$ in $t\in[\nu-1,\nu+T-1]_{\mathbb{Z}_{\nu-1}}$.

Proof. From (H2) and $f(0)\neq 0$ we obtain that $S:P\to P$ is nondecreasing and

0 is not a fixed point of S. For $l \in (0,1)$ and $y \in P$, by (H2) we have

$$\begin{aligned}(Sly)(t) &= \sum_{s=0}^{T} \frac{G(t,s)}{\Gamma(\nu)} \phi_q \left(\sum_{r=0}^{T} \frac{G(s+\nu-1,r)}{\Gamma(\nu)} f(ly(r+\nu-1)) \right) \\ &\geq \sum_{s=0}^{T} \frac{G(t,s)}{\Gamma(\nu)} \phi_q \left(\sum_{r=0}^{T} \frac{G(s+\nu-1,r)}{\Gamma(\nu)} (\alpha(l))^{p-1} f(y(r+\nu-1)) \right) \\ &= \alpha(l) \sum_{s=0}^{T} \frac{G(t,s)}{\Gamma(\nu)} \phi_q \left(\sum_{r=0}^{T} \frac{G(s+\nu-1,r)}{\Gamma(\nu)} f(y(r+\nu-1)) \right) \\ &= \alpha(l)(Sy)(t),\ t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}.\end{aligned}$$

Therefore, $S(ly) \geq \alpha(l)Sy$, for $y \in P$, $l \in (0,1)$.

Let $h(t) = \sum_{s=0}^{T} \frac{G(t,s)}{\Gamma(\nu)} \phi_q \left(\sum_{r=0}^{T} \frac{G(s+\nu-1,r)}{\Gamma(\nu)} \right)$, for $t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}$. Then from Lemma 2.2 we have

$$\begin{aligned}&\sum_{s=0}^{T} \frac{(\nu+T)^{\underline{\nu-1}}(\nu+T-s-1)^{\underline{\nu-1}}}{\Gamma(\nu)(\Gamma(\nu)-(\nu+T)^{\underline{\nu-1}})} \phi_q \left(\sum_{r=0}^{T} \frac{G(s+\nu-1,r)}{\Gamma(\nu)} \right) \\ &\leq h(t) \leq \sum_{s=0}^{T} \frac{C^*(\nu+T-s-1)^{\underline{\nu-1}}}{\Gamma(\nu)-(\nu+T)^{\underline{\nu-1}}} \phi_q \left(\sum_{r=0}^{T} \frac{G(s+\nu-1,r)}{\Gamma(\nu)} \right), \\ &\text{for } t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}.\end{aligned}$$

For convenience let

$$\begin{aligned}\kappa_1 &= \sum_{s=0}^{T} \frac{(\nu+T)^{\underline{\nu-1}}(\nu+T-s-1)^{\underline{\nu-1}}}{\Gamma(\nu)(\Gamma(\nu)-(\nu+T)^{\underline{\nu-1}})} \phi_q \left(\sum_{r=0}^{T} \frac{G(s+\nu-1,r)}{\Gamma(\nu)} \right), \\ \kappa_2 &= \sum_{s=0}^{T} \frac{C^*(\nu+T-s-1)^{\underline{\nu-1}}}{\Gamma(\nu)-(\nu+T)^{\underline{\nu-1}}} \phi_q \left(\sum_{r=0}^{T} \frac{G(s+\nu-1,r)}{\Gamma(\nu)} \right).\end{aligned}$$

Then for $t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}$, we have

$$\begin{aligned}(Sh)(t) &= \sum_{s=0}^{T} \frac{G(t,s)}{\Gamma(\nu)} \phi_q \left(\sum_{r=0}^{T} \frac{G(s+\nu-1,r)}{\Gamma(\nu)} f(h(r+\nu-1)) \right) \\ &\leq \sum_{s=0}^{T} \frac{G(t,s)}{\Gamma(\nu)} \phi_q \left(\sum_{r=0}^{T} \frac{G(s+\nu-1,r)}{\Gamma(\nu)} f(\kappa_2) \right) \\ &= \phi_q(f(\kappa_2))h(t),\end{aligned}$$

and

$$\begin{aligned}(Sh)(t) &= \sum_{s=0}^{T} \frac{G(t,s)}{\Gamma(\nu)} \phi_q \left(\sum_{r=0}^{T} \frac{G(s+\nu-1,r)}{\Gamma(\nu)} f(h(r+\nu-1)) \right) \\ &\geq \sum_{s=0}^{T} \frac{G(t,s)}{\Gamma(\nu)} \phi_q \left(\sum_{r=0}^{T} \frac{G(s+\nu-1,r)}{\Gamma(\nu)} f(\kappa_1) \right) \\ &= \phi_q(f(\kappa_1))h(t).\end{aligned}$$

Let $P_h = \{y \in \mathcal{E} : \phi_q(f(\kappa_1))h(t) \le y(t) \le \phi_q(f(\kappa_2))h(t),\ t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}\}$. Then $Sh \in P_h$. From Lemma 2.3, there exist $u_0, v_0 \in P_h$ and $l \in (0,1)$ such that

$$lv_0 \le u_0 < v_0,\ u_0 \le Su_0 \le Sv_0 \le v_0, \tag{3.1}$$

and S has a unique fixed point in P_h, denoted by $\overline{y}$. We have proved that (1.1) has a unique positive solution in P_h. Next, from (3.1) note all the conditions of Lemma 2.4 are satisfied with $D = [u_0, v_0] \subset P_h$. Consequently, for any $y_0 \in D(y_0 \in P\backslash\{0\})$, by the monotonicity of S, we have

$$S^n u_0 \le S^n y_0 \le S^n v_0,\ \text{for } n \in \mathbb{N}.$$

If we let $y_{n+1} = Sy_n$ then by induction $y_n = S^n y_0$, $n = 0, 1, 2, ...$ Therefore, from $\lim_{n\to\infty} S^n u_0 = \lim_{n\to\infty} S^n v_0 = \overline{y}$ we have

$$y_n(t) = \sum_{s=0}^{T} \frac{G(t,s)}{\Gamma(\nu)} \phi_q \left(\sum_{r=0}^{T} \frac{G(s+\nu-1, r)}{\Gamma(\nu)} f(y_{n-1}(r+\nu-1)) \right) \to \overline{y}(t),$$

uniformly in $t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}$. This completes the proof. $\square$

Theorem 3.2. *Suppose that (H1)-(H3) hold. Then* (1.1) *has a unique positive solution* y^* *in* $P \setminus \{0\}$. *Moreover, for any* $y_0 \in P \setminus \{0\}$, *constructing successively the sequence (n=0,1,2,..)*

$$y_{n+1}(t) = \sum_{s=0}^{T} \frac{G(t,s)}{\Gamma(\nu)} \phi_q \left(\sum_{r=0}^{T} \frac{G(s+\nu-1, r)}{\Gamma(\nu)} f(y_n(r+\nu-1)) \right),\ t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}},$$

we have that $y_n(t)$ *converges uniformly to* $y^*(t)$ *in* $t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}$.

Proof. Step 1. Problem (1.1) has a positive solution.

From (H3) we see that Sy is nonincreasing in y. Note that, for all $\tau \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}$, then $\widetilde{\tau} = \tau - \nu + 1 \in [0, T]_{\mathbb{Z}}$. Hence, from (H3) we have

$$\begin{aligned}
S\left(\frac{1}{\varphi(\tau)} y\right)(t) &= \sum_{s=0}^{T} \frac{G(t,s)}{\Gamma(\nu)} \phi_q \left[\sum_{r=0}^{T} \frac{G(s+\nu-1, r)}{\Gamma(\nu)} f\left(\frac{1}{\varphi(\widetilde{\tau}+\nu-1)} y(r+\nu-1) \right) \right] \\
&\ge \sum_{s=0}^{T} \frac{G(t,s)}{\Gamma(\nu)} \left[\sum_{r=0}^{T} \frac{G(s+\nu-1, r)}{\Gamma(\nu)} (\omega(\widetilde{\tau}+\nu-1))^{p-1} f\left(y(r+\nu-1)\right) \right]^{\frac{1}{p-1}} \\
&= \omega(\widetilde{\tau}+\nu-1) \sum_{s=0}^{T} \frac{G(t,s)}{\Gamma(\nu)} \phi_q \left[\sum_{r=0}^{T} \frac{G(s+\nu-1, r)}{\Gamma(\nu)} f\left(y(r+\nu-1)\right) \right] \\
&= \omega(\tau)(Sy)(t),\ \tau \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}, \widetilde{\tau} \in [0, T]_{\mathbb{Z}},
\end{aligned} \tag{3.2}$$

for $y \in P$, $t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}$. Let $L = \sum_{s=0}^{T} \frac{C^*(\nu+T-s-1)^{\underline{\nu-1}}}{\Gamma(\nu)-(\nu+T)^{\underline{\nu-1}}} \phi_q \left(\sum_{r=0}^{T} \frac{G(s+\nu-1, r)}{\Gamma(\nu)} \right)$. Then $L > 0$. Since $f(y) > 0$ if $y > 0$, and from Lemma 2.2, for $t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}$, we have

$$\frac{(\nu+T)^{\underline{\nu-1}}}{C^*\Gamma(\nu)} \phi_q(f(L))L \le S(L) = \sum_{s=0}^{T} \frac{G(t,s)}{\Gamma(\nu)} \phi_q \left(\sum_{r=0}^{T} \frac{G(s+\nu-1, r)}{\Gamma(\nu)} f(L) \right) \le \phi_q(f(L))L. \tag{3.3}$$

Therefore, we can choose a sufficiently small number $e \in (0,1)$ such that

$$eL \leq S(L) \leq \frac{L}{e}. \tag{3.4}$$

As a result, there exists $\tau_0 \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}$ such that $\varphi(\tau_0) = e$, and then we have

$$\varphi(\tau_0)L \leq S(L) \leq \frac{L}{\varphi(\tau_0)}. \tag{3.5}$$

Note that $\frac{\omega(\tau_0)}{\varphi(\tau_0)} > 1$, and we can take a sufficiently large positive integer k such that

$$\left[\frac{\omega(\tau_0)}{\varphi(\tau_0)}\right]^k \geq \frac{1}{\varphi(\tau_0)}, \text{ and } \left[\frac{\varphi(\tau_0)}{\omega(\tau_0)}\right]^k \leq \varphi(\tau_0). \tag{3.6}$$

Define $u_0 = [\varphi(\tau_0)]^k L,\ v_0 = [\varphi(\tau_0)]^{-k} L$. Then we have

$$u_0 = [\varphi(\tau_0)]^{2k} v_0 < v_0, \text{ and } u_0 \geq \lambda v_0 \text{ if } \lambda \in (0, [\varphi(\tau_0)]^{2k}] \in (0,1). \tag{3.7}$$

From the monotonicity of S, we have $Sv_0 \leq Su_0$. Moreover, from (3.2), (3.6) and (H3) we have

$$\begin{aligned} Sv_0 &= S\left([\varphi(\tau_0)]^{-k}L\right) = S\left(\frac{1}{\varphi(\tau_0)}[\varphi(\tau_0)]^{-k+1}L\right) \geq \omega(\tau_0)S\left([\varphi(\tau_0)]^{-k+1}L\right) \geq \cdots \\ &\geq [\omega(\tau_0)]^k S(L) \geq [\omega(\tau_0)]^k \varphi(\tau_0)L \geq [\varphi(\tau_0)]^k L = u_0. \end{aligned} \tag{3.8}$$

On the other hand, from (3.2) and (H3) we obtain

$$Sy = S\left(\frac{1}{\varphi(\tau)}\varphi(\tau)y\right) \geq \omega(\tau)S(\varphi(\tau)y), \text{ and } S(\varphi(\tau)y) \leq \frac{1}{\omega(\tau)}Sy. \tag{3.9}$$

Thus, from (3.6) we have

$$\begin{aligned} Su_0 &= S([\varphi(\tau_0)]^k L) = S(\varphi(\tau_0)[\varphi(\tau_0)]^{k-1}L) \leq \frac{1}{\omega(\tau_0)}S([\varphi(\tau_0)]^{k-1}L) \leq \cdots \\ &\leq \frac{1}{[\omega(\tau_0)]^k}S(L) \leq \frac{1}{[\omega(\tau_0)]^k}\frac{L}{\varphi(\tau_0)} \leq [\varphi(\tau_0)]^{-k}L = v_0. \end{aligned} \tag{3.10}$$

Therefore, we can construct successively the sequences

$$u_n = Sv_{n-1},\ v_n = Su_{n-1},\ n = 1, 2, ... \tag{3.11}$$

From the monotonicity of S, we have $u_1 = Sv_0 \leq Su_0 = v_1$. By induction, we obtain $u_n \leq v_n$ for $n = 1, 2, ...$ Moreover, from (3.8), (3.10), we know that the sequences $\{u_n\}_{n=1}^{\infty}, \{v_n\}_{n=1}^{\infty}$ satisfy the inequalities:

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0. \tag{3.12}$$

Note that $u_0 \geq \lambda v_0$ if $\lambda \in (0, [\varphi(\tau_0)]^{2k}] \in (0,1)$, and thus $u_n \geq u_0 \geq \lambda v_0 \geq \lambda v_n$, $n = 1, 2, ...$ Let

$$\lambda_n = \sup\{\lambda > 0 : u_n \geq \lambda v_n\}, n = 1, 2, ... \tag{3.13}$$

Then we have $u_n \geq \lambda_n v_n$ and thus $u_{n+1} \geq u_n \geq \lambda_n v_n \geq \lambda_n v_{n+1}$, $n = 1, 2, ...$ Note that $\lambda_{n+1} = \sup\{\lambda > 0 : u_{n+1} \geq \lambda v_{n+1}\}$, so $\lambda_{n+1} \geq \lambda_n$, i.e., $\{\lambda_n\}_{n=1}^{\infty}$ is an increasing sequence with $\lambda_n \in (0,1]$ for $n = 1, 2, \ldots$. Let $\lambda^* = \lim_{n\to\infty} \lambda_n$. Now we show that $\lambda^* = 1$. Indeed, if $\lambda^* \in (0,1)$, from (H4) there exists $\tau^* \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}$ such that $\varphi(\tau^*) = \lambda^*$. Now, we consider the following two possible cases.

Case 1. There is a $N \in \mathbb{N}$ such that $\lambda_n = \lambda^*$ for $n \geq N$.

Hence, for $n \geq N$, from (3.2) we have

$$u_{n+1} = Sv_n \geq S\left(\frac{1}{\lambda^*}u_n\right) = S\left(\frac{1}{\varphi(\tau^*)}u_n\right) \geq \omega(\tau^*)Su_n = \omega(\tau^*)v_{n+1}.$$

Note the definition of λ_n and we have

$$\lambda_{n+1} = \lambda^* \geq \omega(\tau^*) > \varphi(\tau^*) = \lambda^*,$$

and this is a contradiction.

Case 2. For all $n \in \mathbb{N}$, $\lambda_n < \lambda^*$.

This implies $\frac{\lambda_n}{\lambda^*} \in (0,1)$, and there exists $\mu_n \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}$ such that $\varphi(\mu_n) = \frac{\lambda_n}{\lambda^*}$. Consequently, from (3.2) we have

$$\begin{aligned} u_{n+1} = Sv_n \geq S\left(\frac{1}{\lambda_n}u_n\right) &= S\left(\frac{1}{\frac{\lambda_n}{\lambda^*}}\frac{1}{\lambda^*}u_n\right) = S\left(\frac{1}{\varphi(\mu_n)}\frac{1}{\varphi(\tau^*)}u_n\right) \\ &\geq \omega(\mu_n)\omega(\tau^*)Su_n = \omega(\mu_n)\omega(\tau^*)v_{n+1}. \end{aligned}$$

Note the definition of λ_n and we have

$$\lambda_{n+1} \geq \omega(\mu_n)\omega(\tau^*) > \varphi(\mu_n)\omega(\tau^*) = \frac{\lambda_n}{\lambda^*}\omega(\tau^*).$$

Let $n \to \infty$, so

$$\lambda^* \geq \frac{\lambda^*}{\lambda^*}\omega(\tau^*) > \varphi(\tau^*) = \lambda^*,$$

and this is also a contradiction.

Combining the above two cases we have $\lim_{n\to\infty} \lambda_n = 1$.

Finally, we prove that the two sequences $\{u_n\}_{n=1}^{\infty}$, $\{v_n\}_{n=1}^{\infty}$ are convergent, and we first show that $\{u_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{E}$. Indeed, from (3.12) we obtain

$$\begin{aligned} 0 &\leq u_{n+m}(t) - u_n(t) \leq v_n(t) - \lambda_n v_n(t) = (1-\lambda_n)v_n(t) \\ &\leq (1-\lambda_n)v_0(t),\ \forall m \in \mathbb{N}, t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}. \end{aligned}$$

This implies that

$$\|u_{n+m} - u_n\| \leq (1-\lambda_n)\|v_0\|,$$

i.e., $\{u_n\}$ is a Cauchy sequence in $\mathcal{E}$ since $\lambda_n \to 1\,(n \to \infty)$. Consequently, there exists $y^* \in P\backslash\{0\}$ such that $\lim_{n\to\infty} u_n(t) = y^*(t)$ for $t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}$. On the other hand, for all $n \in \mathbb{N}$, we have

$$v_n(t) - u_n(t) \leq v_n(t) - \lambda_n v_n(t) \leq (1-\lambda_n)v_0(t),\ \text{for } t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}.$$

This implies that $\{v_n\}_{n=1}^{\infty}$ converges to the same limit as $\{u_n\}_{n=1}^{\infty}$, i.e., $\lim_{n\to\infty} v_n(t) = y^*(t)$ for $t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}$. From the monotonicity of S, we have

$$u_{n+1}(t) = (Sv_n)(t) \leq (Sy^*)(t) \leq (Su_n)(t) \leq v_{n+1}.$$

Let $n \to \infty$. Then $Sy^* = y^*$, i.e., y^* is a solution of (1.1). Moreover, $y^*(t) = (Sy^*)(t) \geq (Sv_n)(t) = u_{n+1}(t) \geq u_0(t) \equiv [\varphi(\tau_0)]^k L > 0$, so y^* is also a positive solution.

Step 2. Problem (1.1) has a unique solution.

Suppose that (1.1) has two positive solutions, $y^*, x^* \in P\backslash\{0\}$ with $y^*(t) \neq x^*(t)$ for $t \in [0,1]$. From step 1, we have that y^*, x^* have positive upper and lower bounds, so there exists $\eta \in (0,1]$ such that

$$\eta y^*(t) \leq x^*(t) \leq \frac{1}{\eta} y^*(t), \text{ for } t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}.$$

Let

$$\eta_0 = \sup\left\{\eta \in (0,1] : \eta y^*(t) \leq x^*(t) \leq \frac{1}{\eta} y^*(t), \text{ for } t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}\right\}.$$

We claim $\eta_0 = 1$. If false, we have $\eta_0 \in (0,1)$ and there is a $\tau_1 \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}$ such that $\varphi(\tau_1) = \eta_0$. Consequently, from (3.2) and (3.9) we have

$$x^* = Sx^* \geq S\left(\frac{1}{\eta_0} y^*\right) = S\left(\frac{1}{\varphi(\tau_1)} y^*\right) \geq \omega(\tau_1) S y^* = \omega(\tau_1) y^*,$$

and

$$x^* = Sx^* \leq S\left(\eta_0 y^*\right) = S\left(\varphi(\tau_1) y^*\right) \leq \frac{1}{\omega(\tau_1)} S y^* = \frac{1}{\omega(\tau_1)} y^*.$$

As a result, we have

$$\eta_0 y^* = \varphi(\tau_1) y^* < \omega(\tau_1) y^* \leq x^* \leq \frac{1}{\omega(\tau_1)} y^* < \frac{1}{\varphi(\tau_1)} y^* = \frac{1}{\eta_0} y^*.$$

This contradictsthe definition of η_0. Thus $\eta_0 = 1$. Therefore, (1.1) has a unique solution.

Step 3. We establish an iterative sequence, which converges uniformly to the unique positive solution of (1.1).

For any $y_0 \in P\backslash\{0\}$, we can choose a sufficiently small number $\hat{e} \in (0,1)$ such that

$$\hat{e} L \leq y_0 \leq \frac{L}{\hat{e}}. \tag{3.14}$$

From (H3) there exists $\tau_2 \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}$ such that $\varphi(\tau_2) = \hat{e}$. Consequently, we have

$$\varphi(\tau_2) L \leq y_0 \leq \frac{L}{\varphi(\tau_2)}. \tag{3.15}$$

This implies that there exists a $k \in \mathbb{N}$ large enough such that

$$\left[\frac{\omega(\tau_2)}{\varphi(\tau_2)}\right]^k \geq \frac{1}{\varphi(\tau_2)}.$$

Define

$$\hat{u}_0 = [\varphi(\tau_2)]^k L, \ \hat{v}_0 = \frac{L}{[\varphi(\tau_2)]^k}.$$

Consequently, we find $\hat{u}_0 < y_0 < \hat{v}_0$, and let

$$\hat{u}_n = S\hat{v}_{n-1}, \ \hat{v}_n = S\hat{u}_{n-1},$$

and

$$y_n(t) = \sum_{s=0}^{T} \frac{G(t,s)}{\Gamma(\nu)} \phi_q \left(\sum_{r=0}^{T} \frac{G(s+\nu-1,r)}{\Gamma(\nu)} f(y_{n-1}(r+\nu-1)) \right),$$

for $t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}$, $n = 0,1,2,\ldots$. By induction, we get $\hat{u}_n \le y_n \le \hat{v}_n$ for $n = 0,1,2,\ldots$. Similarly with the above two steps, there exists $\hat{y} \in P\backslash\{0\}$ such that

$$\lim_{n\to\infty} \hat{u}_n = \lim_{n\to\infty} \hat{v}_n = \hat{y}, \text{ and } S\hat{y} = \hat{y}.$$

Note the uniqueness of positive solutions, and we have $y^* = \hat{y}$, and thus

$$y_{n+1}(t) = \sum_{s=0}^{T} \frac{G(t,s)}{\Gamma(\nu)} \phi_q \left(\sum_{r=0}^{T} \frac{G(s+\nu-1,r)}{\Gamma(\nu)} f(y_n(r+\nu-1)) \right) \to y^*(t),$$

uniformly in $t \in [\nu-1, \nu+T-1]_{\mathbb{Z}_{\nu-1}}$. This completes the proof. □

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