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UPPER SEMICONTINUITY OF PULLBACK ATTRACTORS FOR MULTI-VALUED RANDOM COCYCLE

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Abstract In this paper we study the upper semicontinuity of random attractors for multi-valued random cocycle when small random perturbations approach zero or small perturbation for random cocycle is considered. Furthermore, we consider the upper semicontinuity of random attractors for multi-valued random cocycle under the condition which the metric dynamical systems is ergodic.

Keywords Pullback attractor, ergodicity, upper semicontinuity.

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1. Introduction

The asymptotic behaviour of dynamical systems is one of the most important problems of modern mathematical physics and the theory has been greatly developed over the last decade or so. In recent years, there is an increasing interest in the study of multi-valued systems, see [2, 3, 5, 10–12, 17–20]. The attempts to extend the notion of global attractor to the non-autonomous case led to the concept of the so-called the theory of pullback (or cocycle) attractors, which has been developed for both the non-autonomous and random dynamical systems (see [1] [6] [8] [9] [13]). The concept of pullback attractors for random dynamical systems was introduced by Crauel and Flandoli in [6, 7], as a generalization of the classical concept of the global attractor for many models in Physics, Chemistry and Biology.

For single-valued case, the upper semicontinuity of attractors for small random perturbation of dynamical systems was considered by [4, 15, 16]. For multi-valued case, the upper semicontinuity of pullback attractors for multi-valued process has been studied in [18] and the upper semicontinuity of pullback attractors for multi-valued noncompact random dynamical systems was considered in [19].

In the present paper, we consider the upper semicontinuity of pullback attractors for multi-valued random cocycle. First, we consider the upper semicontinuity of random attractors for multi-valued random cocycle when small random perturbations approach zero, which can be considered as generalization of single-valued case in [15]. Next, we consider the upper semicontinuity of random attractors for multi-valued random cocycle when small perturbation for random cocycle is considered.

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This can be considered as generalization of single-valued case in [15]. Finally, if invariant measure P of the metric dynamical systems θ_t is ergodic, then we can obtain the upper semicontinuity of random attractors under weaker conditions. These results are new even for single-valued case.

The paper is organized as follows. In Section 2, we recall some basic notations and definitions, and give some results about the existence of random attractors for multi-valued random cocycle. In Section 3, we establish the upper semicontinuity of random attractors when small random perturbations approach zero. In section 4, we establish the upper semicontinuity of random attractors when small perturbation for random cocycle is considered.

2. Preliminaries.

In this section, we recall the theory of attractors for autonomous multi-valued semi-dynamical systems and multi-valued random cocycle.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $(X, \|\cdot\|_X)$ a Banach space with Borel σ-algebra $\mathcal{B}(X)$. The Hausdorff semi-distance between two nonempty subsets A and B of X is defined by

$$dist(A, B) = \sup_{a\in A} \inf_{b\in B} \|a - b\|.$$

Let 2^X be the collection of all subsets of X. First we recall some notations of attractors for multi-valued semiflow.

Definition 2.1. A family of mapping $\phi(t) : X \to 2^X$, $t \in \mathbb{R}^+$, is called an autonomous multi-valued semiflow if it satisfies the following conditions:

1. $\phi(0)x = \{x\}, \ \forall x \in X$;
2. $\phi(s + t)x = \phi(s) \circ \phi(t)x, \quad \forall s, t \in \mathbb{R}^+, \quad x \in X$.

Definition 2.2. A nonempty compact subset $\mathcal{A}_0$ of X is called a global attractor for the autonomous multi-valued semiflow $\phi(t)$ if it satisfies:

1. $\mathcal{A}_0$ is an invariant set, that is
$$\phi(t)\mathcal{A}_0 = \mathcal{A}_0, \ \forall t \in \mathbb{R}^+;$$
2. $\mathcal{A}_0$ attracts each bounded subset B of X, that is
$$\lim_{t\to\infty} dist(\phi(t)B, \mathcal{A}_0) = 0.$$

Next, we recall some notations of attractors for multi-valued random cocycle.

Definition 2.3. $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ is called a metric dynamical system if $\theta : \mathbb{R}\times\Omega \to \Omega$ is $(B(\mathbb{R})\times\mathcal{F}, \mathcal{F})$ measurable, θ_0 is the identity on Ω, $\theta_{s+t} = \theta_t \circ \theta_s$ for all $s, t \in \mathbb{R}$ and $\theta_t P = P$ for all $t \in \mathbb{R}$.

Definition 2.4. Let $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ be a metric dynamical system. A multi-valued mapping

$$\phi : \mathbb{R}^+ \times \Omega \times X \to 2^X,$$

is called a multi-valued random cocycle over a metric dynamical system $(\Omega,\mathcal{F},P,(\theta_t)_{t\in\mathbb{R}})$ if for all $\omega \in \Omega$ and $t, s \in \mathbb{R}^+$, the following conditions are satisfied:

1. $\phi : \mathbb{R}^+ \times \Omega \times X, \quad (t, \omega, x) \mapsto \phi(t, \omega, x)$, which is $B(\mathbb{R}) \times \mathcal{F} \times \mathcal{B}(X)$ measurable;
2. $\phi(0, \omega, \cdot)$ is the identity on X;
3. $\phi(t + s, \omega, \cdot) = \phi(t, \theta_s \omega, \cdot) \circ \phi(s, \omega, \cdot)$.

For the composition of multi-valued mappings, for every nonempty $V \subset X$, $\phi(t, \omega, V)$ is defined by

$$\phi(t, \omega, V) = \bigcup_{x \in V} \phi(t, \omega, x).$$

Definition 2.5 (see [3])**.** A multi-valued random cocycle ϕ is said to be upper semi-continuous if for all $t \in \mathbb{R}^+$ and $\omega \in \Omega$ it follows that for given $x \in X$ and a neighbourhood of $\phi(t, \omega, x)$, $\mathcal{O}(\phi(t, \omega, x))$, there exists $\delta > 0$ such that if $d(x, y) < \delta$ then

$$\phi(t, \omega, y) \subset \mathcal{O}(\phi(t, \omega, x)).$$

Similarly, ϕ is called lower semi-continuous if for all $t \in \mathbb{R}^+$ and $\omega \in \Omega$, for given $x_n \to x (n \to \infty)$ and $y \in \phi(t, \omega, x)$, there exist $y_n \in \phi(t, \omega, x_n)$ such that $y_n \to y$. It is said to be continuous if it is upper and lower semi-continuous.

Definition 2.6. Let $\mathcal{D}$ be a collection of random subsets of X. Then $\mathcal{D}$ is called inclusion-closed if $D = \{D(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$ and $\tilde{D} = \{\tilde{D}(\omega)\}_{\omega \in \Omega}$ with $\tilde{D}(\omega) \subset D(\omega)$ for all $\omega \in \Omega$ imply that $\tilde{D} \in \mathcal{D}$.

Definition 2.7. Let $\mathcal{D}$ be a collection of random subsets of X, and $\{K(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$. Then $\{K(\omega)\}_{\omega \in \Omega}$ is called a $\mathcal{D}$-pullback absorbing set for ϕ if for every $B \in \mathcal{D}$ and $P - a.e. \omega \in \Omega$, there exists $T(B, \omega) > 0$ such that

$$\phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subset K(\omega) \quad \text{for all} \quad t \geq T(B, \omega).$$

Definition 2.8. Let $\mathcal{D}$ be a collection of random subsets of X. A multi-valued random cocycle ϕ is called $\mathcal{D}$-pullback asymptotically upper semi-compact in X, if for $P - a.e. \omega \in \Omega$ and every family $\{B(\omega)\} \in \mathcal{D}$, any sequence $\{\phi(t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^{\infty}$ has a convergent subsequence in X whenever $t_n \to +\infty$, and $x_n \in B(\theta_{-t_n}\omega)$.

Definition 2.9. Let $\mathcal{D}$ be a collection of random subsets of X. Then a random set $\{\mathcal{A}(\omega)\}_{\omega \in \Omega}$ is called a $\mathcal{D}-$random attractor (or $\mathcal{D}-$pullback attractor) for the multi-valued random cocycle ϕ if the following conditions are satisfied, for $P - a.e. \omega \in \Omega$,

(1) $\mathcal{A}(\omega)$ is compact and $\omega \mapsto d(x, \mathcal{A}(\omega))$ is measurable for every $x \in X$;

(2) $\{\mathcal{A}(\omega)\}$ is invariant, that is,

$$\phi(t, \omega, A(\omega)) = A(\theta_t \omega), \ \forall t \geq 0;$$

(3) $\{A(\omega)\}$ (pullback) attracts every set in $\mathcal{D}$, that is, for every $B = \{B(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$,

$$\lim_{t \to \infty} dist(\phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)), \mathcal{A}(\omega)) = 0.$$

Then we have the following existence result for a random attractor of a multi-valued random cocycle, which can be proved following the proof of related results in [12, 18].

Proposition 2.1. *Let $\mathcal{D}$ be a collection of random subsets of X and ϕ a continuous multi-valued random cocycle on X over $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$. Suppose that $\{K(\omega)\}_{\omega\in\Omega}$ is a closed random absorbing set for ϕ in $\mathcal{D}$ and ϕ is $\mathcal{D}$-pullback asymptotically upper-semi compact in X. Then ϕ has a unique $\mathcal{D}$-random attractor $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ which is given by*

$$\mathcal{A}(\omega) = \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau} \phi(t, \theta_{-t}\omega, K(\theta_{-t}\omega))}.$$

3. Upper semicontinuity of attractors for multi-valued random cocycle.

In this section, we establish the upper semicontinuity of random attractors when small random perturbations approach zero. Let $(X, \|\cdot\|)$ be a Banach space and ϕ be an autonomous multi-valued semiflow defined on X. Given $\epsilon > 0$, suppose ϕ_ϵ is a multi-valued random cocycle over a metric system $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$. We further suppose that for $P-a.e.\ \omega \in \Omega$, $t \geq 0, \epsilon_n \to 0,$ and $x_n, x \in X$ with $x_n \to x$, the following holds:

$$\lim_{n\to\infty} dist(\phi_{\epsilon_n}(t, \omega, x_n), \phi(t)x) = 0. \tag{3.1}$$

Let $\mathcal{D}$ be a collection of random subsets of X. Given $\epsilon > 0$, suppose that ϕ_ϵ has a random attractor $\mathcal{A}_\epsilon = \{\mathcal{A}_\epsilon(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and a random absorbing set $E_\epsilon = \{E_\epsilon(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ such that for deterministic positive constant c and for $P-a.e.\ \omega \in \Omega$,

$$\limsup_{\epsilon\to 0} \|E_\epsilon(\omega)\|_X \leq c, \tag{3.2}$$

where $\|E_\epsilon(\omega)\|_X = \sup_{x\in E_\epsilon(\omega)} \|x\|_X$. We assume that there exists $\epsilon_0 > 0$ such that for $P-a.e.\ \omega \in \Omega$,

$$\bigcup_{0<\epsilon\leq\epsilon_0} \mathcal{A}_\epsilon(\omega) \quad \text{is precompact in } X. \tag{3.3}$$

Let $\mathcal{A}_0$ be the global attractor of ϕ in X, which means that $\mathcal{A}_0$ is compact and invariant with respect to ϕ and attracts every bounded subset of X uniformly. Then we have the following theorem.

Theorem 3.1. *Suppose the conditions (3.1),(3.2),(3.3) hold. Then for $P-a.e.\ \omega \in \Omega$,*

$$dist(\mathcal{A}_\epsilon(\omega), \mathcal{A}_0) \to 0, \quad as\ \ \epsilon \to 0. \tag{3.4}$$

Proof. If the conclusion is not true, then there exists a measurable set $\Omega_1 \subset \Omega$ with $P(\Omega_1) > 0$ such that for every $\omega \in \Omega_1$,

$$dist(\mathcal{A}_\epsilon(\omega), \mathcal{A}_0) \not\to 0, \quad \text{as } \ \epsilon \to 0.$$

Thus for every $\omega \in \Omega_1$, there exist a positive number $\delta > 0$ and a sequence $\{x_n\}_{n=1}^\infty$ with $x_n \in \mathcal{A}_{\epsilon_n}(\omega)$ and $\epsilon_n \to 0$ such that

$$dist(x_n, \mathcal{A}_0) \geq \delta. \tag{3.5}$$

Since $\bigcup_{0<\epsilon\leq\epsilon_0} \mathcal{A}_\epsilon$ is precompact in X, it follows that there are $y_0 \in X$ and a subsequence of $\{x_n\}_{n=1}^\infty$ (we still denote by $\{x_n\}_{n=1}^\infty$) such that

$$\lim_{n\to\infty} x_n = y_0.$$

Next let us prove $y_0 \in \mathcal{A}_0$. In order to do that, we take a sequence $\{t_m\}_{m=1}^{\infty}$ with $t_m \to \infty$. By the invariance of $\mathcal{A}_{\epsilon_n}$, we have

$$\phi_{\epsilon_n}(t_1, \theta_{-t_1}\omega, \mathcal{A}_{\epsilon_n}(\theta_{-t_1}\omega)) = \mathcal{A}_{\epsilon_n}(\omega),$$

and we find that there exists a subsequence $\{x_{1,n}\}_{n=1}^{\infty}$ with $x_{1,n} \in \mathcal{A}_{\epsilon_n}(\theta_{-t_1}\omega)$ such that

$$x_n \in \phi_{\epsilon_n}(t_1, \theta_{-t_1}\omega, x_{1,n}), \quad \forall n \geq 1.$$

By (3.3) again, there exist $y_1 \in X$ and a subsequence of $\{x_{1,n}\}_{n=1}^{\infty}$ (we still denote by $\{x_{1,n}\}_{n=1}^{\infty}$) such that

$$\lim_{n\to\infty} x_{1,n} = y_1.$$

Since $x_{1,n} \in \mathcal{A}_{\epsilon_n}(\theta_{-t_1}\omega)$ and $A_{\epsilon_n}(\theta_{-t_1}\omega) \subset E_{\epsilon_n}(\theta_{-t_1}\omega)$, by (3.2), we have

$$\limsup_{n\to\infty} \|x_{1,n}\|_X \leq \limsup_{n\to\infty} \|E_{\epsilon_n}(\theta_{-t_1}\omega)\|_X \leq c.$$

Thus we have

$$\|y_1\|_X \leq c.$$

On the other hand, it follows from (3.1)

$$\lim_{n\to\infty} dist(\phi_{\epsilon_n}(t_1, \theta_{-t_1}\omega, x_{1,n}), \phi(t_1)y_1) = 0.$$

It follows that

$$\lim_{n\to\infty} d(x_n, \phi(t_1)y_1) \leq \lim_{n\to\infty} dist(\phi_{\epsilon_n}(t_1, \theta_{-t_1}\omega, x_{1,n}), \phi(t_1)y_1) = 0.$$

Hence we have

$$y_0 \in \overline{\phi(t_1)y_1}.$$

Similarly, for each $m \geq 2$, repeating the above procedure, we can find that there is $y_m \in X$ such that

$$y_0 \in \overline{\phi(t_m)y_m}, \quad \forall m \geq 2,$$

and

$$\|y_m\|_X \leq c, \ \forall m \geq 2.$$

Letting $t_m \to \infty$, we have

$$dist(y_0, \mathcal{A}_0) \leq \limsup_{t_m\to\infty} [dist(y_0, \phi(t_m)y_m) + dist(\phi(t_m)B(c), \mathcal{A}_0)] = 0,$$

where $B(c) = \{x \in X : \|x\|_X \leq c\}$. This implies $y_0 \in \mathcal{A}_0$, since $\mathcal{A}_0$ is compact. Therefore we have

$$dist(x_n, \mathcal{A}_0) \leq \|x_n - y_0\| \to 0, \quad \text{as}\, n \to \infty,$$

a contradiction with (3.5). The proof is complete. □

Remark 3.1. The idea of proof of theorem 3.1 is similar to that of the proof of theorem in [15] for single value random cocycle. In [19], they also considered the upper semicontinuity of multi-valued random noncompact dynamical systems, and they supposed that the attractors are contained in a finite dimension subspace. Furthermore, if let X be a complete metric space with metric d, and instead the condition (3.2) in theorem by the following condition : There exist a bounded set B and $\epsilon_1 > 0$ such that for $P - a.e.\ \omega \in \Omega$,

$$\bigcup_{0<\epsilon\leq\epsilon_1} E_\epsilon(\omega) \subset B. \tag{3.6}$$

Therefore we have the following result. Suppose the conditions (3.1), (3.3),(3.6) hold, then for $P - a.e.\ \omega \in \Omega$,

$$dist(\mathcal{A}_\epsilon(\omega), \mathcal{A}_0) \to 0, \quad \text{as} \ \ \epsilon \to 0.$$

Next we suppose that P is an ergodic measure with respect to θ_t. Then condition (3.2) can be replaced by the following condition: There is a random map $c(\omega) : \Omega \to \mathbb{R}^+$ such that for $P - a.e.\ \omega \in \Omega$,

$$\limsup_{\epsilon\to 0} \|E_\epsilon(\omega)\|_X \leq c(\omega). \tag{3.7}$$

Theorem 3.2. *Suppose that P is an ergodic with respect to θ_t and the conditions (3.1), (3.3), (3.7) hold. Then for $P - a.e.\ \omega \in \Omega$,*

$$dist(\mathcal{A}_\epsilon(\omega), \mathcal{A}_0) \to 0, \quad as \ \ \epsilon \to 0. \tag{3.8}$$

Proof. If it is not the case, then there exists a measurable set $\Omega_1 \subset \Omega$ with $P(\Omega_1) > 0$ such that for every $\omega \in \Omega_1$,

$$dist(\mathcal{A}_\epsilon(\omega), \mathcal{A}_0) \not\to 0, \quad \text{as} \ \ \epsilon \to 0.$$

There exist a measurable set $\Omega_2 \subset \Omega_1$ with $P(\Omega_2) > 0$ and a determined positive constant C such that for every $\omega \in \Omega_2$,

$$c(\omega) \leq C.$$

Thus for every $\omega \in \Omega_2$, there exists a positive number $\delta > 0$ and a sequence $\{x_n\}_{n=1}^\infty$ with $x_n \in \mathcal{A}_{\epsilon_n}(\omega)$ and $\epsilon_n \to 0$ such that

$$dist(x_n, \mathcal{A}_0) \geq \delta. \tag{3.9}$$

Since $\bigcup_{0<\epsilon\leq\epsilon_0} \mathcal{A}_\epsilon$ is precompact in X, it follows that there are $y_0 \in X$ and a subsequence of $\{x_n\}_{n=1}^\infty$ (we still denote by $\{x_n\}_{n=1}^\infty$) such that

$$\lim_{n\to\infty} x_n = y_0.$$

Let us prove $y_0 \in \mathcal{A}_0$. By the ergodicity of P with respect to θ_t, and Poincare recurrence theorem (see [14]), for $P - a.e.\ \omega \in \Omega_2$, we can take a sequence $\{t_m\}_{m=1}^\infty$ with $t_m \to \infty$ such that $\theta_{-t_m}\omega \in \Omega_2$. Thus

$$\limsup_{m\to\infty} \|E_{\epsilon_n}(\theta_{-t_m}\omega)\|_X \leq c(\theta_{-t_m}\omega) \leq C, \ \ \forall\ m \geq 1.$$

Next we can follow the line of the proof of theorem 3.1 to obtain a contradiction. □

4. Upper semicontinuity of attractors for multi-valued random cocycle

In this section, we discuss the upper semicontinuity of pullback attractors of a family of multi-valued random cocycles on a Banach space X. Suppose Λ is a metric space. Given $\lambda \in \Lambda$, let ϕ_λ be a continuous multi-valued random cocycle on X over $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$. Suppose there exists $\lambda_0 \in \Lambda$ such that for very $t \in \mathbb{R}^+$, $\omega \in \Omega$, $\lambda_n \in \Lambda$ with $\lambda_n \to \lambda_0$, and x_n, $x \in X$ with $x_n \to x$, the following holds:

$$\lim_{n\to\infty} dist(\phi_{\lambda_n}(t,\omega,x_n), \phi_{\lambda_0}(t,\omega,x)) = 0. \tag{4.1}$$

For each $\lambda \in \Lambda$, let $\mathcal{D}_\lambda$ be a collection of families of nonempty subsets of X. Suppose there is a random map $R_{\lambda_0} : \Omega \to \mathbb{R}^+$ such that the family

$$B = \{B(\omega) = \{x \in X : \|x\|_X \le R_{\lambda_0}(\omega),\, \omega \in \Omega\} \text{ belongs to } \mathcal{D}_{\lambda_0}. \tag{4.2}$$

Suppose further that for each $\lambda \in \Lambda$, Φ_λ has a $\mathcal{D}_\lambda$ pullback attractor $\mathcal{A}_\lambda \in \mathcal{D}_\lambda$ and a $\mathcal{D}_\lambda$−pullback absorbing set $E_\lambda \in \mathcal{D}_\lambda$ such that for $P-a.e.$ $\omega \in \Omega$,

$$\limsup_{\lambda\to\lambda_0} \|K_\lambda(\omega)\|_X \le R_{\lambda_0}(\omega), \tag{4.3}$$

where $\|S\|_X = \sup_{x\in S}\|x\|_X$ for a subset S of X. We finally assume that for $P-a.e.$ $\omega \in \Omega$,

$$\bigcup_{\lambda\in\Lambda} \mathcal{A}_\lambda(\omega) \quad \text{is precompact in } X. \tag{4.4}$$

Theorem 4.1. *Suppose the conditions (4.1) –(4.4) hold. Then for $P-a.e.$ $\omega \in \Omega$,*

$$dist(\mathcal{A}_\lambda(\omega), \mathcal{A}_{\lambda_0}(\omega)) \to 0, \quad \text{as } \lambda \to \lambda_0. \tag{4.5}$$

Proof. Suppose the conclusion is not true, then there exists a measurable set $\Omega_1 \subset \Omega$ with $P(\Omega_1) > 0$ such that for every $\omega \in \Omega_1$,

$$dist(\mathcal{A}_\lambda(\omega), \mathcal{A}_{\lambda_0}(\omega)) \not\to 0, \quad \text{as } \lambda \to \lambda_0.$$

Thus for every $\omega \in \Omega_1$, there exist a positive number $\delta > 0$ and a sequence $\lambda_n \to \lambda_0$ such that for all $n \in \mathbb{N}$,

$$dist(\mathcal{A}_{\lambda_n}(\omega), \mathcal{A}_{\lambda_0}(\omega)) \ge \delta. \tag{4.6}$$

Therefore we can find a sequence $\{x_n\}_{n=1}^\infty$ with $x_n \in \mathcal{A}_{\lambda_n}(\omega)$ such that

$$dist(x_n, \mathcal{A}_{\lambda_0}(\omega)) \ge \delta \quad \text{for all } n \in \mathbb{N}.$$

Since $\bigcup_{\lambda\in\Lambda} \mathcal{A}_\lambda(\omega)$ is precompact in X, it follows that there are $y_0 \in X$ and a subsequence of $\{x_n\}_{n=1}^\infty$ (we still denote by $\{x_n\}_{n=1}^\infty$) such that

$$\lim_{n\to\infty} x_n = y_0.$$

Next let us prove $y_0 \in \mathcal{A}_{\lambda_0}(\omega)$. In order to do that, we take a sequence $\{t_m\}_{m=1}^\infty$ with $t_m \to \infty$. By the invariance of $\mathcal{A}_{\lambda_n}$, for every $n \in \mathbb{N}$,

$$\phi_{\lambda_n}(t_1, \theta_{-t_1}\omega, \mathcal{A}_{\lambda_n}(\theta_{-t_1}\omega)) = \mathcal{A}_{\lambda_n}(\omega),$$

we find that there exists a subsequence $\{x_{1,n}\}_{n=1}^{\infty}$ with $x_{1,n} \in \mathcal{A}_{\lambda_n}(\theta_{-t_1}\omega)$ such that,

$$x_n \in \phi_{\lambda_n}(t_1, \theta_{-t_1}\omega, x_{1,n}), \quad \forall n \geq 1.$$

By (4.4) again, there exists $y_1 \in X$ and a subsequence of $\{x_{1,n}\}_{n=1}^{\infty}$ (we still denote by $\{x_{1,n}\}_{n=1}^{\infty}$) such that

$$\lim_{n\to\infty} x_{1,n} = y_1.$$

Since $x_{1,n} \in \mathcal{A}_{\lambda_n}(\theta_{-t_1}\omega)$ and $A_{\lambda_n}(\theta_{-t_1}\omega) \subset E_{\lambda_n}(\theta_{-t_1}\omega)$, by (4.3), we have

$$\limsup_{n\to\infty} \|x_{1,n}\|_X \leq \limsup_{n\to\infty} \|E_{\lambda_n}(\theta_{-t_1}\omega)\|_X \leq R_{\lambda_0}(\theta_{-t_1}\omega).$$

Thus we have

$$\|y_1\|_X \leq R_{\lambda_0}(\theta_{-t_1}\omega).$$

On the other hand, it follows from (4.1)

$$\lim_{n\to\infty} dist(\phi_{\lambda_n}(t_1, \theta_{-t_1}\omega, x_{1,n}), \phi_{\lambda_0}(t_1, \theta_{-t_1}\omega, y_1) = 0.$$

It follows that

$$\lim_{n\to\infty} dist(x_n, \phi_{\lambda_0}(t_1, \theta_{-t_1}\omega, y_1)) \leq \lim_{n\to\infty} dist(\phi_{\lambda_n}(t_1, \theta_{-t_1}\omega, x_{1,n}), \phi_{\lambda_0}(t_1, \theta_{-t_1}\omega, y_1)) = 0.$$

Thus

$$dist(y_0, \phi_{\lambda_0}(t_1, \theta_{-t_1}\omega, y_1)) = 0.$$

Hence we have

$$y_0 \in \overline{\phi_{\lambda_0}(t_1, \theta_{-t_1}\omega, y_1)}.$$

Similarly, for each $m \geq 2$, repeating the above procedure, we can find that there is $y_m \in X$ such that

$$y_0 \in \overline{\phi_{\lambda_0}(t_m, \theta_{-t_m}\omega, y_m)}, \quad \forall m \geq 2,$$

and

$$\|y_m\|_X \leq R_{\lambda_0}(\theta_{-t_m}\omega), \ \forall m \geq 2.$$

Letting $t_m \to \infty$, we have

$$\begin{aligned} dist(y_0, \mathcal{A}_{\lambda_0}(\omega)) &\leq \limsup_{t_m\to\infty} dist(\phi_{\lambda_0}(t_m, \theta_{-t_m}\omega, y_m), \mathcal{A}_{\lambda_0}(\omega)) \\ &\leq \limsup_{t_m\to\infty} dist(\phi_{\lambda_0}(t_m, \theta_{-t_m}\omega, B(\theta_{-t_m}\omega)), \mathcal{A}_{\lambda_0}(\omega)) = 0. \end{aligned}$$

This implies $y_0 \in \mathcal{A}_{\lambda_0}(\omega)$, since $\mathcal{A}_{\lambda_0}(\omega)$ is compact. Therefore we have

$$dist(x_n, \mathcal{A}_{\lambda_0}(\omega)) \leq \|x_n - y_0\| \to 0, \quad \text{as}\, n \to \infty,$$

a contradiction with (4.6). The proof is complete. □

Remark 4.1. In theorem 4.1, pullback absorbing sets $\{E_\lambda(\omega)\}$ may depend on $\lambda \in \Lambda$. It is not necessary to assume that $\{E_\lambda(\omega)\}$ are independence of λ.

In the definition of random attractors (definition 2.9), it is required that a random attractor attracts every set in $\mathcal{D}$. If we require weaker condition that a random attractor only attracts every bounded set (non random) $B \in X$ as in paper [4], whether do we have the upper semicontinuity of random attractors for multi-valued random cocycles? In this case, we also have the following theorem.

Theorem 4.2. *Suppose that P is ergodic with respect to θ_t and the conditions (4.1) (4.3) (4.4) hold. Then for $P-a.e.$ $\omega \in \Omega$,*

$$dist(\mathcal{A}_\lambda(\omega), \mathcal{A}_{\lambda_0}(\omega)) \to 0, \quad as \ \ \lambda \to \lambda_0. \tag{4.7}$$

Proof. Suppose it is not the case, then there exists a measurable set $\Omega_1 \subset \Omega$ with $P(\Omega_1) > 0$ such that for every $\omega \in \Omega_1$,

$$dist(\mathcal{A}_\lambda(\omega), \mathcal{A}_{\lambda_0}(\omega)) \not\to 0, \quad \text{as} \ \ \lambda \to \lambda_0.$$

There exist a measurable set $\Omega_2 \subset \Omega_1$ with $P(\Omega_2) > 0$ and a determined positive constant R such that for every $\omega \in \Omega_2$,

$$R_{\lambda_0}(\omega) \leq R.$$

Thus for every $\omega \in \Omega_2$, there exist a positive number $\delta > 0$ and a sequence $\lambda_n \to \lambda_0$ such that for all $n \in \mathbb{N}$,

$$dist(\mathcal{A}_{\lambda_n}(\omega), \mathcal{A}_{\lambda_0}(\omega)) \geq \delta.$$

Therefore we can find a sequence $\{x_n\}_{n=1}^\infty$ with $x_n \in \mathcal{A}_{\lambda_n}(\omega)$ such that for all $n \in \mathbb{N}$,

$$dist(x_n, \mathcal{A}_{\lambda_0}(\omega)) \geq \delta. \tag{4.8}$$

Since $\bigcup\limits_{\lambda \in \Lambda} \mathcal{A}_\lambda(\omega)$ is precompact in X, it follows that there are $y_0 \in X$ and a subsequence of $\{x_n\}_{n=1}^\infty$ (we still denote by $\{x_n\}_{n=1}^\infty$) such that

$$\lim_{n\to\infty} x_n = y_0.$$

Next let us prove $y_0 \in \mathcal{A}_{\lambda_0}(\omega)$. By the ergodicity of P with respect to θ_t, and Poincare recurrence theorem (see [14]), for $P-a.e.$ $\omega \in \Omega_2$, we can take a sequence $\{t_m\}_{m=1}^\infty$ with $t_m \to \infty$ such that $\theta_{-t_m}\omega \in \Omega_2$.

We can follow the line of the proof of theorem 4.1 and find a sequence $\{y_m\}_{n=1}^\infty$ such that

$$\|y_m\| \leq R_{\lambda_0}(\theta_{-t_m}\omega) \leq R, \quad \forall m \geq 1$$

and

$$y_0 \in \overline{\phi_{\lambda_0}(t_m, \theta_{-t_m}\omega, y_m)}, \quad \forall m \geq 1.$$

Letting $t_m \to \infty$, we have

$$\begin{aligned} dist(y_0, \mathcal{A}_{\lambda_0}(\omega)) &\leq \limsup_{t_m\to\infty} dist(\phi_{\lambda_0}(t_m, \theta_{-t_m}\omega, y_m), \mathcal{A}_{\lambda_0}(\omega)) \\ &\leq \limsup_{t_m\to\infty} dist(\phi_{\lambda_0}(t_m, \theta_{-t_m}\omega, B(R)), \mathcal{A}_{\lambda_0}(\omega)) = 0. \end{aligned}$$

Where $B(R) = \{x \in X : \|x\| \leq R\}$. This implies $y_0 \in \mathcal{A}_{\lambda_0}(\omega)$, since $\mathcal{A}_{\lambda_0}(\omega)$ is compact. Therefore we have

$$dist(x_n, \mathcal{A}_{\lambda_0}(\omega)) \leq \|x_n - y_0\| \to 0, \quad \text{as}\, n \to \infty,$$

a contradiction with (4.8). The proof is complete.

□

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