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LIMIT POINT, STRONG LIMIT POINT AND DIRICHLET CONDITIONS FOR DISCRETE HAMILTONIAN SYSTEMS*

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Abstract This paper deals with discrete Hamiltonian systems with a singular endpoint. The limit point condition, the strong limit point condition and the Dirichlet condition are studied based on asymptotic behaviors or square summabilities in the maximal domains. The equivalence between the limit point and strong limit point conditions is established for a class of such systems; and for degenerated Hamiltonian system, the three conditions are shown to imply each other.

Keywords Discrete Hamiltonian system, linear relation, limit point case, strong limit point case,Dirichlet condition.

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1. Introduction

The classification and determination of limit-point and limit circle for differential operators are one of the important branches of spectral theory ([2,3,9,12–16,18–22,25,35,36,38]) since the foundation work of H. Weyl [37]. The theory of discrete Hamiltonian systems has been extensively researched since the early 1960s. Discrete Hamiltonian systems originate from the discretization of continuous Hamiltonian systems, and from discrete processes acting in accordance with the Hamiltonian principle, such as discrete physical problems, discrete control problems and variational problems of sum integrals. Results on discrete Hamiltonian systems are summarized in Ahlbrandt and Peterson [1]. The spectral theory for singular discrete systems was first studied by Atkinson [5]. His work was followed by many scholars (c.f. [6–8,10,11,23,26–34] and references cited therein). In this paper, we study the discrete Hamiltonian systems of the form

$$Ly := J\Delta y(t) - Q(t)R(y)(t) = \lambda W(t)R(y)(t) \tag{1.1}$$

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for $t \in \mathbb{Z}_+ = \{0, 1, 2, \cdots, \}$, where Δ denotes the forward difference operator, $\Delta y(t) = y(t+1) - y(t)$, $y(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$ is $\mathbb{C}^{2n}$ valued (column) function on $\mathbb{Z}_+$ and $x(t), u(t)$ are the first and last $\mathbb{C}^n$ components of $y(t)$, $R(y)(t)$ is the partial right shift operator, i.e.,

$$R(y)(t) = \begin{bmatrix} x(t+1) \\ u(t) \end{bmatrix} \text{ and } J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix},$$
$$Q(t) = \begin{bmatrix} -C(t) & A^*(t) \\ A(t) & B(t) \end{bmatrix}, W(t) = \begin{bmatrix} W_1(t) & 0 \\ 0 & W_2(t) \end{bmatrix}.$$

Here I_n is the $n \times n$ identity matrix; $Q(t)$ and $W(t)$ are $2n \times 2n$ Hermitian matrix valued functions; $W_1 \geq 0$ and $W_2 \geq 0$ are $n \times n$ semi-positive definite matrix valued functions; and $\lambda \in \mathbb{C}$ is the spectral parameter. Moreover, as usual, A^* stands for the complex conjugate of the transpose of A. With the symbols mentioned above, (1.1) can be re-written into its separated form

$$\begin{cases} \Delta x(t) = A(t)x(t+1) + B(t)u(t) + \lambda W_2(t)u(t), \\ \Delta u(t) = C(t)x(t+1) - A^*(t)u(t) - \lambda W_1(t)x(t+1), t \in \mathbb{Z}_+. \end{cases} \tag{1.2}$$

For the special case $W_2(t) \equiv 0$, we call the Hamiltonian system (1.2) a semi-degenerated Hamiltonian system.

To ensure the existence, uniqueness and continuation of the solutions of any initial value problem for (1.1), we always assume that the following condition holds.

$$(A1) \qquad I_n - A(t) \text{ is invertible in } \mathbb{Z}_+.$$

We now introduce the function space. Since W is only semi-positive definite, the space

$$\mathcal{L}_W^2 = \Big\{ y = \{y(t)\}_{t=0}^{\infty} \subset \mathbb{C}^{2n} | \sum_{t=0}^{\infty} R(y)^*(t)W(t)R(y)(t) < +\infty \Big\} \tag{1.3}$$

of square summable functions only has a semi-norm

$$\|y\| = \Big(\sum_{t=0}^{\infty} R(y)^*(t)W(t)R(y)(t) \Big)^{1/2}, \qquad \forall\, y \in \mathcal{L}_W^2 \tag{1.4}$$

and a semi-definite inner product

$$\langle y, z \rangle = \sum_{t=0}^{\infty} R(z)^*(t)W(t)R(y)(t), \qquad \forall\, y,\, z \in \mathcal{L}_W^2. \tag{1.5}$$

So, the most natural Hilbert space for the study of (1.1) is the quotient space

$$l_W^2 = \mathcal{L}_W^2 / \mathcal{N}_W, \quad \text{where } \mathcal{N}_W = \{ f \in \mathcal{L}_W^2 : \ \|f\| = 0 \}, \tag{1.6}$$

with an obvious norm and an obvious inner product, still denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$. Obviously, for $y(t)=\begin{bmatrix}x(t)\\ u(t)\end{bmatrix}$, one has $y(\cdot)\in l^2_W$ if and only if $x(\cdot+1)\in l^2_{W_1}$ and $u(\cdot)\in l^2_{W_2}$ simultaneously. Note that for each semi-positive definite $n\times n$ matrix valued function M on $\mathbb{Z}_+$, we can define $\mathcal{L}^2_M$ and l^2_M, similar to $\mathcal{L}^2_W$ and l^2_W, but the norm is defined by $\|x\|=\left(\sum_{t=0}^{\infty}x^*(t)M(t)x(t)\right)^{1/2}$, without the partial right shift operator.

In this paper, we study the relationship between the strong limit point condition and limit point condition, as well as between the Dirichlet condition and strong limit point condition for general discrete Hamiltonian systems. First, for the class of discrete Hamiltonian systems whose potential function Q is bounded from below by their weighted function W, the equivalence between the limit point and strong limit point conditions is established. See Theorem 3.1. Second, for a class of semi-degenerated discrete Hamiltonian systems, the limit point condition, strong limit point condition and Dirichlet condition are shown to imply each other, under some assumptions on the coefficient matrices Q and associated Dirac systems. See Theorem 3.2. Finally, an example is given to illustrate the importance of the hypothesis.

2. Preliminary Knowledge

In this section, following the line introduced by E.A. Coddington [8], M. Lesch and M.Malamud [21], and Yuming Shi [30] for spectral theory of linear relations (or subspaces), we introduce some basic knowledge on linear relation (or linear subspace) firstly.

Let X be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$. Let T be a linear relation in $X^2:=X\times X$. The domain of T, the range of T, and the kernel of T are defined respectively by

$$\begin{aligned}\mathcal{D}(T)&:=\{x\in X:(x,f)\in T\text{ for some }f\in X\},\\ \mathcal{R}(T)&:=\{f\in X:(x,f)\in T\text{ for some }x\in X\},\\ \mathcal{N}(T)&:=\{x\in X:(x,0)\in T\}.\end{aligned}$$

For any $\lambda\in\mathbb{C}$, denote

$$T-\lambda I=\{(x,f-\lambda x):(x,f)\in T\}.$$

For two linear relations $T,S\in X^2$, if $T\cap S=\{0\}$, the sum of T and S is defined as

$$T\dot{+}S=\{(x+y,f+g):(x,f)\in T,(y,g)\in S\}.$$

A linear relation T is called closed if T is a closed subspace in X^2. The adjoint of T is defined by

$$T^*:=\{(y,f)\in X^2:\langle y,g\rangle=\langle f,x\rangle\text{ for all }\ (x,g)\in T\}.$$

A linear relation $T\subseteq X^2$ is called Hermitian if $T\subseteq T^*$, and it is called self-adjoint if $T=T^*$.

Lemma 2.1 ([4]). *Let T be a linear relation in X^2.*

(1) *T^* is a closed linear relation;*

(2) *$T^* = (\overline{T})^*$, and $T^{**} = \overline{T}$, where $\overline{T}$ is the closure of T;*

(3) *$\mathcal{N}(T^*) = \mathcal{R}(T)^\perp = \mathcal{R}(\overline{T})^\perp$.*

(4) *Let T be a closed linear relation and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then $\mathcal{R}(T - \lambda I)$ is closed.*

Definition 2.1 ([30, Definition 2.2]). Let X be a Hilbert space and T a linear relation in X^2. $\mathcal{R}(T-\lambda I)^\perp$ is called the deficiency space of T and λ, and $\dim(\mathcal{R}(T-\lambda I)^\perp)$ is called the deficiency index of T and λ.

By Lemma 2.1, we have that

$$\mathcal{R}(T - \lambda I)^\perp = \mathcal{R}(\overline{T} - \lambda I)^\perp = \mathcal{N}(T^* - \overline{\lambda} I) = \{y : (y, \overline{\lambda} y) \in T^*\}.$$

This implies the deficiency indices of T and $\overline{T}$ with the same λ are equal. For convenience, we denote

$$M_\lambda(T) := \{(y, \lambda y) \in T^*\}, d_\lambda(T) := \dim(\mathcal{D}(M_\lambda(T)), d_\pm(T) := d_{\pm i}(T).$$

By Definition 2.1, $d_\lambda(T)$ is the deficiency index of T.

Lemma 2.2 ([21, Corollary 2.23]). *Let T be an Hermitian linear relation in X^2. Then $d_\lambda(T)$ is constant in the upper and lower half-planes.*

So for an Hermitian linear relation T, we call $d_\pm(T)$ the positive and negative deficiency indices of T. The deficiency indices of a closed Hermitian linear relation are crucial in the investigation of its spectra since the deficiency indices determine the number of linear independent self-adjoint boundary conditions that one needs to get a self-adjoint extension of the Hermitian linear relation.

Lemma 2.3 ([21, Proposition 2.22]). *Let T be a closed Hermitian linear relation in X^2. Then for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$,*

$$T^* = T \dot{+} M_\lambda(T) \dot{+} M_{\overline{\lambda}}(T).$$

We now define linear relations H and H_{00} on $l^2_W \times l^2_W$ related to the Hamiltonian system (1.1) as follows:

$$\{y, f\} \in H \Leftrightarrow y, f \in \mathcal{L}^2_W,\ Ly(t) = W(t)R(f)(t),$$
$$\{y, f\} \in H_{00} \Leftrightarrow y(0) = 0, \exists\, n \in \mathbb{Z}_+ \text{ such that } y(t) \equiv 0, t \geq n + 1 \text{ and } \{y, f\} \in H.$$

Clearly, H and H_{00} are linear subspaces in $l^2_W \times l^2_W$, we call H and H_{00} the maximal subspace and pre-minimal subspace corresponding to L, and $\overline{H_{00}} := H_0$ is called the minimal subspace corresponding to L. It can be verified (see [26, Lemma 3.6, Theorem 3.1]) that H_{00} is an Hermitian linear relation in $l^2_W \times l^2_W$, and $H^*_{00} = H^*_0 = H$.

Next we introduce the following **definiteness condition**

$(A2)$ There exists $n_0 \in \mathbb{Z}_+$ such that for all $\lambda \in \mathbb{C}$ and non-trivial solution y of (1.1),

$$\sum_{t=0}^{m} R(y)^*(t)W(t)R(y)(t) > 0, \qquad m \geq n_0.$$

For semi-degenerated Hamiltonian systems, the definiteness condition (A2) holds if and only if the matrix pair (A, B) satisfies the following condition: for each $k \geq n_0$,

$$B(t)u(t) = 0, t \in [1, k] \cap \mathbb{Z}_+, \text{ and } \Delta u(t) = -A^*(t)u(t), t \in [0, k] \cap \mathbb{Z}_+$$

always implies $u(t) \equiv 0, t \in [0, k] \cap \mathbb{Z}_+$.

Since H_0 is Hermitian, by Lemma 2.2, for any $\lambda \in \mathbb{C}$ with $\operatorname{Im} \lambda > 0$ (resp., < 0), the dimension of the deficiency space $R(H_0 - \bar{\lambda})^{\perp}$ is independent of the selection of λ. Under the definiteness condition, it was proved in [26, Corollary 5.1] that the deficiency index of H_0 (i.e., H_{00}) equals to the number of linearly independent square summable solutions of (1.1). So we have that for $\lambda \in \mathbb{C} \setminus \mathbb{R}$, the dimension of the deficiency space $M_\lambda(H_0) := \{(y, \lambda y) \in H\}$, i.e., the deficiency index of H_0 equals to the dimension of

$$\mathcal{M}_\lambda := \{y \in \mathcal{L}^2_W : Ly(t) = \lambda W(t)R(y)(t) \text{ for t} \in \mathbb{Z}_+\}. \tag{2.1}$$

By [29, Theorem 4.1], we know that $n \leq d_\pm \leq 2n$, and

$$d_+ = \dim\{y \in \mathcal{L}^2_W : Ly(t) = iW(t)R(y)(t) \text{ for } t \in \mathbb{Z}_+\},$$
$$d_- = \dim\{y \in \mathcal{L}^2_W : Ly(t) = -iW(t)R(y)(t) \text{ for } t \in \mathbb{Z}_+\}.$$

Followed the Weyl-Titchmarsh classification of second order differential expressions, the Hamiltonian system (1.1) is said to be of the *limit point* type at $+\infty$ if $d_\pm = n$, and of the *limit circle* type at $+\infty$ if $d_\pm = 2n$.

Theorem 2.1. *Assume that (1.1) fulfills (A1) and the definiteness condition (A2). Then the minimal deficiency index of H_0 is invariant under bounded perturbation with respect to the weighted matrix. Namely,* (1.1) *is of the limit point type at $+\infty$ if and only if the following Hamiltonian system*

$$\hat{L}y := J\Delta y(t) - \tilde{Q}(t)R(y)(t) = \lambda W(t)R(y)(t) \tag{2.2}$$

is of the limit point type at $+\infty$, where

$$\tilde{Q}(t) = \begin{bmatrix} -C(t) + C_0W_1(t) & A^*(t) \\ A(t) & B(t) + B_0W_2(t) \end{bmatrix} = Q(t) + \begin{bmatrix} C_0I_n & 0 \\ 0 & B_0I_n \end{bmatrix} W(t),$$

B_0 and C_0 are constants.

Proof. Since the perturbation is $K(t) = \operatorname{diag}(\mathrm{C_0W_1}, \mathrm{B_0W_2})$, we define the linear relation (linear operator) by multiplying the matrix $K(t)$, denoted it by K, here B_0 and C_0 are constants. For $y \in \mathcal{D}(H)$, there exists an $f \in \mathcal{L}^2_W$ such that $\{y, f\} \in H$, i.e., $Ly(t) = W(t)R(f)(t)$, and hence

$$\begin{aligned}\hat{L}y(t) = Ly(t) + K(t)R(y)(t) =& W(t)[R(f)(t) + \operatorname{diag}(C_0I_n, B_0I_n)R(y)(t)] \\ =& W(t)[R(f + \operatorname{diag}(C_0I_n, B_0I_n)y)(t)],\end{aligned}$$

so we obtain $\{y, f + \operatorname{diag}(C_0I_n, B_0I_n)y\} \in H + K$, and this means $y \in \mathcal{D}(H + K)$. This proves $\mathcal{D}(H) \subset \mathcal{D}(H + K)$. Similarly, we deduce that $\{y, f\} \in H + K$ implies $\{y, f - \operatorname{diag}(C_0I_n, B_0I_n)y\} \in H$. So $\mathcal{D}(H + K) \subset \mathcal{D}(H)$. Then the domains of linear relations H and $H + K$ are the same, i.e., $\mathcal{D}(H) = \mathcal{D}(H + K)$. If there exists a sequence $\{y_n\} \subset \mathcal{D}(H_{00})$ such that $\lim_{n\to\infty} y_n = y$, we know that there exists $f_n \in \mathcal{L}^2_W$ such that $\{y_n, f_n\} \in H$, since $\mathcal{D}(H) = \mathcal{D}(H+K)$, we obtain $\{y_n, f_n + \mathrm{diag}(C_0 I_n, B_0 I_n) y_n\} \in H + K$, so $\{f_n\}$ is a Cauchy sequence if and only if $\{f_n + \mathrm{diag}(C_0 I_n, B_0 I_n) y_n\}$ is a Cauchy sequence. This proves $\mathcal{D}(H_0) = \mathcal{D}(H_0 + K)$. By Lemma 2.3, we know that $H = H_0 \dot{+} M_i(H_0) \dot{+} M_{-i}(H_0)$ and $H + K = (H_0 + K) \dot{+} M_i((H_0+K)) \dot{+} M_{-i}((H_0+K))$. So $\dim\{M_i(H_0) \dot{+} M_{-i}(H_0)\} = \dim\{M_i((H_0 + K)) \dot{+} M_{-i}((H_0+K))\}$. By (2.1) we deduce that $d_+(H_0) + d_-(H_0) = d_+(H_0+K) + d_-(H_0+K)$, this together with $n \le d_\pm \le 2n$ implies that $d_\pm(H_0) = d_\pm(H_0+K) = n$. This completes the proof. □

It was shown [29, Theorem 6.15] that (1.1) is of the limit point type at $+\infty$ if and only if for any given $y, z \in \mathcal{D}(H)$,

$$\lim_{t\to+\infty} z^*(t) J y(t) = 0. \tag{2.3}$$

It is easy to verify that (2.3) is equivalent to for each pair of $\begin{bmatrix} x_1 \\ u_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ u_2 \end{bmatrix} \in \mathcal{D}(H)$,

$$\lim_{t\to+\infty} [x_1^*(t) u_2(t) - u_1^*(t) x_2(t)] = 0. \tag{2.4}$$

The further classification of limit point type is the *strong limit point* type and the *weak limit point* type, which was given by Everitt, et. al. in [12, 13] for high order scalar differential equations and deeply studied in [14, 15]. Recently, the concept of strong limit point type has been generalized to general Hamiltonian differential systems by Qi and Chen [22], and to discrete Hamiltonian systems by Sun and Shi [32]. Here, we use the same definition for discrete Hamiltonian system (1.1). We call system (1.1) is of the *strong limit point* type at $+\infty$ if for each pair of $\begin{bmatrix} x_1 \\ u_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ u_2 \end{bmatrix} \in \mathcal{D}(H)$,

$$\lim_{t\to+\infty} x_1^*(t) u_2(t) = 0. \tag{2.5}$$

By definition, the strong limit point condition at $+\infty$ implies the limit point condition at $+\infty$; but the converse is not true in general. In paper [32, Theorem 2.1], the authors obtained that the strong limit point condition at $+\infty$ is equivalent to that for any $\begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{D}(H)$,

$$\lim_{t\to+\infty} x^*(t) u(t) = 0. \tag{2.6}$$

The discrete Hamiltonian system (1.1) is said to satisfy the *Dirichlet condition* if for every $\begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{D}(H)$,

$$\sum_{t=0}^{+\infty} (|x^*(t+1) C(t) x(t+1)| + |u^*(t) B(t) u(t)|) < +\infty. \tag{2.7}$$

The strong limit point condition and Dirichlet condition are closely related, and for $2n$ order differential equations, these relationships have been investigated by Everitt [16] (see the references therein). However, to our best knowledge, there is no related result for the Dirichlet condition of the discrete Hamiltonian system.

3. Main Results

In this section, we will give the relation among the limit point type, the strong limit point type and the Dirichlet condition, some equivalent results are obtained based on the asymptotic behaviors or square summabilities of functions in the maximal domains. First, we give a lemma which will be used in the proof of the main results.

For two $n \times n$ Hermitian matrices M_1 and M_2, $M_1 \sim M_2$ means that $k_1 M_1 \leq M_2 \leq k_2 M_1$ for some constants $k_1, k_2 > 0$.

Lemma 3.1. *Let M_1 and M_2 be semi-positive-definite matrices and $M_1 \sim M_2$. Then M_1 and M_2 are diagonalizable simultaneously.*

Proof. Since $M_1 \sim M_2$, we obtain that $\operatorname{rank} M_1 = \operatorname{rank} M_2 \triangleq r$, where $0 < r \leq n$. M_1 is Hermitian, there exists a nonsingular matrix P such that

$$P^* M_1 P = \begin{bmatrix} E_r & 0 \\ 0 & 0 \end{bmatrix}, P^* M_2 P \triangleq \begin{bmatrix} M_r & N \\ N^* & S \end{bmatrix},$$

where $M_r = (m_{ij})_{r\times r}$, $N = (n_{ij})_{r\times(n-r)}$, $S = (s_{ij})_{(n-r)\times(n-r)}$. Since $k_1 M_1 \leq M_2 \leq k_2 M_1$ for some constants $k_1, k_2 > 0$, we have

$$k_1 \begin{bmatrix} E_r & 0 \\ 0 & 0 \end{bmatrix} \leq \begin{bmatrix} M_r & N \\ N^* & S \end{bmatrix} \leq k_2 \begin{bmatrix} E_r & 0 \\ 0 & 0 \end{bmatrix}$$

and $k_1 E_r \leq M_r \leq k_2 E_r$, $k_1 \leq m_{11} \leq k_2$. So $\operatorname{rank} M_r = r$ and $S \equiv 0$. We show that $N \equiv 0$. If it is not true, then there exists an $n_{ij} \in \mathbb{C}$ such that $n_{ij} \neq 0$. Define $\alpha = (1, 0, \cdots, 0, x, 0, \cdots 0)^T$, where x is the $(j+r)$-th component, then we get

$$k_1 \leq \alpha^* \begin{bmatrix} M_r & N \\ N^* & 0 \end{bmatrix} \alpha \leq k_2,$$

i.e., $k_1 \leq m_{11} + 2\operatorname{Re}(n_{ij} x^*) \leq k_2$. This together with $k_1 \leq m_{11} \leq k_2$ implies $|2\operatorname{Re}(n_{ij} x^*)| \leq k_2 - k_1$, which is impossible for arbitrary $x \in \mathbb{C}$. So $P^* M_2 P = \begin{bmatrix} M_r & 0 \\ 0 & 0 \end{bmatrix}$ and M_r is Hermitian, then there exists a unitary matrix U_r such that $U_r^* M_r U_r = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_r)$. Take $Q = \begin{bmatrix} U_r & 0 \\ 0 & E_{n-r} \end{bmatrix}$. Then

$$\begin{aligned} Q^* P^* M_2 P Q &= \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_r, 0, \cdots, 0), \\ Q^* P^* M_1 P Q &= Q^* \begin{bmatrix} E_r & 0 \\ 0 & 0 \end{bmatrix} Q = \begin{bmatrix} U_r^* E_r U_r & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} E_r & 0 \\ 0 & 0 \end{bmatrix} \\ &= \operatorname{diag}(1, 1, \cdots, 1, 0, \cdots, 0). \end{aligned}$$

This completes the proof of this lemma. □

The following theorem is about the relationship between the limit point condition and strong limit point condition.

Theorem 3.1. *Assume that (1.1) fulfills (A1) and the definiteness condition (A2), and satisfies the following condition:*

$$B \geq B_0 W_2 \text{ and } C \geq C_0 W_1 \text{ on } \mathbb{Z}_+ \text{ for some constants } B_0 \text{ and } C_0. \tag{3.1}$$

Then, (1.1) *is of the limit point type at* $+\infty$ *if and only if it is of the strong limit point type at* $+\infty$.

Proof. The sufficiency is proved by definition, only the necessity requires a proof. Since under the definiteness condition, we know by Theorem 2.1 that the bounded perturbation of the Hamiltonian systems with respect to the weight matrix W does not change the minimal deficiency indices, without loss of generality, we may suppose that $B_0 \geq 1$ and $C_0 \geq 1$, for otherwise we can replace (1.1) by its bounded perturbation

$$\begin{cases} \Delta x(t) = A(t)x(t+1) + [B(t) - (B_0 - 1)W_2(t)]u(t) + \lambda W_2(t)u(t), \\ \Delta u(t) = [C(t) - (C_0 - 1)W_1(t)]x(t+1) - A^*(t)u(t) - \lambda W_1(t)x(t+1), \end{cases} \tag{3.2}$$

which is of the limit point type at $+\infty$ if and only if so is (1.1). Firstly, we want to use the transformation $T_\theta : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ for $\theta \in (0, \pi/4)$ defined by

$$T_\theta y = \begin{bmatrix} I_n & 0 \\ 0 & -\mathrm{i}\,\mathrm{e}^{\mathrm{i}\,\theta} I_n \end{bmatrix} y \tag{3.3}$$

to get a new discrete Hamiltonian systems $L_\theta y(t) = \lambda W_\theta R(y)(t)$ associated with (1.1), and prove that they are also of the limit point type at $+\infty$.

Fix a $\theta \in (0, \pi/4)$. For a solution $y = \begin{bmatrix} x \\ u \end{bmatrix}$ of $Ly(t) = \mathrm{i}\, W(t)R(y)(t)$, set

$$y_\theta = \begin{bmatrix} x_\theta \\ u_\theta \end{bmatrix} = T_\theta y. \tag{3.4}$$

Then,

$$\begin{cases} \Delta x_\theta(t) = A(t)x_\theta(t+1) + B_\theta(t)u_\theta(t) + iW_{2,\theta}(t)u_\theta(t), \\ \Delta u_\theta(t) = C_\theta(t)x_\theta(t+1) - A^*(t)u_\theta(t) - iW_{1,\theta}(t)x_\theta(t+1), \end{cases} \tag{3.5}$$

where

$$B_\theta = B\sin\theta - W_2\cos\theta, \qquad W_{2,\theta} = B\cos\theta + W_2\sin\theta, \tag{3.6}$$

$$C_\theta = C\sin\theta - W_1\cos\theta, \qquad W_{1,\theta} = C\cos\theta + W_1\sin\theta. \tag{3.7}$$

By the condition (3.1) and our assumption B_0, $C_0 \geq 1$,

$$W_{1,\theta} \geq W_1, \quad W_{2,\theta} \geq W_2 \qquad \text{on } \mathbb{Z}_+. \tag{3.8}$$

Now, we consider the discrete Hamiltonian system

$$L_\theta y(t) := J\Delta y(t) - Q_\theta(t)R(y)(t) = \lambda W_\theta(t)R(y)(t), \tag{3.9}$$

where

$$Q_\theta = \begin{bmatrix} -C_\theta & A^* \\ A & B_\theta \end{bmatrix}, \quad W_\theta = \begin{bmatrix} W_{1,\theta} & 0 \\ 0 & W_{2,\theta} \end{bmatrix} \geq W. \tag{3.10}$$

So, $y_\theta(t)$ is a solution of $L_\theta y(t) = \mathrm{i}\, W_\theta(t)R(y)(t)$. Conversely, by reversing the above discussions, one shows that if y_θ is a solution of $L_\theta y(t) = \mathrm{i}\, W_\theta(t)R(y)(t)$, then $y = (T_\theta)^{-1}y_\theta$ is a solution of $Ly(t) = \mathrm{i}\, W(t)R(y)(t)$.

Assume that $y_{1,\theta}$, ..., $y_{m,\theta}$ are linearly independent solutions of $L_\theta y(t) = \mathrm{i}\, W_\theta(t) R(y)(t)$ in $\mathcal{L}^2_{W_\theta}$, where $m \in \mathbb{N}$. Then, $y_k = (T_\theta)^{-1}y_{k,\theta}$, $k = 1$, ..., m, are linearly independent solutions of $Ly(t) = \mathrm{i}\, W(t)R(y)(t)$; and they are in $\mathcal{L}^2_W$, since $W_\theta \geq W$. Thus, the positive deficiency index of (3.9) is at most $d_+ = n$, and hence equals $d_+ = n$.

Similarly, using $T_{\pi-\theta}$, one shows that the negative deficiency index of (3.9) is also $d_- = n$. Hence, (3.9) is of the limit point type at $+\infty$, too.

Moreover, the above discussions about the positive deficiency index of (3.9) imply that if y_1, ..., y_n are linearly independent solutions of $Ly(t) = \mathrm{i}\, W(t)R(y)(t)$ in $\mathcal{L}^2_W$, then $T_\theta y_1$, ..., $T_\theta y_n$ are linearly independent solutions of $L_\theta y(t) = \mathrm{i}\, W_\theta(t)R(y)(t)$ in $\mathcal{L}^2_{W_\theta}$. Hence,

$$y \in \mathcal{M}_i \quad \Longrightarrow \quad T_\theta y \in \mathcal{M}_i(L_\theta, W_\theta) \subset \mathcal{D}_1(L_\theta, W_\theta), \tag{3.11}$$

where $\mathcal{M}_i$ is defined by (2.1), it is the null space of $L - \mathrm{i}\, W$ in $\mathcal{L}^2_W$, and $\mathcal{M}_i(L_\theta, W_\theta)$ is for $L_\theta - \mathrm{i}\, W_\theta$, $\mathcal{D}_1(L_\theta, W_\theta)$ is the domain of the maximal linear relation generated by L_θ on $l^2_{W_\theta}$. Similarly,

$$y \in \mathcal{M}_{-i} \quad \Longrightarrow \quad T_\theta y \in \mathcal{M}_i(\widetilde{L}_\theta, \widetilde{W}_\theta) \subset \mathcal{D}_1(\widetilde{L}_\theta, \widetilde{W}_\theta), \tag{3.12}$$

where $\mathcal{M}_{-i}$ has an obvious meaning, $\widetilde{L}_\theta y = J\Delta y - \widetilde{Q}_\theta R(y)$, and

$$\widetilde{Q}_\theta = \begin{bmatrix} -\widetilde{C}_\theta & A^* \\ A & \widetilde{B}_\theta \end{bmatrix}, \quad \widetilde{W}_\theta = \begin{bmatrix} \widetilde{W}_{1,\theta} & 0 \\ 0 & \widetilde{W}_{2,\theta} \end{bmatrix}, \tag{3.13}$$

with

$$\widetilde{B}_\theta = B\sin\theta + W_2\cos\theta, \qquad \widetilde{W}_{2,\theta} = B\cos\theta - W_2\sin\theta, \tag{3.14}$$

$$\widetilde{C}_\theta = C\sin\theta + W_1\cos\theta, \qquad \widetilde{W}_{1,\theta} = C\cos\theta - W_1\sin\theta. \tag{3.15}$$

Note that $W_\theta \geq W \geq \widetilde{W}_\theta$, and for $r > 0$ sufficiently small, we obtain $\widetilde{W}_\theta \geq rW_\theta$ (Since $B_0 \geq 1$ and $C_0 \geq 1$ and $\theta \in (0, \pi/4)$, we choose $0 < r \leq \min\{\frac{C_0\cot\theta - 1}{\cot\theta + 1}, \frac{B_0\cot\theta - 1}{\cot\theta + 1}\}$). So

$$0 \leq rW_\theta(t) \leq \widetilde{W}_\theta(t) \leq W(t) \leq W_\theta(t), \qquad t \in \mathbb{Z}_+. \tag{3.16}$$

Moreover, we obtain $\widetilde{B}_\theta - B_\theta = 2\cos\theta W_2$ and $C_\theta - \widetilde{C}_\theta = -2\cos\theta W_1$.

Now, we want to prove $\mathcal{D}_1(\widetilde{L}_\theta, \widetilde{W}_\theta) = \mathcal{D}_1(L_\theta, W_\theta)$. For $y \in \mathcal{D}_1(\widetilde{L}_\theta, \widetilde{W}_\theta)$, we obtain $y \in \mathcal{L}^2_{\widetilde{W}_\theta}$ and there exists $f \in \mathcal{L}^2_{\widetilde{W}_\theta}$ such that $\widetilde{L}_\theta y(t) = \widetilde{W}_\theta(t)R(f)(t)$. By (3.16), we deduce that $y, f \in \mathcal{L}^2_{W_\theta}$ and

$$\begin{aligned} L_\theta y(t) &= \widetilde{L}_\theta y(t) + \begin{bmatrix} C_\theta - \widetilde{C}_\theta & 0 \\ 0 & \widetilde{B}_\theta - B_\theta \end{bmatrix}(t)R(y)(t) \\ &= \widetilde{W}_\theta(t)R(f)(t) + 2\cos\theta\,\mathrm{diag}\,(-I_n, I_n)W(t)R(y)(t). \end{aligned} \tag{3.17}$$

Since $W_\theta \sim \widetilde{W}_\theta$, by Lemma 3.1, there exists a nonsingular matrix-valued function $U(\cdot)$ on $\mathbb{Z}_+$ such that $U^*W_\theta U$ and $U^*\widetilde{W}_\theta U$ are diagonal matrices, namely, $U^*W_\theta U = \operatorname{diag}(w_1, w_2, \cdots, w_{2n})$ and $U^*\widetilde{W}_\theta U = \operatorname{diag}(\tilde{w}_1, \tilde{w}_2, \cdots, \tilde{w}_{2n})$, where $w_i, \tilde{w}_i \geq 0$ for $1 \leq i \leq 2n$. By (3.16), we obtain $rU^*W_\theta U \leq U^*\widetilde{W}_\theta U \leq U^*W_\theta U$, so $rw_i \leq \tilde{w}_i \leq w_i$, and there exist $\lambda_i : r \leq \lambda_i \leq 1$ (λ_i can be selected as an arbitrary number lying $[r, 1]$ when $w_i = 0$) such that $\tilde{w}_i = \lambda_i w_i$ for $1 \leq i \leq 2n$. Then

$$\begin{aligned} \widetilde{W}_\theta R(f) &= U^{*-1}U^*\widetilde{W}_\theta U U^{-1}R(f) \\ &= U^{*-1}\operatorname{diag}(\tilde{w}_1, \cdots, \tilde{w}_{2n})U^{-1}R(f) \\ &= U^{*-1}\operatorname{diag}(w_1, \cdots, w_{2n})\operatorname{diag}(\lambda_1, \cdots, \lambda_{2n})U^{-1}R(f) \\ &= W_\theta U\operatorname{diag}(\lambda_1, \cdots, \lambda_{2n})U^{-1}R(f). \end{aligned} \tag{3.18}$$

Since $r \leq \lambda_i \leq 1$ for $1 \leq i \leq 2n$ and $f \in \mathcal{L}^2_{W_\theta}$, we obtain $f_1 = U\operatorname{diag}(\lambda_1, \cdots, \lambda_{2n})U^{-1}f \in \mathcal{L}^2_{W_\theta}$ and $\widetilde{W}_\theta R(f) = W_\theta R(f_1)$. Similarly, $W_\theta \sim W$ means that there exists $f_2 \in \mathcal{L}^2_{W_\theta}$ such that $W_\theta R(f_2) = 2\cos\theta\operatorname{diag}(-I_n, I_n)WR(y)$. So $\{y, f_1 + f_2\} \in L_\theta$ and $y \in \mathcal{D}_1(L_\theta, W_\theta)$. This proves $\mathcal{D}_1(\widetilde{L}_\theta, \widetilde{W}_\theta) \subset \mathcal{D}_1(L_\theta, W_\theta)$. Similarly, we can obtain $\mathcal{D}_1(L_\theta, \widetilde{W}_\theta) \subset \mathcal{D}_1(\widetilde{L}_\theta, W_\theta)$. So $\mathcal{D}_1(\widetilde{L}_\theta, \widetilde{W}_\theta) = \mathcal{D}_1(L_\theta, W_\theta)$.

Hence (3.11) and (3.12) yield that

$$y \in \mathcal{M}_{\pm i} \quad \Longrightarrow \quad T_\theta y \in \mathcal{D}_1(L_\theta, W_\theta). \tag{3.19}$$

Secondly, we want to prove

$$y \in \mathcal{D}(H_0) \quad \Longrightarrow \quad T_\theta y \in \mathcal{D}_1(L_\theta, W_\theta). \tag{3.20}$$

Since H_0 is the closure of linear relation H_{00}. Let $y(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathcal{D}(H_0)$, then there exists $f(t) = \begin{bmatrix} g(t) \\ l(t) \end{bmatrix} \in \mathcal{L}^2_W$ such that $Ly(t) = W(t)R(f)(t)$, and there exists $(y_j, f_j) \in H_{00}$ such that $y_j \to y$, $f_j \to f$ in $\mathcal{L}^2_W$ as $j \to +\infty$. Set

$$\begin{bmatrix} x_{j,k}(t) \\ u_{j,k}(t) \end{bmatrix} = y_j - y_k, \begin{bmatrix} g_{j,k}(t) \\ l_{j,k}(t) \end{bmatrix} = f_j - f_k, \widetilde{W} = \begin{bmatrix} C & 0 \\ 0 & B \end{bmatrix},$$

where $j, k \in \mathbb{Z}_+$. Since $\operatorname{supp} y_j$ is compact for $j = 1, 2, \cdots$, and is contained in $\mathbb{Z}_+ \backslash \{0\}$, we obtain

$$\begin{aligned} &\sum_{t=0}^{+\infty}\left[x^*_{j,k}(t+1)C(t)x_{j,k}(t+1) + u^*_{j,k}(t)B(t)u_{j,k}(t)\right] \\ =&\sum_{t=0}^{+\infty}\left[x^*_{j,k}(t+1)u_{j,k}(t+1) - x^*_{j,k}(t)u_{j,k}(t)\right] \\ &+\sum_{t=0}^{+\infty}\left[x^*_{j,k}(t+1)W_1(t)g_{j,k}(t+1) - l^*_{j,k}(t)W_2(t)u_{j,k}(t)\right] \\ =&\sum_{t=0}^{+\infty}\left[x^*_{j,k}(t+1)W_1(t)g_{j,k}(t+1) - l^*_{j,k}(t)W_2(t)u_{j,k}(t)\right]. \end{aligned} \tag{3.21}$$

Since $(y_j, f_j) \in H_{00}$, the right-hand side of (3.21) tends to zero as $j, k \to +\infty$, which together with B, $C \geq 0$ on $\mathbb{Z}_+$ yields that $\{y_j\}_{j=1}^{+\infty}$ is a Cauchy sequence in $\mathcal{L}^2_{\widetilde{W}}$, hence $y_j \to \tilde{y}$ in $\mathcal{L}^2_{\widetilde{W}}$ as $j \to +\infty$. Since $y \in \mathcal{L}^2_{\widetilde{W}}$ implies $y \in \mathcal{L}^2_W$ and $y_j \to y$ in $\mathcal{L}^2_W$, we know that $y = \tilde{y}$, so $y(\cdot) \in l^2_{\widetilde{W}}$, i.e., $x(\cdot + 1) \in l^2_C$, and $u(\cdot) \in l^2_B$. Moreover, we deduce that there exist $\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in l^2_{W_\theta}$, such that $y_\theta = \begin{bmatrix} x_\theta \\ u_\theta \end{bmatrix} = T_\theta y$ satisfies

$$\Delta x_\theta(t) = A(t)x_\theta(t+1) + B_\theta(t)u_\theta(t) + \mathrm{i}\, W_{2,\theta}(t)u_\theta(t) + W_2(f_2(t) + \mathrm{e}^{-\mathrm{i}\,\theta}\, u_\theta(t)), \quad (3.22)$$

$$\begin{aligned} \Delta u_\theta(t) &= C_\theta(t+1)x_\theta(t+1) - A^*(t)u_\theta(t) - \mathrm{i}\, W_{1,\theta}(t)x_\theta(t+1) \\ &\quad + W_1(t)\,\mathrm{e}^{\mathrm{i}\,\theta}(x_\theta(t+1) + \mathrm{i}\, f_1(t+1)), \end{aligned} \quad (3.23)$$

where $W_{1,\theta}$ and $W_{2,\theta}$ are defined in (3.6) and (3.7), respectively. Set $\delta = \min\{B_0, C_0\}$ and note that

$$\frac{\sqrt{2}}{2}C \leq C\cos\theta \leq W_{1,\theta} \leq (1 + 1/\delta)C \text{ and } \frac{\sqrt{2}}{2}B \leq B\cos\theta \leq W_{1,\theta} \leq (1 + 1/\delta)B$$

on $\mathbb{Z}_+$. One sees that $f_1(\cdot + 1) \in \mathcal{L}^2_{W_1}$, $f_2(\cdot) \in \mathcal{L}^2_{W_2}$, $x_\theta(\cdot + 1) = x(\cdot + 1) \in \mathcal{L}^2_C$, and $u_\theta(\cdot) = -\mathrm{i}\,\mathrm{e}^{\mathrm{i}\,\theta}\, u(t) \in \mathcal{L}^2_B$. Thus, $y_\theta \in \mathcal{L}^2_{W_\theta}$. Similar to the proof of the equality (3.18), we know that there exist $g_1(\cdot + 1) \in \mathcal{L}^2_{W_{1,\theta}}$ and $g_2(\cdot) \in \mathcal{L}^2_{W_{2,\theta}}$ such that

$$\begin{aligned} W_{1,\theta}(t)g_1(t+1) &= W_1(t)\,\mathrm{e}^{\mathrm{i}\,\theta}(x_\theta(t+1) + \mathrm{i}\, f_1(t+1)), \\ W_{2,\theta}(t)g_2(t) &= W_2(f_2(t) + \mathrm{e}^{-\mathrm{i}\,\theta}\, u_\theta(t)). \end{aligned} \quad (3.24)$$

If we set $f_{1,\theta}(t) = \mathrm{i}\, x_\theta(t) - g_1(t)$ and $f_{2,\theta}(t) = \mathrm{i}\, u_\theta(t) + g_2(t)$, then $f_\theta(t) := \begin{bmatrix} f_{1,\theta}(t) \\ f_{1,\theta}(t) \end{bmatrix} \in \mathcal{L}^2_{W_\theta}$. Furthermore, (3.22) and (3.23) can be rewritten as

$$\begin{aligned} \Delta x_\theta(t) &= A(t)x_\theta(t+1) + B_\theta(t)u_\theta(t) + W_{2,\theta}(t)f_{2,\theta}(t) \\ \Delta u_\theta(t) &= C_\theta(t+1)x_\theta(t+1) - A^*(t)u_\theta(t) - W_{1,\theta}(t)f_{1,\theta}(t+1), \end{aligned} \quad (3.25)$$

i.e., $L_\theta y_\theta(t) = W_\theta(t)R(f_\theta)(t)$, and hence $y_\theta \in \mathcal{D}_1(L_\theta, W_\theta)$. This establishes (3.20).

Finally, we complete the proof using some asymptotic behavior of the elements in the maximal domain $\mathcal{D}_1(L_\theta, W_\theta)$.

By Lemma 2.3, $H = H_0 \dotplus \mathcal{M}_i \dotplus \mathcal{M}_{-i}$, and hence (3.19) and (3.20) yield

$$y \in \mathcal{D}(H) \quad \Longrightarrow \quad T_\theta y \in \mathcal{D}_1(L_\theta, W_\theta). \quad (3.26)$$

Since (3.9) is of the limit point type, we have

$$\lim_{t \to +\infty} y_\theta^*(t)\, J\, z_\theta(t) = 0 \qquad \forall\, y_\theta,\, z_\theta \in \mathcal{D}_1(L_\theta, W_\theta). \quad (3.27)$$

Now, let $y = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathcal{D}(H)$. Choose θ_1, $\theta_2 \in (0, \pi/4)$ such that $\theta_1 \neq \theta_2$. Set

$$y_{1,\theta} = T_{\theta_1} y, \quad y_{2,\theta} = T_{\theta_2} y. \quad (3.28)$$

Then, (3.26) and (3.27) imply that as $t \to +\infty$,

$$e^{2\,i\,\theta_1}\, x^*(t)u(t) + u^*(t)x(t) = -\,i\,e^{i\,\theta_1}\, y^*_{1,\theta}(t)\, J\, y_{1,\theta}(t) \longrightarrow 0, \tag{3.29}$$

$$e^{2\,i\,\theta_2}\, x^*(t)u(t) + u^*(t)x(t) = -\,i\,e^{i\,\theta_2}\, y^*_{2,\theta}(t)\, J\, y_{2,\theta}(t) \longrightarrow 0. \tag{3.30}$$

Hence, $x^*(t)u(t) \to 0$ as $t \to +\infty$. Therefore, (1.1) is of the strong limit point type at $+\infty$. This completes the proof. □

The following is a direct corollary of Theorem 3.1, applied to semi-degenerated discrete Hamiltonian systems

$$\begin{cases} \Delta x(t) = A(t)x(t+1) + B(t)u(t), \\ \Delta u(t) = C(t)x(t+1) - A^*(t)u(t) - \lambda W_1(t)x(t+1), t \in \mathbb{Z}_+. \end{cases} \tag{3.31}$$

Corollary 3.1. *Assume that (3.31) fulfills (A1) and the definiteness condition (A2), and satisfies the following condition:*

$$B \geq 0 \text{ and } C \geq C_0 W_1 \text{ on } \mathbb{Z}_+ \text{ for some constant } C_0. \tag{3.32}$$

Then, (3.31) *is of the limit point type at* $+\infty$ *if and only if it is of the strong limit point type at* $+\infty$*.*

Next, we establish the equivalence of the strong limit point condition and Dirichlet condition for Hamiltonian system (3.31).

Theorem 3.2. *Assume that (3.31) fulfills (A1) and the definiteness condition (A2),* B *and* C *satisfy* (3.32)*; and the Dirac system*

$$\begin{cases} \Delta x(t) = A(t)x(t+1) + B(t)u(t) + \lambda B(t)u(t), \\ \Delta u(t) = C(t)x(t+1) - A^*(t)u(t) - \lambda W_1(t)x(t+1), t \in \mathbb{Z}_+ \end{cases} \tag{3.33}$$

associated with (3.31) is of the limit point type at $+\infty$*. Then,*

(i) *for each* $y(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathcal{D}(H)$ *such that* $u \in \mathcal{L}^2_B$*, one has that* $x^*(t)u(t) \to 0$ *as* $t \to +\infty$*; and*

(ii) *for (3.31), each of the following three conditions at* $+\infty$ *implies the other two at* $+\infty$*: the limit point condition, the strong limit point condition, and the Dirichlet condition.*

Note that the weight function in the Dirac system (3.33) is $V = \mathrm{diag}(W_1, B)$, in stead of W; and hence (3.33) is studied in $\mathcal{L}^2_V$ and l^2_V, neither $\mathcal{L}^2_W$ nor l^2_W.

Proof. For simplicity, we denote the maximum domain relating to the Dirac system (3.33) by $\mathcal{D}_1(V)$, where $V = \mathrm{diag}(W_1, B)$. Note that the weight function of system (3.31) is $\mathrm{diag}(W_1, 0)$, so the Dirac system (3.33) satisfies the definiteness condition (A2).

(i) Since (3.33) is of the limit point type at $+\infty$, the assumptions on B and C together with Theorem 3.1 imply that (3.33) is actually of the strong limit point type at $+\infty$. So, for each $y(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathcal{D}_1(V)$,

$$\lim_{t \to +\infty} x^*(t)u(t) = 0. \tag{3.34}$$

If $y(t)=\begin{bmatrix}x(t)\\u(t)\end{bmatrix}\in\mathcal{D}(H)$ such that $u\in\mathcal{L}^2_B$, then $y(t)=\begin{bmatrix}x(t)\\u(t)\end{bmatrix}\in\mathcal{L}^2_V$ and there exists $f(t)=\begin{bmatrix}f_1(t)\\f_2(t)\end{bmatrix}\in\mathcal{L}^2_W$ such that

$$\begin{cases}\Delta x(t)=A(t)x(t+1)+B(t)u(t),\\ \Delta u(t)=C(t)x(t+1)-A^*(t)u(t)-W_1(t)f_1(t+1), t\in\mathbb{Z}_+\end{cases}\tag{3.35}$$

for $W_2=0$ now; so,

$$\begin{cases}\Delta x(t)=A(t)x(t+1)+B(t)u(t)+W_2(t)\tilde{f}_2(t),\\ \Delta u(t)=C(t)x(t+1)-A^*(t)u(t)-W_1(t)f_1(t+1), t\in\mathbb{Z}_+\end{cases}\tag{3.36}$$

with $\tilde{f}_2(t)\equiv 0$, and hence $y(t)=\begin{bmatrix}x(t)\\u(t)\end{bmatrix}\in\mathcal{D}_1(V)$ since $\begin{bmatrix}f_1(t)\\\tilde{f}_2(t)\end{bmatrix}\in\mathcal{L}^2_V$. Thus, (3.34) is true.

(ii) For (3.31), the equivalence between the limit point condition and strong limit point condition is implied by Theorem 3.1. So, we need only to show that (3.31) is of the strong limit point type at $+\infty$ if and only if it satisfies the Dirichlet condition at $+\infty$.

Necessity. Let $y(t)=\begin{bmatrix}x(t)\\u(t)\end{bmatrix}\in\mathcal{D}(H)$. Then, there exists $f(t)=\begin{bmatrix}f_1(t)\\f_2(t)\end{bmatrix}\in\mathcal{L}^2_W$ such that (3.35) is valid. Thus, the strong limit point condition (3.34) yields

$$\sum_{t=0}^{+\infty}(x^*(t+1)C(t)x(t+1)+u^*(t)B(t)u(t))=\langle f,y\rangle-x^*(0)u(0)<+\infty.\tag{3.37}$$

So,

$$\begin{aligned}&\sum_{t=0}^{+\infty}(|x^*(t+1)C(t)x(t+1)|+|u^*(t)B(t)u(t)|)\\ \le&\sum_{t=0}^{+\infty}[x^*(t+1)(C(t)-C_0W_1(t))x(t+1)+|C_0|x^*(t+1)W_1(t)x(t+1)+u^*(t)B(t)u(t)]\\ =&\sum_{t=0}^{+\infty}(x^*(t+1)C(t)x(t+1)+u^*(t)B(t)u(t))+(|C_0|-C_0)\sum_{t=0}^{+\infty}x^*(t+1)W_1(t)x(t+1)\\ <&+\infty.\end{aligned}\tag{3.38}$$

Hence, (3.31) satisfies the Dirichlet condition.

Sufficiency. Let $y(t)=\begin{bmatrix}x(t)\\u(t)\end{bmatrix}\in\mathcal{D}(H)$. Then, the Dirichlet condition implies

$$\sum_{t=0}^{+\infty}[u^*(t)B(t)u(t)+x^*(t+1)(C(t)-C_0W_1(t))x(t+1)]<+\infty,\tag{3.39}$$

and hence $\sum_{t=0}^{+\infty} u^*(t)B(t)u(t) < +\infty$, i.e., $u \in \mathcal{L}_B^2$. Thus, (3.34) is true by (i). Therefore, (3.31) is of the strong limit point type at $+\infty$. This completes the proof. □

As an application, we now consider the Sturm-Liouville difference equation

$$-\Delta(p(t-1)\Delta x(t-1)) + q(t)x(t) = \lambda\omega(t)x(t) \quad \text{on } \mathbb{Z}_+, \tag{3.40}$$

where $p(t)$, $q(t)$ and $\omega(t)$ are $n\times n$ Hermitian matrix valued functions on $\mathbb{Z}_+$, $p(t)$ and $\omega(t)$ are positive definite. System (3.40) can be transformed into a semi-degenerated Hamiltonian system of the form

$$\begin{cases} \Delta x(t) = p^{-1}(t)u(t), \\ \Delta u(t) = q(t)x(t+1) - \lambda\omega(t+1)x(t+1) \end{cases} \tag{3.41}$$

with $A(t) \equiv 0, B(t) = p^{-1}(t), C(t) = q(t+1)$ and $W_1(t) = \omega(t+1)$. Now we consider the Dirac system related to (3.41) as follows

$$\begin{cases} \Delta x(t) = p^{-1}(t)u(t) + \lambda p^{-1}(t)u(t), \\ \Delta u(t) = q(t)x(t+1) - \lambda\omega(t+1)x(t+1). \end{cases} \tag{3.42}$$

Since the weighted function is $V(t) = \mathrm{diag}(\omega(t+1), p^{-1}(t))$, (3.42) can be viewed as a bounded perturbation system with respect to the weighted function of the Dirac system

$$\begin{cases} \Delta x(t) = \lambda p^{-1}(t)u(t), \\ \Delta u(t) = q(t)x(t+1) - \lambda\omega(t+1)x(t+1). \end{cases} \tag{3.43}$$

Both the systems (3.42) and (3.43) have the same deficiency indices. At $\lambda = 0$, we can solve (3.43), and obtain one of its fundamental solution matrix

$$\Phi(t) = \begin{bmatrix} 0 & I_n \\ I_n & \sum_{s=0}^{t-1} q(s) \end{bmatrix}.$$

So if for all $j \in \{1, 2, \cdots, n\}$, one has

$$\sum_{t=0}^{+\infty}(p^{-1})_{jj}(t) = +\infty, \text{or } \sum_{t=0}^{+\infty}\left[\omega_{jj}(t+1) + \sum_{s=0}^{t-1} q_j^*(s)p^{-1}(t)\sum_{s=0}^{t-1} q_j(s)\right] = +\infty, \tag{3.44}$$

where $(p^{-1})_{jj}$ is the (j,j)-entry of p^{-1}, q_j is the j-th column of q, etc, then the first or the last n-solutions of $\Phi(t)$ are not in l_V^2, thus we have (3.43) is of limit point type at $+\infty$, so does (3.42). Now we obtain the following corollary by Theorem 3.2.

Corollary 3.2. *Assume that $q(t) \geq q_0\omega(t)$ for some constant q_0. Furthermore,* (3.44) *holds for all $j \in \{1, 2, \cdots, n\}$. Then for system* (3.40)*, each of the following three conditions at $+\infty$ yields the other two at $+\infty$: the limit point condition, the strong limit point condition, and the Dirichlet condition.*

We remark that in Theorem 3.2, (3.31) is in general not of the limit point type at $+\infty$. For an example, see Example 3.1.

Example 3.1. The scalar Sturm-Liouville equation

$$-\Delta^2 x(t-1) = \lambda 2^{-t} x(t) \quad \text{on } \mathbb{Z}_+ \tag{3.45}$$

satisfies (3.44), but is of the limit circle type at $+\infty$.

Actually, $\sum_{t=0}^{+\infty} p^{-1}(t) = +\infty$, for now $p = 1$; and the linearly independent solutions 1 and t of the Sturm-Liouville equation at $\lambda = 0$ are both in l^2_w, since now $w(t) = 2^{-t}$.

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