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HOPF BIFURCATION ANALYSIS OF A DENSITY PREDATOR-PREY MODEL WITH CROWLEY-MARTIN FUNCTIONAL RESPONSE AND TWO TIME DELAYS*

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Abstract In this paper, a delayed density dependent predator-prey model with Crowley-Martin functional response and two time delays for the predator is considered. By analyzing the corresponding characteristic equations, the local stability of each of the feasible equilibria of the system is addressed and the existence of Hopf bifurcation at the coexistence equilibrium is established. With the help of normal form method and center manifold theorem, some explicit formulas determining the direction of Hopf bifurcation and the stability of bifurcating period solutions are derived. Finally, numerical simulations are given to illustrate the theoretical results.

Keywords Predator-prey system, density dependence, Hopf bifurcations, time-delay.

MSC(2010) 70K45, 70K50.

1. Introduction

Since the pioneer work of Lotka and Volterra [5], the Lotka-Volterra system has been extensively investigated. A general two dimensional predator-prey model is given by the following system

$$\begin{aligned}\dot{x}(t) &= Xf(X) - g(X,Y)Y,\\ \dot{y}(t) &= cg(X,Y)Y - dY,\end{aligned} \tag{1.1}$$

where X and Y denote prey and predator densities at time t, respectively. The term $f(X)$ stands for the prey growth rate in the absence of predators, while the term $g(X,Y)$ denotes the average feeding rate of a predator (i.e. the functional response of predators to prey density). The parameters c and d denote the efficiency of predators to convert the consumed prey into predator's new offspring and predator's mortality rate respectively. Crowley-Martin [2] assumed that predator's

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predation will decrease due to high predator density (interference among the predator individual) even when prey density is high (presence of handling or searching of prey by predator individual) [1, 8, 12]. There are a few literatures available on predator-prey model with Crowley-Martin functional response [1–3, 8, 12, 13]. The Crowley-Martin functional response is predator dependent. The per capita feeding rate for predator y in this formulation is

$$\eta(x,y)=\frac{bx}{(1+a_1x)(1+b_1y)}.$$

In 2016, Syed Abbas and Swati Tyagi devoted their attention to the bifurcating phenomenons of a predator-prey system with a single time delay [10]. The system is described by the following form

$$\begin{aligned}\dot{X}&=X(A-BX-\frac{CY}{A_1+B_1X+C_1Y+B_1C_1XY}),\\ \dot{Y}&=Y(-D-EX)+\frac{FX(T-\bar{\tau})Y(T-\bar{\tau})}{A_1+B_1X(T-\bar{\tau})+C_1Y(T-\bar{\tau})+B_1C_1X(T-\bar{\tau})Y(T-\bar{\tau})}.\end{aligned}\tag{1.2}$$

Let $X=Ax/B, y=Y, t=AT, \tau=\bar{\tau}/A$. System (1.2) is reduced to the following dimensionless form of a delayed predator-prey system:

$$\begin{aligned}\dot{x}(t)&=x(t)-x^2(t)-\frac{cx(t)y(t)}{1+a_1x(t)+b_1y(t)+c_1x(t)y(t)},\\ \dot{y}(t)&=-dy(t)-ey^2(t)+\frac{cx(t-\tau)y(t-\tau)}{1+a_1x(t-\tau)+b_1y(t-\tau)+c_1x(t-\tau)y(t-\tau)},\end{aligned}\tag{1.3}$$

where $c=C/AA_1, a_1=AB_1/A_1B, c_1=AB_1C_1/A_1B, d=D/A, e=E/A, f=FA/AA_1B$. X and Y denote prey and predator densities at time t, respectively. All the parameters A,B,C,D,E,F,A_1,B_1,C_1 in the system are assumed to be only positive values and will be considered as constants throughout discussion. Because the initial conditions are positive, the food of predator is assumed to be partial dependent on the prey of the system. In [10], by choosing τ as bifurcation parameter, the authors showed that when τ passes through the critical value, the positive equilibrium lost its stability and the system exhibited Hopf bifurcation.

In this paper, we consider the following system

$$\begin{aligned}\dot{x}(t)&=x(t)-x^2(t)-\frac{cx(t-\tau_1)y(t-\tau_1)}{1+a_1x(t-\tau_1)+b_1y(t-\tau_1)+c_1x(t-\tau_1)y(t-\tau_1)},\\ \dot{y}(t)&=-dy(t)-ey^2(t)+\frac{cx(t-\tau_1)y(t-\tau_1)}{1+a_1x(t-\tau_2)+b_1y(t-\tau_2)+c_1x(t-\tau_2)y(t-\tau_2)}.\end{aligned}\tag{1.4}$$

This paper is organized as follows: In Section 2, the local stability of each of the feasible equilibria for system (1.4) is discussed and the existence of Hopf bifurcation at the coexistence equilibrium is established. In Section 3, the formulas for determining the direction of Hopf bifurcation and the stability of bifurcating period solutions are derived. In Section 4, numerical simulations are presented. A brief conclusion is given in Section 5.

2. Local stability and Hopf bifurcation

In this section, we shall discuss the local stability of each of the feasible equilibria of system (1.4) and the existence of Hopf bifurcation at the coexistence equilibrium. It is easy to see that system (1.4) has a trivial equilibrium $E_1^*(0,0)$ and a predator-extinction equilibrium $E_2^*(1,0)$. Further, if the following holds:

$$(H_1)(1+a_1x^*)(1-x^*) > 0, c-(b_1+c_1x^*)(1-x^*) > 0,$$

system (1.4) has a coexistence equilibrium $E_3^*(x^*,y^*)$, where x^* is a root of the following quintic equrtion about z

$$\zeta z^5 + \xi z^4 + \mu z^3 + \beta z^2 + \gamma z + \delta = 0,$$

where

$$\begin{aligned}
&\zeta = c_1^2 f,\\
&\xi = 2c_1^2 f(b_1^2 - c),\\
&\mu = a_1c(a_1e - c_1d) + f(b_1^2 + c_1^2 + 2cc_1 - 4b_1c_1),\\
&\beta = c((-c_1)d + a_1d(c_1-b_1)) - a_1ce(a_1+2) + f(2b_1c - 2cc_1 - 2b_1^2 + 2b_1c_1),\\
&\gamma = cd(c_1-b_1) + a_1cd(b_1-c) - ce(1+2a_1) + f(c^2 - b_1^2 - 2b_1c),\\
&\delta = cd(b_1-c) - ce,\\
&y^* = \frac{(1+a_1x^*)(1-x^*)}{c-(b_1+c_1x^*)(1-x^*)}.
\end{aligned}$$

We now study the local stability of the trivial equilibrium $E_1^*(0,0)$ and the predator-extinction equilibrium $E_2^*(1,0)$. The characteristic equation of system (1.4) at $E_1^*(0,0)$ takes the form

$$(\lambda-1)(\lambda+d) = 0. \tag{2.1}$$

Hence, $E_1^*(0,0)$ is always unstable since system (1.4) has a positive root $\lambda = 1$.

The characteristic equation of system (1.4) at $E_2^*(1,0)$ is of the form

$$(\lambda+1)(\lambda+d-\frac{f}{1+a_1}e^{-\lambda\tau_2}) = 0, \tag{2.2}$$

where $\lambda = -1$ is a negative eigenvalue. Hence the root of (2.2) is determined by the following equation:

$$\lambda + d - \frac{f}{1+a_1}e^{-\lambda\tau_2} = 0. \tag{2.3}$$

By analyzing (2.3), when $\tau_2 = 0$ and $d > \frac{f}{1+a_1}$, the equilibrium $E_2^*(1,0)$ is locally asymptotically stable. Let$\lambda = i\omega$, we get

$$\begin{aligned}
&\omega(1+a_1) = -f\sin\omega\tau_2,\\
&d(1+a_1) = f\cos\omega\tau_2,
\end{aligned}$$

that is, $\omega^2 = (\frac{f}{1+a_1})^2 - d^2$. If the $\frac{f}{1+a_1} > d$, (2.3) has a positive root ω. Therefore, there is a positive constant τ', such that for $\tau_2 > \tau'$, $E_2^*(1,0)$ is unstable.

According the above discussions, we obtain the following results.

Theorem 2.1. *For system (1.4),*

(i) the trivial equilibrium $E_1^*(0,0)$ is always unstable;

(ii) if $\tau_2 > \tau'$, the predator-extinction equilibrium $E_2^*(1,0)$ is unstable; if $\tau_2 = 0, d > \frac{f}{1+a_1}$, E_2^* is locally asymptotically stable.

In the following, we will consider the local stability of the coexistence equilibrium E_3^* and the existence of Hopf bifurcations at E_3^*.

Remark 2.1. The sufficient conditions for the global asymptotic stability of the positive equilibrium solution E_3^* of the non-delayed system (1.2) implies that the interior equilibrium solution E_3^* of the delayed system (1.4) is globally asymptotic stable if E_3^* of the non-delayed system is globally asymptotic stable and conditions of the Ref [10] hold.

In this section, we shall study the direction of Hopf bifurcation and stability of the periodic solutions bifurcating from the steady state E_3^*, by using normal form method and center manifold theorem introduced by Hassard et al. [4]. Throughout this section, without loss of generality, we always assume that system (1.4) undergoes Hopf bifurcation at the steady state E_3^* for one of the critical values.

Let $x_1(t) = x(t) - x^*, y_1(t) = y(t) - y^*$, and denote $u = (x_1(t), y_1(t))$. Still denote $x_1(t), y_1(t)$ by $x(t), y(t)$, respectively. Using Taylor expansion to expand the system (1.4) at the positive equilibrium E_3^*(Proof in Ref [10]), we have

$$\begin{aligned}\dot{x}(t) &= a_{11}x(t-\tau_1)+a_{12}y(t-\tau_1)+a_{13}x(t)+\sum_{i+j+l\geq 2} f_1^{(ijl)}x^i(t-\tau_1)y^j(t-\tau_1)x^l(t),\\ \dot{y}(t) &= a_{21}x(t-\tau_2)+a_{22}y(t-\tau_2)+b_{23}y(t)+\sum_{i+j+l\geq 2} f_2^{(ijl)}x^i(t-\tau_2)y^j(t-\tau_2)y^l(t),\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}
a_{11} &= -\frac{cy^*(1+b_1y^*)}{(1+a_1x^*+b_1y^*+c_1x^*y^*)^2},\\
a_{12} &= -\frac{cx^*(1+a_1x^*)}{(1+a_1x^*+b_1y^*+c_1x^*y^*)^2},\\
a_{13} &= 1-2x^*,\\
b_{11} &= \frac{fy^*(1+b_1y^*)}{(1+a_1x^*+b_1y^*+c_1x^*y^*)^2},\\
b_{12} &= \frac{fx^*(1+a_1x^*)}{(1+a_1x^*+b_1y^*+c_1x^*y^*)^2},\\
b_{13} &= -d-2ey^*,\\
f_1^{(ijl)} &= \frac{1}{i!j!l!}\frac{\partial_{i+j+l}f_1}{\partial x^i(t-\tau_1)\partial y^j(t-\tau_1)\partial x^l(t}\ |_{(x^*,y^*)},\\
f_2^{(ijl)} &= \frac{1}{i!j!l!}\frac{\partial_{i+j+l}f_2}{\partial x^i(t-\tau_2)\partial y^j(t-\tau_2)\partial y^l(t)}\ |_{(x^*,y^*)},\\
f_1 &= x(t)-x^2(t)-\frac{cx(t-\tau_1)y(t-\tau_1)}{1+a_1x(t-\tau_1)+b_1y(t-\tau_1)+c_1x(t-\tau_1)y(t-\tau_1)},\\
f_2 &= -dy(t)-y^2(t)+\frac{fx(t-\tau_2)y(t-\tau_2)}{1+a_1x(t-\tau_2)+b_1y(t-\tau_2)+c_1x(t-\tau_2)y(t-\tau_2)}.
\end{aligned}$$

Then the linearized system of the corresponding equation at E_3^* is as follows:

$$\begin{aligned}\dot{x}(t) &= a_{11}x(t-\tau_1) + a_{12}y(t-\tau_1) + a_{13}x(t),\\ \dot{y}(t) &= a_{21}x(t-\tau_2) + a_{22}y(t-\tau_2) + b_{23}y(t).\end{aligned} \tag{2.5}$$

The characteristic equation of system (1.3) is

$$\lambda^2 + A\lambda + H + e^{-\lambda\tau_1}(B\lambda + C) + e^{-\lambda\tau_2}(D\lambda + E) + Fe^{-\lambda(\tau_1+\tau_2)} = 0. \tag{2.6}$$

Case 1: $\tau_1 = \tau_2 = 0$.
The characteristic Eq.(2.6) is

$$\lambda^2 + (A + B + D)\lambda + C + E + F + H = 0. \tag{2.7}$$

By Routh-Hurwitz criterion, if

$$(H11): \quad A + B + D > 0,\ C + E + F + H > 0,$$

all roots of Eq.(2.6) with $\tau_1 = \tau_2 = 0$ have negative real parts. Namely, the equilibrium point E_3^* is locally asymptotically stable when the condition (H11) is satisfied.
Case 2: $\tau_1 = 0,\ \tau_2 > 0$.
When $\tau_1 = 0$, then (2.6) becomes

$$\lambda^2 + \lambda(A + B) + C + H + e^{-\lambda\tau_2}(D\lambda + E + F) = 0. \tag{2.8}$$

For $\omega_2 > 0$, let $i\omega_2$ be a root of (2.8), and separating real and imaginary parts, we have

$$\begin{aligned}D\omega_2\sin(\omega_2\tau_2) + (E + F)\cos(\omega_2\tau_2) &= \omega_2^2 - C - H,\\ D\omega_2\cos(\omega_2\tau_2) - (E + F)\sin(\omega_2\tau_2) &= -\omega_2(A + B),\end{aligned} \tag{2.9}$$

which leads to

$$\omega_2^4 + e_1\omega_2^2 + e_2 = 0, \tag{2.10}$$

where $e_1 = (A + B)^2 - 2(C + H) - D^2, e_2 = (C + H)^2 - (E + F)^2$. Let $\omega_2^2 = v_1$, then (2.10) becomes

$$v_1^2 + e_1v_2 + e_2 = 0. \tag{2.11}$$

Denote

$$f_1(v_1) = v_1^2 + e_1v_1 + e_2. \tag{2.12}$$

Since $f_1(0) = e_2, \lim_{v_1\to+\infty} f_1(v_1) = +\infty$, and from (2.12), we have

$$f_1'(v_1) = 2v_1 + e_1. \tag{2.13}$$

After similar discussions about the roots of (2.13) as in [9], we have the following lemma.

Lemma 2.1. *For the polynomial Eq.(2.11), we have the following results:*

(H21) If $(A+B)^2 - 2(C+H) - D^2 < 0$, *(2.11) has at least one positive root;*

(H22) If $(A+B)^2 - 2(C+H) - D^2 \geq 0$, *(2.11) has at least one positive root if and only if there exists* $z^* > 0$, *such that* $G'(z^*) = 0$ *and* $G(z^*) \geq 0$.

Suppose that (2.11) has positive roots. Without loss of generality, we assume that it has two positive roots, which are denoted as v_{11} and v_{12}. Then (2.10) has two positive roots $\omega_{2k} = \sqrt{v_{1k}}, k = 1, 2$. The corresponding critical value of time delay $\tau_{2k}^{(j)}$ is

$$\tau_{2k}^{(j)} = \frac{1}{\omega_{2k}}\{\arccos \frac{(\omega_{2k}^2 - C - H)(E+F) - \omega_{2k}^2 D(A+B)}{\omega_{2k}^2 D^2 + (E+F)^2} + 2j\pi\}, \tag{2.14}$$

where $\pm\omega_{2k}$ is a pair of purely imaginary roots of (2.8) with $\tau_2 = \tau_{2k}^{(j)}$, and let $\tau_{20} = \min_{k\in\{1,2\}}\{\tau_{2k}^{(0)}\}$, $\omega_{20} = \omega_{2k_0}$.

According to the Hopf bifurcation theorem [6, 7, 11, 14], we need to verify the transversality condition. Differentiating (2.8) with respect to τ_2, and noticing that λ is a function of τ_2, we can obtain

$$(\frac{d\lambda}{d\tau_2})^{-1} = \frac{(2\lambda + A + B)e^{\lambda\tau_2}}{\lambda(D\lambda + E + F)} + \frac{D}{(D\lambda + E + F)} - \frac{\tau_2}{\lambda}, \tag{2.15}$$

which leads to

$$Re(\frac{d\lambda}{d\tau_2})^{-1} = Re(\frac{(2\lambda + A + B)e^{\lambda\tau_2}}{\lambda(D\lambda + E + F)} + \frac{D}{(D\lambda + E + F)})_{\lambda=i\omega_{20}} + Re(\frac{\tau_2}{\lambda})_{\lambda=i\omega_{20}}.$$

Noting that $\{\frac{d(Re\lambda)}{d\tau_2}\}_{\lambda=i\omega_{20}}$ and $\{Re(\frac{d\lambda}{d\tau_2})^{-1}\}_{\lambda=i\omega_{20}}$ have the same sign, we have

$$\{\frac{d(Re\lambda)}{d\tau_2}\}_{\lambda=i\omega_{20}} = \{Re(\frac{d\lambda}{d\tau_2})^{-1}\}_{\lambda=i\omega_{20}}.$$

Therefore, $\{\frac{d(Re\lambda)}{d\tau_2}\}_{\lambda=i\omega_{20}} \neq 0$ if the following condition holds:

(H23) $2\omega_{20}^2 + (A+B)^2 - 2(C+H) - D^2 \neq 0$.

By the above discussion, we have the following results.

Theorem 2.2. *For system(1.4),* $\tau_1 = 0$,

(i) if (H21) holds, the positive equilibrium $E_3^*(x^*, y^*)$ *is asymptotically stable for all* $\tau_2 \geq 0$;

(ii) if (H22) and (H23) holds, the positive equilibrium $E_3^*(x^*, y^*)$ *is asymptotically stable for all* $\tau_2 \in [0, \tau_{20})$ *and unstable for* $\tau_2 > \tau_{20}$. *Furthermore, system (1.4) undergoes a Hopf bifurcation at the positive equilibrium* $E_3^*(x^*, y^*)$ *when* $\tau_2 = \tau_{20}$.

Case 3: $\tau_2 = 0, \tau_1 > 0$.

The calculation is very similar to Case 2, we have the following results.

Theorem 2.3. *For system (1.4),* $\tau_2 = 0$, *the positive equilibrium* $E_3^*(x^*, y^*)$ *is asymptotically stable for all* $\tau_1 \in [0, \tau_{10})$ *and unstable for* $\tau_1 > \tau_{10}$. *Furthermore, system (1.4) undergoes a Hopf bifurcation at the positive equilibrium* $E_3^*(x^*, y^*)$ *when* $\tau_1 = \tau_{10}$, *where* τ_{10} *represents the minimum critical value of time delay* τ_1 *for the occurrence of Hopf bifurcation when* $\tau_2 = 0$.

Case 4: $\tau_1 = \tau_2 = \tau = 0$.

Theorem 2.4. *For system (1.4), $\tau_1 = \tau_2 = \tau = 0$, the positive equilibrium $E_3^*(x^*, y^*)$ is asymptotically stable for all $\tau \in [0, \tau_0)$ and unstable for $\tau > \tau_0$. Furthermore, system (1.2) undergoes a Hopf bifurcation at the positive equilibrium $E_3^*(x^*, y^*)$ when $\tau = \tau_0$, where τ_0 represents the minimum critical value of time delay τ for the occurrence of Hopf bifurcation.*

Case 5: $\tau_1 > 0, \tau_2 \in [0, \tau_{20})$ and $\tau_1 \neq \tau_2$.

We consider (2.6) with τ_2 in its stable interval, and τ_1 is regarded as the parameter. Let $i\omega_1(\omega_1 > 0)$ be the root of (2.6), then we can obtain

$$\begin{aligned} &E_{11}\sin(\omega_1\tau_1) + E_{12}\cos(\omega_1\tau_1) = E_{13}, \\ &E_{11}\cos(\omega_1\tau_1) - E_{12}\sin(\omega_1\tau_1) = E_{14}. \end{aligned} \tag{2.16}$$

$$\begin{aligned} E_{11} &= \omega_1 B - F\sin(\omega_1\tau_2), \\ E_{12} &= C + F\cos(\omega_1\tau_2), \\ E_{13} &= \omega_1^2 - H - \omega_1 D\sin(\omega_1\tau_2) - E\cos(\omega_1\tau_2), \\ E_{14} &= E\sin(\omega_1\tau_2) - \omega_1 D\cos(\omega_1\tau_2) - \omega_1 A. \end{aligned}$$

From (2.16), we can get

$$\omega_1^4 + e_1\omega_1^2 + e_2 + (e_3\omega_1^2 + e_4)\cos\omega_1\tau_2 + (e_5\omega_1^3 + e_6\omega_1)\sin\omega_1\tau_2 = 0, \tag{2.17}$$

where $e_1 = A^2 + D^2 - B^2 - 2H, e_2 = H^2 + E^2 - C^2 - F^2, e_3 = 2AD - 2E, e_4 = 2HE - 2CF, e_5 = -2D, e_6 = 2HD + 2BF - 2AE$.

In order to give the main results, Suppose that Eq.(2.17) has at least finite positive root, we denote the positive roots of (2.17) as $\omega_2^{(1)}$, $\omega_2^{(2)}$, $\omega_2^{(3)}$ and $\omega_2^{(4)}$. For every $\omega_2^{(i)}(i = 1, 2, 3, 4)$ the corresponding critical value of time delay $\tau_{1i}^{(j)}(j = 1, 2)$ is

$$\tau_{1i}^{(j)} = \frac{1}{\omega_1}\arccos\{\frac{E_{12}E_{13} + E_{11}E_{14}}{E_{11}^2 + E_{12}^2} + 2j\pi\}_{\omega_1 = i\omega_1^j}, i = 1, 2, 3, 4; j = 0, 1, 2, \cdots.$$

Let $\tau_{10}' = \min\{\tau_{1i}^{(0)} | i = 1, 2, ...6; j = 0, 1, 2...\}$, ω_{10}' is the corresponding root of (2.17) with τ_{10}'.

In the following, we differentiate both sides of (2.6) with respect to τ_1 to verify the transversality condition. Taking the derivative of λ with respect to τ_1 in (2.6) and substituting $\lambda = i\omega_{10}'$, we get

$$Re(\frac{d\lambda}{d\tau_1})^{-1}_{\lambda = i\omega_{10}} = Re(\frac{P + Qi}{M + Ni}) = Re(\frac{PM + QN}{M^2 + N^2}),$$

where

$$\begin{aligned} P =& A - B\tau_1\omega_{10}\sin\omega_{10}\tau_1 + (B - C\tau_1)\cos\omega_{10}\tau_1 + (-D\omega_{10}\tau_2 + (\tau_1 + \tau_2) \\ & F\sin\omega_{10}\tau_1)\sin\omega_{10}\tau_2 + (D - E\tau_2 - (\tau_1 + \tau_2)F\cos\omega_{10}\tau_1)\cos\omega_{10}\tau_2, \\ Q =& 2\omega_{10} + (C\tau_1 - B)\sin\omega_{10}\tau_1 - B\tau_1\omega_{10}\cos\omega_{10}\tau_1 + (-D\omega_{10}\tau_2 + (\tau_1 + \tau_2) \\ & F\sin\omega_{10}\tau_1)\cos\omega_{10}\tau_2 + (-D + E\tau_2 + (\tau_1 + \tau_2)F\cos\omega_{10}\tau_1)\sin\omega_{10}\tau_2, \\ M =& \omega_{10}C\sin\omega_{10}\tau_1 - \omega_{10}^2 B\cos\omega_{10}\tau_1 + \omega_{10}F\sin\omega_{10}\tau_1\cos\omega_{10}\tau_2 \end{aligned}$$

$$
\begin{aligned}
&+\omega_{10}F\sin\omega_{10}\tau_1\cos\omega_{10}\tau_2,\\
N=&\omega_{10}C\cos\omega_{10}\tau_1+\omega_{10}^2B\sin\omega_{10}\tau_1+\omega_{10}F\cos\omega_{10}\tau_1\cos\omega_{10}\tau_2\\
&-\omega_{10}F\sin\omega_{10}\tau_1\sin\omega_{10}\tau_2.
\end{aligned}
$$

As can be seen from the above formula, it is obvious that the transversality condition $\{\frac{d(Re\lambda)}{d\tau_1}\}_{\lambda=i\omega_{10}}>0.$ for the occurrence of Hopf bifurcation is well satisfied provided $PM+QN>0$.

By the discussion above, we have the following results.

Theorem 2.5. *For system (1.4), $\tau_1>0, \tau_2\in[0,\tau_{20})$ and $\tau_1\neq\tau_2$. Suppose that the conditions Eq.(2.17) has at least finite positive root and $\{\frac{d(Re\lambda)}{d\tau_1}\}_{\lambda=i\omega_{10}}>0$ hold, the positive equilibrium $E_3^*(x^*,y^*)$ is asymptotically stable for all $\tau_1\in[0,\tau_{10})$ and unstable for $\tau_1>\tau_{10}$. Furthermore, system (1.4) undergoes a Hopf bifurcation at the positive equilibrium $E_3^*(x^*,y^*)$ when $\tau_1=\tau_{10}$.*

Case 6: $\tau_2>0, \tau_1\in[0,\tau_{10})$ and $\tau_1\neq\tau_2$.

We consider (2.6) with τ_1 in its stable interval, and τ_2 is regarded as a parameter. The calculation is very similar to Case 5, we can obtain the following theorem.

Theorem 2.6. *For system (1.4), $\tau_2>0, \tau_1\in[0,\tau_{10})$ and $\tau_1\neq\tau_2$, the positive equilibrium $E_3^*(x^*,y^*)$ is symptotically stable for all $\tau_2\in[0,\tau_{20})$ and unstable for $\tau_2>\tau_{20}$. Furthermore, the system (1.4) undergoes a Hopf bifurcation at the positive equilibrium $E_3^*(x^*,y^*)$ when $\tau_2=\tau_{20}$, where τ_{20} represents the minimum critical value of time delay τ_2 for the occurrence of Hopf bifurcation when $\tau_1\in[0,\tau_{10})$.*

3. Direction and stability of Hopf bifurcation

By the above discussions, we have shown that the system (1.4) undergoes Hopf bifurcation for different combinations of τ_1 and τ_2. Now, we shall study the direction of Hopf bifurcation and the stability of bifurcating periodic solutions of system (1.4) with respect to τ_2 and $\tau_1\in[0,\tau_{10})$. The theoretical approach applied is based on the normal form theory and center manifold theorem [4]. It is considered that system (1.4) undergoes Hopf bifurcation at $\tau_2=\tau_{20}', \tau_1\in[0,\tau_{10})$. Without loss of generality, we assume that $\tau_{20}'>\tau_1'$.

Let $\tau_2=\tau_{20}'+\mu, \mu\in R, t=s\tau_2, x(s\tau_2)=\hat{x}(s), y(s\tau_2)=\hat{y}(s)$, denote $x=\hat{x}, y=\hat{y}$ and $t=s$, then system (1.4) can be written as a functional differential equation (FDE) in $C=C([-1,0],R^2)$:

$$
\dot{U}(t)=(\tau_{20}'+\mu)(B_1U(t)+B_2U(t-\frac{\tau_1'}{\tau_2})+B_3U(t-1)+f(x,y)), \tag{3.1}
$$

where $U(t)=(x(t),y(t))^T, f=(f_1,f_2)^T$, and

$$
\begin{aligned}
f_1&=\sum_{i+j+k\geq 2}\frac{1}{i!j!k!}f_{ijl}^1x^i(t-\frac{\tau_1'}{\tau_2})y^j(t-\frac{\tau_1'}{\tau_2})x^l(t),\\
f_2&=\sum_{i+j+k\geq 2}\frac{1}{i!j!k!}f_{ijl}^2x^i(t-1)y^j(t-1)y^l(t).
\end{aligned}
$$

Denote

$$
L_\mu\phi=(\tau_{20}'+\mu)(B_1\phi(0)+B_2\phi(-\frac{\tau_1'}{\tau_2})+B_3\phi(-1)),
$$

$$\phi = (\phi_1, \phi_2)^T \epsilon((-1, 0], R^2), \tag{3.2}$$

where

$$B_1 = \begin{bmatrix} a_{13} & 0 \\ 0 & a_{23} \end{bmatrix}, B_2 = \begin{bmatrix} a_{11} & a_{12} \\ 0 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 \\ a_{21} & a_{22} \end{bmatrix}.$$

Hence, by the Riesz representation theorem, there exists a 2×2 matrix function $\eta(\theta, \mu)$ of bounded variation for $\theta \in [-1, 0]$, such that $L_\mu \phi$ is the form

$$L_\mu \phi = \int_{-1}^{0} d\eta\,(\theta, \mu)\,\phi(\theta), \qquad \text{for} \quad \phi \in C. \tag{3.3}$$

$\delta(\theta)$ is the Dirac delta function. In fact, we can choose

$$\eta(\theta, \mu) = \begin{cases} (\tau'_{20} + \mu)(B_1 + B_2 + B_3), & \theta = 0, \\ (\tau'_{20} + \mu)(B_2 + B_3), & \theta \in [-\frac{\tau'_1}{\tau_2}, 0), \\ (\tau'_{20} + \mu)B_3, & \theta \in [-1, -\frac{\tau'_1}{\tau_2}), \\ 0, & \theta = -1, \end{cases} \tag{3.4}$$

where

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \theta \in [-1, 0), \\ \int_{-1}^{0} d\eta(s, \mu)\phi(s), & \theta = 0, \end{cases} \tag{3.5}$$

and

$$R\phi = \begin{cases} 0, & \theta \in [-1, 0), \\ h(\mu, \Phi), & \theta = 0, \end{cases} \tag{3.6}$$

where

$$h(\mu, \Phi) = (\tau'_{20} + \mu) \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}, \tag{3.7}$$

where

$$h_1 = \sum_{i+j+k \geq 2} \frac{1}{i!j!k!} f^1_{ijl} \phi^i_1(-\frac{\tau'_1}{\tau_2}) \phi^j_2(-\frac{\tau'_1}{\tau_2}) \phi^l_1(0),$$
$$h_2 = \sum_{i+j+k \geq 2} \frac{1}{i!j!k!} f^2_{ijl} \phi^i_1(-1) \phi^j_2(-1) \phi^l_2(0).$$

Then Eq.(3.1) can be written as

$$\dot{u}_t = A(\mu)u_t + Ru_t. \tag{3.8}$$

Assume that $q(\theta)$ is the eigenvector of $A(0)$ corresponding to $i\omega_0\tau'_{20}$, then $A(0)q(\theta) = i\omega_0\tau'_{20}q(\theta)$. It follows from the definition of $A(0)$ that

$$\begin{pmatrix} i\omega_0 - a_{11}e^{-i\omega_0\tau'_1} - a_{13} & -b_{11}e^{-i\omega_0\tau'_1} \\ -b_{11}e^{-i\omega_0\tau'_{20}} & i\omega_0 - b_{12}e^{-i\omega_0\tau'_{20}} - b_{13} \end{pmatrix} q(\theta) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus, we can easily compute $q(\theta) = (1, \alpha)^T e^{i\omega_0\tau'_{20}\theta}$, where

$$\alpha = \frac{i\omega_0 - a_{11}e^{-i\omega_0\tau'_1} - a_{13}}{b_{11}e^{-i\omega_0\tau'_1}}. \tag{3.9}$$

Similarly, it can be verified that $q^*(s) = D(1, \alpha^*)e^{i\omega_0\tau'_{20}s}$ is the eigenvector corresponding to the eigenvalue $-i\omega_0\tau'_{20}$ of A^*, which is the adjoint operator of $A(0)$, where

$$\alpha^* = \frac{-i\omega_0 - a_{11}e^{-i\omega_0\tau'_1} - a_{13}}{b_{11}e^{-i\omega_0\tau'_{20}}}. \tag{3.10}$$

In order to assure the bilinear inner product $< q^*(s), q(\theta) >= 1$, we have

$$\begin{aligned}\langle q^*(\theta), q(\theta)\rangle &= \bar{D}(1, \bar{\alpha^*})(1, \alpha)^T \\ &\quad - \int_{-1}^0 \int_{\xi=0}^\theta \bar{D}(1, \bar{\alpha^*})e^{-i(\xi-\theta)\omega_0\tau'_{20}}d\eta(\theta)(1, \alpha)^T e^{i\xi\omega_0\tau'_{20}}d\xi \\ &= \bar{D}(1 + \alpha\bar{\alpha^*} - \int_{-1}^0 (1, \bar{\alpha^*})\theta e^{i\theta\omega_0\tau'_{20}}d\eta(\theta)(1, \alpha)^T) \\ &= \bar{D}(1 + \alpha\bar{\alpha^*} + (b_{12}\bar{\alpha^*} + b_{13}\alpha\bar{\alpha^*})\tau'_{20}e^{-i\omega_0\tau'_{20}} \\ &\quad + \tau'_1(a_{11} + a_{12})e^{-i\omega_0\tau'_1}).\end{aligned} \tag{3.11}$$

Therefore, we can choose $\bar{D}$ as

$$\begin{aligned}\bar{D} &= [(1 + \alpha\bar{\alpha^*} + (b_{12}\bar{\alpha^*} + b_{13}\alpha\bar{\alpha^*})\tau'_{20}e^{-i\omega_0\tau'_{20}} + \tau'_1(a_{11} + a_{12})e^{-i\omega_0\tau'_1})]^{-1} \\ &= \frac{1}{Re(D^{-1}) + iIm(D^{-1})}.\end{aligned}$$

We can get $< q^*(s), \bar{q}(\theta) >= 0$. In what follows, we will obtain the coordinates to describe the center manifold C_0 at $\tau = \tau_0$. Noticing that $u_t(\theta) = (x_t(\theta), y_t(\theta))^T = zq(\theta) + \bar{z}\bar{q}(\theta) + W(t, \theta)$, we have

$$\begin{aligned}x_{1t}(0) =& z + \bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + W_{30}^{(1)}(0)\frac{z^3}{6} \\ &+ W_{21}^{(1)}(0)\frac{z^2\bar{z}}{2} + \cdots, \\ x_{2t}(0) =& z\alpha + \bar{z}\bar{\alpha} + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2} \\ &+ W_{30}^{(2)}(0)\frac{z^3}{6} + W_{21}^{(2)}(0)\frac{z^2\bar{z}}{2} + \cdots, \\ x_{1t}(-1) =& ze^{-i\omega_k\tau_k} + \bar{z}e^{i\omega_k\tau_k} + W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\bar{z} + W_{02}^{(1)}(-1)\frac{\bar{z}^2}{2} \\ &+ W_{30}^{(1)}(-1)\frac{z^3}{6} + W_{21}^{(1)}(-1)\frac{z^2\bar{z}}{2} + \cdots, \\ x_{2t}(-1) =& z\alpha e^{-i\omega_k\tau_k} + \bar{z}\bar{\alpha}e^{i\omega_k\tau_k} + W_{20}^{(2)}(-1)\frac{z^2}{2} + W_{11}^{(2)}(-1)z\bar{z} + W_{02}^{(2)}(-1)\frac{\bar{z}^2}{2} \\ &+ W_{30}^{(2)}(-1)\frac{z^3}{6} + W_{21}^{(2)}(-1)\frac{z^2\bar{z}}{2} + \cdots.\end{aligned} \tag{3.12}$$

Thus, from Eq.(3.2) we have

$$
\begin{aligned}
g(z,\bar{z}) = \bar{q^*}(0) f_0(z,\bar{z}) = \tau_{20}'\bar{D}(1,\alpha^*)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \\
= \tau_{20}'\bar{D}(k_{11}x_{1t}^2(-\tfrac{\tau_1'}{\tau_{20}'}) + k_{12}x_{1t}(-\tfrac{\tau_1'}{\tau_{20}'})x_{2t}(-\tfrac{\tau_1'}{\tau_{20}'} + k_{13}x_{2t}^2(-\tfrac{\tau_1'}{\tau_{20}'}) + k_{14}x_{1t}^2 \\
+ \bar{\alpha^*}(k_{21}x_{1t}^2(-1) + k_{22}x_{2t}^2(-1) + k_{23}x_{1t}(-1)x_{2t}(-1) + k_{24}x_{2t}^2(-1))),
\end{aligned}
\tag{3.14}
$$

where

$$
\begin{aligned}
h_1 &= k_{11}\phi_1^2(-\frac{\tau_1'}{\tau_{20}'}) + k_{12}\phi_1(\frac{\tau_1'}{\tau_{20}'})\phi_2(-\frac{\tau_1'}{\tau_{20}'}) + k_{13}\phi_2^2(-\frac{\tau_1'}{\tau_{20}'}) + k_{14}\phi_1^2(0), \\
h_1 &= k_{21}\phi_1^2(-1) + k_{22}\phi_2^2 + k_{23}\phi_1(-1)\phi_2(-1) + k_{24}\phi_2^2(0), \\
k_{11} &= \frac{-2cy^*(a_1 + c_1y^* + a_1b_1y^* + b_1c_1(y^*)^2)}{(1 + a_1x^* + b_1y^* + c_1x^*y^*)^3}, \\
k_{12} &= \frac{c(1 + a_1x^* + b_1y^* - c_1x^*y^* + 2a_1b_1x^*y^*)}{(1 + a_1x^* + b_1y^* + c_1x^*y^*)^3}, \\
k_{13} &= \frac{-2cx^*(b_1 + c_1x^* + a_1b_1x^* + a_1c_1(x^*)^2)}{(1 + a_1x^* + b_1y^* + c_1x^*y^*)^3}, \\
k_{14} &= -2, \\
k_{21} &= \frac{f}{c}k_{11}, \\
k_{22} &= \frac{f}{c}k_{12}, \\
k_{23} &= \frac{f}{c}k_{13}, \\
k_{24} &= -2e.
\end{aligned}
$$

Comparing the coefficients with (3.14), we can obtain

$$
\begin{aligned}
g_{20} =& 2\tau_{20}'\bar{D}((k_{11} + k_{12}\alpha + k_{13}\alpha^2)e^{-2i\omega_0\tau_1'} + k_{14} + \bar{\alpha^*}(k_{21} + k_{22}\alpha^2 + k_{23}\alpha)e^{-2i\omega_0\tau_{20}'} \\
&+ k_{24}\alpha^2), \\
g_{11} =& \tau_{20}'\bar{D}(2k_{11} + k_{12}(\alpha + \bar{\alpha}) + 2k_{13}\alpha\bar{\alpha} + 2k_{14} + \bar{\alpha^*}(2k_{21} + 2\alpha\bar{\alpha}k_{22} + (\alpha + \bar{\alpha})k_{23} \\
&+ 2\alpha\bar{\alpha}k_{24})), \\
g_{02} =& 2\tau_{20}'\bar{D}((k_{11} + k_{12}\bar{\alpha} + k_{13}\bar{\alpha}^2)e^{2i\omega_0\tau_1'} + k_{14} + \bar{\alpha^*}(k_{21} + k_{22}\bar{\alpha}^2 \\
&+ k_{23}\bar{\alpha})e^{2i\omega_0\tau_{20}'} + k_{24}\bar{\alpha}^2), \\
g_{21} =& 2\tau_{20}'\bar{D}[(k_{14}2W_{11}^{(1)}(0) + 2k_{21}\bar{\alpha^*}W_{11}^{(1)}(-1)e^{-i\omega_0\tau_{20}'} + \alpha\bar{\alpha^*}k_{23}W_{11}^{(1)}(-1) \\
&e^{-i\omega_0\tau_{20}'} + k_{14}(W_{20}^{(1)}(0) + \bar{\alpha^*}k_{21}W_{20}^{(1)}(-1)e^{i\omega_0\tau_{20}'} + \frac{1}{2}k_{23}W_{20}^{(1)}(-1) \\
&e^{i\omega_0\tau_{20}'} + 2k_{22}W_{11}^{(2)}(-1)e^{-i\omega_0\tau_{20}'}\alpha\bar{\alpha} + k_{23}W_{11}^{(2)}(-1)e^{-i\omega_0\tau_{20}'}\bar{\alpha^*} \\
&+ 2k_{24}W_{11}^{(2)}(0)\alpha\bar{\alpha^*} + k_{22}W_{20}^{(2)}(-1)\bar{\alpha}\bar{\alpha^*} + \frac{1}{2}k_{23}W_{20}^{(2)}(-1)\bar{\alpha^*}e^{-i\omega_0\tau_{20}'} \\
&+ k_{24}W_{20}^{(2)}(0)\bar{\alpha}\bar{\alpha^*}) + 2k_{11}W_{11}^{(3)}(-\frac{\tau_1'}{\tau_{20}'})e^{-i\omega_0\tau_1'0} + k_{12}W_{11}^{(3)}(-\frac{\tau_1'}{\tau_{20}'})
\end{aligned}
$$

$$e^{-i\omega_0\tau'_{10}} + k_{12}W_{11}^{(3)}(-\frac{\tau'_1}{\tau'_{20}})e^{-i\omega_0\tau'_{10}}\alpha + 2k_{13}W_{11}^{(3)}(-\frac{\tau'_1}{\tau'_{20}})\alpha$$
$$+ k_{11}W_{20}^{(3)}(-\frac{\tau'_1}{\tau'_{20}})e^{i\omega_0\tau'_{10}} + \frac{1}{2}k_{12}W_{20}^{(3)}(-\frac{\tau'_1}{\tau'_{20}})e^{i\omega_0\tau'_{10}}$$
$$+ \frac{1}{2}\bar{\alpha}k_{12}W_{20}^{(3)}e^{i\omega_0\tau'_{10}} + \bar{\alpha}k_{13}W_{20}^{(3)}(-\frac{\tau'_1}{\tau'_{20}})e^{i\omega_0\tau'_{10}}].$$

In order to obtain g_{21}, we need to compute $W_{20}(\theta)$ and $W_{11}(\theta)$,

$$\dot{W} = \dot{x}_t - \dot{z}q - \dot{z}\bar{q} = \begin{cases} AW - 2Re\{\bar{q}_*(0)f_0q(\theta)\}, \theta \in [0,1), \\ AW - 2Re\{\bar{q}_*(0)f_0q(\theta)\} + f_0, \theta = 0. \end{cases} \tag{3.15}$$

Substituting the corresponding series into (3.15) and comparing the coefficients, we obtain

$$(A - 2i\tau'_{20}\omega_0)W_{20}(\theta) = -H_{20}(\theta), \quad AW_{11}(\theta) = -H_{11}(\theta). \tag{3.16}$$

For $\theta \in [-1,0)$, from (3.15), we also know that

$$H(z,\bar{z},\theta) = -\bar{q^*}(0)f_0q(\theta) - q^*(0)\bar{f}_0\bar{q}(\theta) = -gq(\theta) - \bar{g}\bar{q}(\theta). \tag{3.17}$$

Comparing the coefficients with (3.17), we have

$$H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), \tag{3.18}$$

and

$$H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \tag{3.19}$$

From (3.16), (3.17) and the definition of A, we can easily get

$$\dot{W}_{20}(\theta) = 2i\tau'_{20}\omega_0W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta). \tag{3.20}$$

Notice that $q(\theta) = (1,\alpha)^Te^{i\theta\omega_0\tau'_0}$, hence

$$W_{20}(\theta) = \frac{ig_{20}}{\tau'_{20}\omega_0}q(0)e^{i\theta\omega_0\tau'_{20}} + \frac{i\bar{g}_{02}}{3\tau'_{20}\omega_0}\bar{q}(0)e^{-i\theta\omega_0\tau'_{20}} + E_1e^{2i\theta\omega_0\tau'_{20}}, \tag{3.21}$$

where $E_1 = (E_1^{(1)}, E_1^{(2)}) \in R^2$ is a constant vector. Similarly, we can obtain

$$W_{11}(\theta) = -\frac{ig_{11}}{\tau'_{20}\omega_0}q(0)e^{i\theta\omega_0\tau'_{20}} + \frac{i\bar{g}_{11}}{\tau'_{20}\omega_0}\bar{q}(0)e^{-i\theta\omega_0\tau'_{20}} + E_2, \tag{3.22}$$

where $E_2 = (E_2^{(1)}, E_2^{(2)}) \in R^2$ is a constant vector.

In what follows, we shall seek appropriate E_1 and E_2. According to the definition of A and (3.16), we have

$$\int_{-1}^{0} d\eta(\theta,0)W_{20}(\theta) = 2i\tau'_{20}\omega_0W_{20}(0) - H_{20}(0). \tag{3.23}$$

and

$$\int_{-1}^{0} d\eta(\theta,0)W_{11}(\theta) = -H_{11}(0), \tag{3.24}$$

where $\eta(\theta)=\eta(\theta,0)$. From (3.15) and (3.17), we have

$$H_{20}(0)=-g_{20}q(0)-\bar{g}_{02}\bar{q}(0)+2\tau_{20}'\begin{bmatrix}(k_{11}+k_{12}\alpha+k_{13}\alpha^2)e^{-2i\omega_0\tau_{10}'}+k_{14}\\(k_{21}+k_{23}\alpha+k_{22}\alpha^2)e^{2i\omega_0\tau_{20}'}+k_{24}\alpha^2\end{bmatrix} \tag{3.25}$$

and

$$H_{11}(0)=-g_{11}q(0)-\bar{g}_{11}\bar{q}(0)+2\tau_{20}'\begin{bmatrix}2k_{11}+k_{12}Re\alpha+k_{12}Re\bar{\alpha}+2k_{13}\alpha\bar{\alpha}+2k_{14}\\2k_{21}+k_{23}Re\alpha+k_{23}Re\bar{\alpha}+2k_{22}\alpha\bar{\alpha}+2k_{24}\alpha\bar{\alpha}\end{bmatrix}. \tag{3.26}$$

Substituting (3.21) and (3.25) into (3.23) and noticing that

$$(i\omega_0\tau_{20}'I-\int_{-1}^{0}e^{i\omega_0\tau_{20}'\theta}d\eta(\theta))q(0)=0,$$

and

$$(-i\omega_0\tau_{20}'I-\int_{-1}^{0}e^{-i\omega_0\tau_{20}'\theta}d\eta(\theta))q(0)=0,$$

we have

$$(2i\omega_0\tau_{20}'I-\int_{-1}^{0}e^{2i\omega_0\tau_{20}'\theta}d\eta(\theta))E_1=2\tau_{20}'\begin{pmatrix}(k_{11}+k_{12}\alpha+k_{13}\alpha^2)e^{-2i\omega_0\tau_{10}'}+k_{14}\\(k_{21}+k_{23}\alpha+k_{22}\alpha^2)e^{2i\omega_0\tau_{20}'}+k_{24}\alpha^2\end{pmatrix}.$$

That is,

$$\begin{aligned}&\begin{pmatrix}2i\omega_0-a_{13}-a_{11}x^*e^{-i\omega_0(-\frac{\tau_1'}{\tau_{20}'})} & a_{12}x^*e^{-i\omega_0(-\frac{\tau_1'}{\tau_{20}'})}\\-a_{21}y^*e^{-i\omega_0\tau_{20}'} & 2i\omega_0-a_{23}-a_{22}y^*e^{-i\omega_0\tau_{20}'}\end{pmatrix}E_1\\&=2\begin{pmatrix}(k_{11}+k_{12}\alpha+k_{13}\alpha^2)e^{-2i\omega_0\tau_{10}'}+k_{14}\\(k_{21}+k_{23}\alpha+k_{22}\alpha^2)e^{2i\omega_0\tau_{20}'}+k_{24}\alpha^2\end{pmatrix}.\end{aligned} \tag{3.27}$$

According to the Crammer's criteria, the solutions of (3.27) are described by

$$\begin{aligned}E_1=2&\begin{pmatrix}2i\omega_0-a_{13}-a_{11}x^*e^{-i\omega_0(-\frac{\tau_1'}{\tau_{20}'})} & a_{12}x^*e^{-i\omega_0(-\frac{\tau_1'}{\tau_{20}'})}\\-a_{21}y^*e^{-i\omega_0\tau_{20}'} & 2i\omega_0-a_{23}-a_{22}y^*e^{-i\omega_0\tau_{20}'}\end{pmatrix}^{-1}\\&\times\begin{pmatrix}(k_{11}+k_{12}\alpha+k_{13}\alpha^2)e^{-2i\omega_0\tau_{10}'}+k_{14}\\(k_{21}+k_{23}\alpha+k_{22}\alpha^2)e^{2i\omega_0\tau_{20}'}+k_{24}\alpha^2\end{pmatrix}.\end{aligned} \tag{3.28}$$

Similarly, substituting (3.22) and (3.26) into (3.24), we have

$$\begin{aligned}E_2=2&\begin{pmatrix}-a_{11}-a_{13} & -a_{12}\\-a_{21} & -a_{22}-a_{23}\end{pmatrix}^{-1}\\&\begin{pmatrix}2k_{11}+k_{12}Re\alpha+k_{12}Re\bar{\alpha}+2k_{13}\alpha\bar{\alpha}+2k_{14}\\2k_{21}+k_{23}Re\alpha+k_{23}Re\bar{\alpha}+2k_{22}\alpha\bar{\alpha}+2k_{24}\alpha\bar{\alpha}\end{pmatrix}.\end{aligned} \tag{3.29}$$

Thus, we can compute the following values

$$\begin{aligned}
c_1(0) &= \frac{i}{2\omega_0\tau'_{20}}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2},\\
\mu_2 &= -\frac{Re\{c_1(0)\}}{Re\{\lambda'(\tau'_{20})\}},\\
\beta_2 &= 2Re\{c_1(0)\},\\
T_2 &= -\frac{Im(c_1(0)) + \mu_2 Im\{\lambda'(\tau'_{20})\}}{\omega_0\tau'_{20}},
\end{aligned}$$

which determines the quantities of bifurcating periodic solutions in the center manifold at the critical value τ_{20}, *i.e.*, μ_2 determines the directions of a Hopf bifurcation: if $Re\{\lambda'(\tau_0)\} > 0$, $\mu_2 > 0$(*resp.* $\mu_2 < 0$), the Hopf bifurcation is supercritical (*resp.* subcritical) and the periodic solutions exist for $\tau > \tau_0(\tau < \tau_0)$. β_2 determines the stability of the bifurcation periodic solutions: the bifurcating periodic solutions are stable(unstable) if $\beta_2 < 0(\beta_2 > 0)$. The period of the bifurcating periodic solutions is determined by the sign of $T_2:$ if $T_2 > 0(T_2 < 0)$, the bifurcating periodic solutions increase(decrease).

4. Numerical simulations

In this section, we present some numerical simulations by using Matlab 7.0 to illustrate the analytical results, and the corresponding wave form and the phase plots of system (1.4) are drawn.

Let $a_1 = 1.05, b_1 = 0.8, c = 2.49, d = 0.1, e = 0.1, c_1 = 0.0005, f = 0.8$. Then, we have the following particular example of system (1.4):

$$\begin{cases}\dot{x}(t) = x(t) - x^2(t) - \frac{2.49x(t-\tau_1)y(t-\tau_1)}{1+1.05x(t-\tau_1)+0.8y(t-\tau_1)+0.005x(t-\tau_1)y(t-\tau_1)},\\ \dot{y}(t) = -0.1y(t) - 0.1y^2(t) + \frac{0.8x(t-\tau_2)y(t-\tau_2)}{1+1.05x(t-\tau_2)+0.8y(t-\tau_2)+0.005x(t-\tau_2)y(t-\tau_2)}.\end{cases} \tag{4.1}$$

System (4.1) has a positive equilibrium $E^*(0.3256, 0.4640)$.

For $\tau_1 = 0, \tau_2 > 0$, we can get $\omega_{20} = 0.1549, \tau_{20} = 6.7756$. From Theorem 2.2, we know that the positive equilibrium E^* is asymptotically stable when $\tau_2 \in [0, \tau_{20})$. When the time delay τ_2 passes through the critical value τ_{20}, the positive equilibrium E^* loses its stability and a Hopf bifurcation occurs, and a family of periodic solutions bifurcate from the positive equilibrium E^*. The corresponding wave form and the phase plots are depicted in Figures 1 and 2.

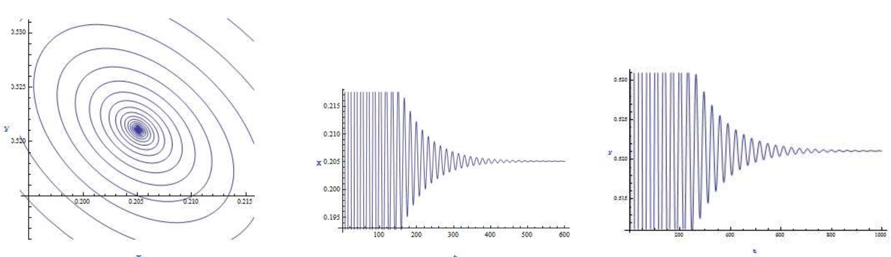

Figure 1. When $\tau_1 = 0, E^*$ is asymptotically stable for $\tau_2 = 2.75 < \tau_{20} = 6.7756$.

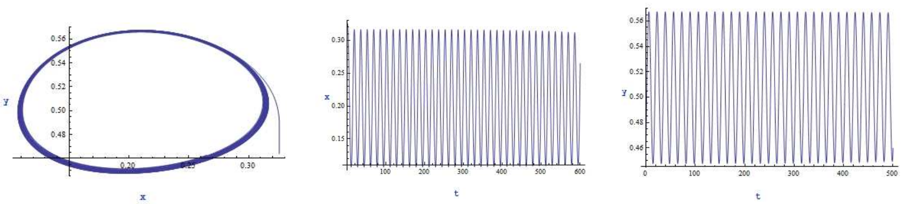

Figure 2. When $\tau_1 = 0, E^*$ undergoes a Hopf bifurcation for $\tau_2 = 3.8 < \tau_{20} = 6.7756$.

For $\tau_2 = 0, \tau_1 = 1.01657 < \tau_{10} = 1.4340$, The positive equilibrium E^* is asymptotically stable. we can get $\omega_{10} = 0.4303$. when the time delay τ_1 passes through the critical value τ_{10}, the positive equilibrium E^* lose its stability and a Hopf bifurcation occurs. The corresponding wave form and the phase plots are depicted in Figures 3 and 4. For $\tau_1 = 0.1$, when $\tau_2 = 7$, According to Theorem 2.5, E^* is asymptotically

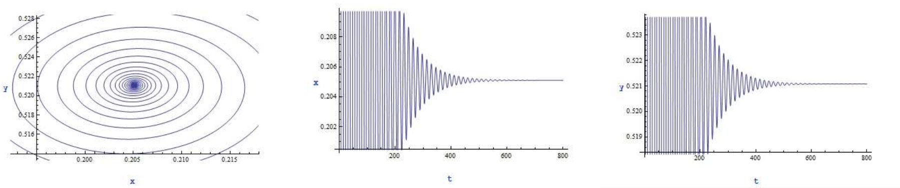

Figure 3. When $\tau_2 = 0, E^*$ is asymptotically stable for $\tau_1 = 0.98 < \tau_{10} = 1.4340$.

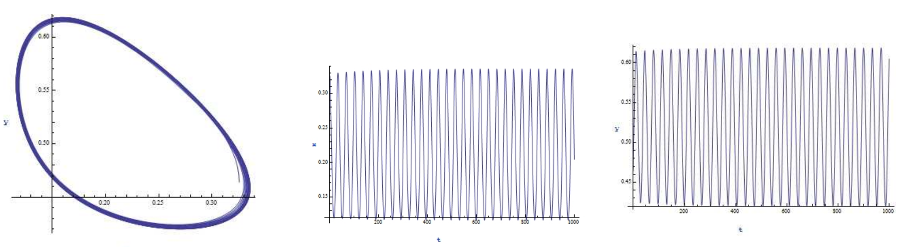

Figure 4. When $\tau_2 = 0, E^*$ undergoes a Hopf bifurcation for $\tau_1 = 1.01657 < \tau_{10} = 1.4340$.

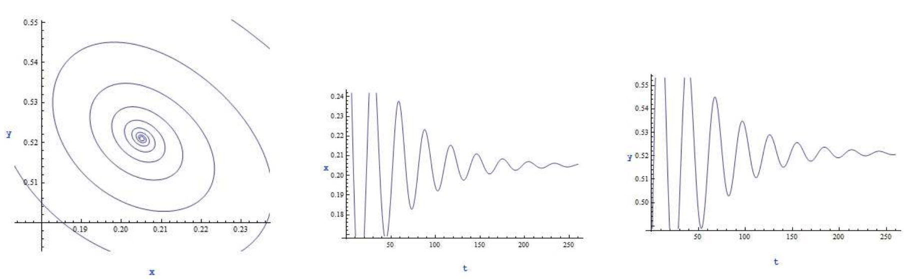

Figure 5. E^* is asymptotically stable for $\tau_1 = 0.10, \tau_2 = 1.9 < \tau_{20} = 6.7756$.

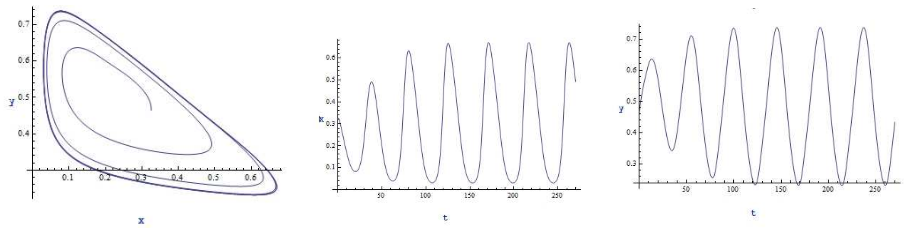

Figure 6. E^* undergoes a Hopf bifurcation for $\tau_1 = 0.1, \tau_2 = 7.0 > \tau_{20} = 6.7756$.

stable and unstable when$\tau_2 > \tau'_{20}$. After the computation of (3.17), we can obtain $c_1(0) = -1.1183e + 03 + 1.0511e + 02i, \mu_2 = 11.9746, \beta_2 = -2.2365e + 03, T_2 = -88.8339$, From Theorem 3.1, the Hopf bifurcation is supercritical, the bifurcating periodic solutions are stable, which can be depicted in Figures 5 and 6. Similarly, for $\tau_1 = 2.8, \tau_2 = 6.7464$, we can obtain $\omega'_{10} = 0.4102, \tau'_{10} = 1.5460$.

5. Conclusions.

In this paper, a density dependent predator-prey model with Crowley-Martin functional response and two time delays is considered. By analyzing the corresponding characteristic equations, we investigate on local stability of each of the feasible equilibria and establish the existence of Hopf bifurcations at the coexistence equilibrium. It has been shown that, the time delay due to the gestation of the predator is marked because the critical value of τ_2 is smaller than that of τ_1 we only consider, respectively. We can obtain that system becomes unstable if time delays are large enough. It shows that the densities of the predator and prey population will keep in an oscillatory case. By applying the normal form theory and center manifold theorem, the explicit formulas which determine the direction of Hopf bifurcation and stability of the bifurcating periodic solution are derived. The numerical results in which the Hopf bifurcation is super critical and the bifurcation periodic solutions are stable are in accord with the theoretical analysis.

In addition, owing to the lack of hunting ability of the immature predator, stage structured. However, some predator species dislike hunting immature preys, or many immature preys are concealed in the caves or nests to keep from being attacked by the predators in the natural world. Because of these, it deserves our attention to explore the influence concerning stage structure for the prey in system (1.4) for further discussion.

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References

[1] E. Beretta and Y. Kuang, *Global analyses in some delayed ratio-dependent predator-prey systems*, Nonlinear Anal.: Theory Methods Appl., 1998, 32(3), 381–408.

[2] P. H. Crowley and E. K. Martin, *Functional responses and interference within*

and between year classes of a dragonfly population, J N Am Benth Soc, 1989, 8(3), 211–221.

[3] Q. Dong, W. Ma and M. Sun, *The asymptotic behaviour of a chemostat model with Crowley-Martin type functional response and time delays*, J Math Chem, 2013, 51(5), 1231–1248.

[4] B. D. Hassard, N. D. Kazarinoff and Y. H. Wan, *Theory and Applications of Hopf Bifurcation*, Cambridge University Press, Cambridge, UK, 1982, 62, 713–714.

[5] A. Lotka, *Elements of mathematical biology*, NewYork: Dover Publications, 1958, 47, 764.

[6] S. M. Li and X. F. Tan, *Stability analysis of an enterprise competitive model with time delay*, JAAC, 2016, 3, 684–689.

[7] S. Ruan and J. Wei, *On the zero of some transcendential functions with applications to stability of delay differential equations with two delays*, Dyn. Contin. DiscreteImpuls. Syst. Ser. A, 2003, 10, 863–874.

[8] G. T. Sklaski and J. F. Gilliam, *Functional responses with predator interference: viable alter native to Holling typeII model*, Ecology, 2001, 82(11), 3083–3092.

[9] Y. Song and J. Wei, *Bifurcation analysis for Chen's system with delayed feedback and its application to Control of chaos*, Chaos, Solitons & Fractals, 2004, 22(1), 75–91.

[10] J. P. Tripathia, S. Tyagia and S. Abbasa, *Global analysis of a delayed density dependent predator-prey model with Crowley-Martin functional response*, Commun. Nonlinear Sci. Numer. Simul, 2016, 30, 45–69.

[11] X. Tian and R. Xu, *Stability analysis of an SEIS epidemic model with nonlinear incidence and time delay*, JAAC, 2014, 4, 405–418.

[12] R. K. Upadhyay and R. K. Naji, *Dynamics of three species food chain model with Crowley-Martin type functional response*, Chaos, Solitons & Fractals, 2009, 42(3), 1337–1346.

[13] R. K. Upadhyay, S. N. Raw and V. Rai, *Dynamic complexities in a tri-trophic food chain model with Holling type II and Crowley-Martin functional response*, Nonlinear Anal.: Model Control, 2010, 15(3), 361–375.

[14] X. Wu and C. Zhang, *Dynamic properties of the Oregonator model with delay*, JAAC, 2012, 2(1), 91–102.