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NONTRIVIAL PERIODIC SOLUTIONS TO A TYPE OF DELAYED RESONANT DIFFERENTIAL EQUATIONS*

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Abstract In this paper, we consider a type of delayed resonant differential equations. We focus on the existence of periodic solutions. Employing the Clark dual, we provide two sets of criteria on the existence of at least one periodic solution. In fact, the periodic solutions are critical points minimizing the dual functional of the coupled Hamiltonian system on certain subspaces of a Banach space.

Keywords Periodic solution, delay differential equation, Hamiltonian system, Clark dual.

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1. Introduction

In 1962, Jones [9] firstly investigated the existence of periodic solutions to a scalar delay differential equation. Since then much has been done for similar equations by employing methods including fixed point theory, Hopf bifurcation theorems, the coincidence degree theorem, and the Poincaré-Bendixson theory. We refer the readers to papers [1, 17, 18, 22], monograph [7], and survey [27].

In 1974, Kaplan and Yorke [10] investigated the delay differential equation,

$$x'(t) = -f(x(t-1)), \qquad x \in \mathbb{R}, \tag{1.1}$$

where $\mathbb{R}$ is the set of all real numbers, and f is odd and continuous such that $xf(x) > 0$ for $x \neq 0$. They introduced a totally new technique, which reduced the existence of periodic solutions of (1.1) to that of an associated planar system of ordinary differential equations. By making use of this technique, it is shown that (1.1) possesses a periodic solution with period 4. In the same paper, they also considered the delay differential equation,

$$x'(t) = -f(x(t-1)) - f(x(t-2)) - \cdots - f(x(t-n+1)), \qquad x \in \mathbb{R}. \tag{1.2}$$

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They claimed that under some appropriate assumptions, the existence of periodic solutions to (1.2) can be obtained by that of an associated system of ordinary differential equations.

In 1978, by making use of the fixed point theory, Nussbaum [22] showed that Kaplan and Yorke's conjecture is valid. In 1998, Li and He [11] tried to make use of Kaplan and Yorke's original idea to reinvestigate (1.2). They transformed the existence of periodic solutions of (1.2) to that of a Hamiltonian system. By making use of Lyapunov center theorem and some known results on convex Hamiltonian systems, Li and He [12] established some results on the existence of periodic solutions to (1.2). For more results, we refer to [11, 13, 16]. In 2006, Fei reduced the existence of periodic solutions of (1.2) to the existence of some symmetric periodic solutions of Hamiltonian systems. By making use of pseudo index theory, Fei [2, 3] proved the existence of multiple periodic solutions to (1.2) with odd number and even number of delays, respectively.

In 2005, Guo and Yu [5] directly built a variational structure for a delay differential system with one delay, where the variational functional contains the delay. They showed that the existence of periodic solutions of delay differential equations is equivalent to that of critical points of the associated variational functional. By making use of pseudo index theory, Guo and Yu [5] obtained some sufficient conditions guaranteeing the existence of multiple periodic solutions. Several years later, Guo and Yu [6] and Zheng and Guo [28] respectively found the equivalence between the existence of periodic solutions of delay differential systems with odd number and even number of delays and that of critical points of associated variational functionals, respectively. Two results were obtained on the existence of multiple periodic solutions.

In 2013, Yu [26] found that the existence of periodic solutions to (1.1) depends on the behavior of $f(x)/x$ at both zero and infinity. Precisely, if

$$\min\{a_1, a_2\} < \frac{\pi}{2}(1+4k) < \max\{a_1, a_2\},$$

then (1.1) possesses a periodic solution with period $4/(1+4k)$, where $a_1 = \lim_{x\to 0} f(x)/x$ and $a_2 = \lim_{x\to\infty} f(x)/x$. In the same paper, results on the existence of periodic solutions to (1.1) when $\min\{a_1, a_2\} < \pi/2$ or $\max\{a_1, a_2\} > \pi/2$ were also established.

In 2016, Ge and Zhang studied the following delay system

$$\dot{x}(t) = -\sum_{i=1}^{n} \nabla F(x(t-i)), x \in \mathbb{R}^N.$$

Here N is a positive integer. They assumed that

$$\nabla F(x) = A_\infty x + o(|x|),\ |x| \to \infty,$$

$$\nabla F(x) = A_0 x + o(|x|),\ |x| \to 0,$$

where $A_\infty, A_0 \in \mathbb{R}^{N\times N}$ are symmetric constant matrices and $|\cdot|$ denotes the norm in $\mathbb{R}^N$. By making use of variational methods, the authors [4] proved the existence of multiplicity of periodic solutions depends only upon the eigenvalues of both matrices A_0 and A_∞. More results on this direction, we refer to [14, 15].

Recently, based on the Kaplan-Yorke method, Han, Xu and Tian studied the existence and multiple periodic solutions of delay differential equation $\dot{x}(t) = bx(t-$

$1) + \epsilon f(x(t), x(t-1), \epsilon)$. By making use of bifurcation theory, they proved the existence of multiple solutions to the delay differential equations with period $4/(4k+1)$ or $4/(4k+3)$. We refer to reference [8].

We mention that, in most of the above references, the delay differential equations are asymptotically linear at both zero and infinity. In 2012, Wu and Wu [23] studied the following boundary value problem,

$$\begin{cases} u'(t) = -\Lambda u(t+r) - f(t, u(t-r)), \\ u(0) = -u(2r), \qquad u(0) = u(4r), \end{cases} \tag{1.3}$$

where $r > 0$ is a given constant and $\Lambda \in (-\frac{\pi}{2r}, \frac{3\pi}{2r})$ is a parameter. By using some results for strongly indefinite functionals, they got some results on the existence of multiple periodic solutions to (1.3) under the non-resonance assumption. For more results in this direction, we refer to [21, 24, 25].

In this article, we are going to investigate a type of delayed resonant differential equations, which can be viewed as a nonlinear perturbation at an arbitrary eigenvalue. Precisely, we investigate the following equation,

$$x'(t) = -\alpha x(t-1) - f(t, x(t-1)), \qquad x \in \mathbb{R}, \tag{1.4}$$

where $\alpha = (-1)^{k-1}(2k-1)\pi/2$ and k is a positive integer. Assume that $f : [0,4] \times \mathbb{R} \to \mathbb{R}$ satisfies

($\mathbf{f}_1$) $f(t,x)$ is odd with respect to x and is 1-periodic with respect to t, *i.e.*,

$$f(t,-x) = -f(t,x), \qquad f(t+1,x) = f(t,x), \qquad (t,x) \in [0,4] \times \mathbb{R};$$

($\mathbf{f}_2$) $F(\cdot, x)$ is measurable for each $x \in \mathbb{R}$ and $F(t,\cdot)$ is convex and continuously differentiable for a.e. $t \in [0,4]$, where $F(t,x) = \int_0^x f(t,s)ds$.

By making change of variables, one can reduce the existence of periodic solutions of (1.4) to that of some symmetric periodic solutions of a coupled Hamiltonian system. By searching the critical points of the dual variational functional, we prove the existence of critical points, which minimize the dual functional on some subspaces of a Banach space.

The rest of this article is organized as follows. In Section 2, we transform the existence of periodic solutions of (1.4) to that of some symmetric periodic solutions of a coupled Hamiltonian system. Then the dual variational functional associated with the Hamiltonian system on some suitable subspaces of a Banach space is established. Then in Section 3, we state and prove the main results.

2. Preliminaries

Denote by $\mathbb{N}^*$, $\mathbb{Z}$, and $\mathbb{R}^+$ the sets of all positive integers, integers, and non-negative real numbers, respectively.

2.1. The coupled Hamiltonian system

Suppose that $x(t)$ is a 4-periodic solution of (1.4) such that $x(t) = -x(t-2)$. Let

$$x_1(t) = x(t), \qquad x_2(t) = x(t-1). \tag{2.1}$$

Denote $y(t) = (x_1(t), x_2(t))^\tau$ and $H(t, y) = F(t, x_1) + F(t, x_2)$, where $^\tau$ denotes the transpose of a vector. Then y satisfies

$$Jy'(t) = \alpha y + \nabla H(t, y), \tag{2.2}$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the standard symplectic matrix. Computing directly, one can verify the following lemma.

Lemma 2.1. *Assume that f satisfies* $(\mathbf{f}_1)$. *Then the following statements hold.*

(1) *Any solution $x(t)$ of* (1.4) *such that $x(t) = -x(t-2)$ will give a solution $y(t) = (x_1(t), x_2(t))^\tau$ of* (2.2), *where $x_1(t), x_2(t)$ are defined by* (2.1). *Moreover, such a solution $y(t)$ possesses a symmetric structure as follows,*

$$x_1(t) = -x_2(t-1), \quad x_2(t) = x_1(t-1). \tag{2.3}$$

(2) *Any solution $y(t)$ of* (2.2) *with the symmetric structure* (2.3) *will give a solution of* (1.4) *by letting $x(t) = x_1(t)$. Moreover, $x(t) = -x(t-2)$.*

To study the existence of periodic solutions of (2.2) with the symmetric structure (2.3), in the following, we provide the dual variational functional.

2.2. The dual variational functional

Let $L^2(0, 4; \mathbb{R}^2)$ be the Banach space of quadratic integrable 4-periodic functions from $\mathbb{R}$ into $\mathbb{R}^2$. For $y \in L^2(0, 4; \mathbb{R}^2)$, it has the following Fourier expansion in the sense that it is convergent under the norm defined on $L^2(0, 4; \mathbb{R}^2)$,

$$y(t) = a_0 + \sum_{j=1}^{\infty} \left[a_j \cos\left(\frac{\pi}{2} jt\right) + b_j \sin\left(\frac{\pi}{2} jt\right) \right], \tag{2.4}$$

where a_0, a_j, $b_j \in \mathbb{R}^2$ for $j \in \mathbb{N}^*$. Define a subset E of $L^2(0, 4; \mathbb{R}^2)$ as follows,

$$E = \left\{ y \in L^2(0, 4; \mathbb{R}^2) \,\middle|\, \begin{array}{l} y(t) = \sum_{j=1}^{\infty} \left[a_j \cos\left(\frac{\pi}{2}(2j-1)t\right) + b_j \sin\left(\frac{\pi}{2}(2j-1)t\right) \right], \\ \text{where } b_j = (-1)^j J a_j \text{ for } j \in \mathbb{N}^* \end{array} \right\}.$$

For any $y \in E$, computing directly, one has

$$\begin{aligned} Jy(t-1) &= J \left\{ \sum_{j=1}^{\infty} \left[a_j \cos\left(\tfrac{\pi}{2}(2j-1)t - j\pi + \tfrac{\pi}{2}\right) + b_j \sin\left(\tfrac{\pi}{2}(2j-1)t - j\pi + \tfrac{\pi}{2}\right) \right] \right\} \\ &= J \sum_{j=1}^{\infty} (-1)^j \left[-a_j \sin\left(\frac{\pi}{2}(2j-1)t\right) + b_j \cos\left(\frac{\pi}{2}(2j-1)t\right) \right] \\ &= \sum_{j=1}^{\infty} \left[-(-1)^j J a_j \sin\left(\frac{\pi}{2}(2j-1)t\right) + (-1)^j J b_j \cos\left(\frac{\pi}{2}(2j-1)t\right) \right] \\ &= -\sum_{j=1}^{\infty} \left[b_j \sin\left(\frac{\pi}{2}(2j-1)t\right) + a_j \cos\left(\frac{\pi}{2}(2j-1)t\right) \right] \end{aligned}$$

$$= -y(t).$$

Consequently, $-Jy(t-1) = y(t)$, that is, every element of E possesses the symmetric structure (2.3).

Before going further, define

$$\mathbf{Z} = \left\{ \omega \in E \,\middle|\, \int_0^4 \omega(t)\sin(\alpha t)dt = \int_0^4 \omega(t)\cos(\alpha t)dt = 0 \right\}.$$

Obviously, $\mathbf{Z}$ is the kernel of the linear operator of $A : E \to \mathbb{R}^4$, where

$$A(w) = \left(\int_0^4 w(t)\sin(\alpha t)dt, \int_0^4 w(t)\cos(\alpha t)dt \right).$$

Therefore, $\mathbf{Z}$ is a closed linear subspace of E. One can easily verify the following property on elements of $\mathbf{Z}$.

Proposition 2.1. *$\omega \in \mathbf{Z}$ if and only if its Fourier series is*

$$\omega(t) = \sum_{\substack{j\in\mathbb{N}^*\\ j\neq k}} \left[a_j \cos\left(\frac{\pi}{2}(2j-1)t\right) + b_j \sin\left(\frac{\pi}{2}(2j-1)t\right) \right], \tag{2.5}$$

where $b_j = (-1)^j Ja_j$ for $j \in \mathbb{N}^* \setminus \{k\}$.

Let us consider the boundary value problem of the linear non-homogeneous ordinary differential system,

$$\begin{cases} Jy'(t) = \alpha y + z(t), \qquad z \in E, \\ y(0) = y(4). \end{cases} \tag{2.6}$$

Proposition 2.2. (2.6) *has a unique solution in $\mathbf{Z}$, denoted by Lz, if and only if $z \in \mathbf{Z}$.*

Proof. On the one hand, suppose that (2.6) has a solution $y \in \mathbf{Z}$. Then it follows from (2.5) that

$$\begin{aligned} Jy' - \alpha y = \sum_{\substack{j\in\mathbb{N}^*\\ j\neq k}} \Big\{ &\Big[\frac{\pi}{2}(2j-1)Jb_j - \alpha a_j\Big] \cos\left(\frac{\pi}{2}(2j-1)t\right) \\ &+ \Big[-\frac{\pi}{2}(2j-1)Ja_j - \alpha b_j \big] \sin\left(\frac{\pi}{2}(2j-1)t\right)\Big]\Big\}. \end{aligned}$$

Set $A_j = \frac{\pi}{2}(2j-1)Jb_j - \alpha a_j$ and $B_j = -\frac{\pi}{2}(2j-1)Ja_j - \alpha b_j$. One can easily check that $(-1)^j JA_j = B_j$ for all $j \in \mathbb{N}^* \setminus \{k\}$. Then Proposition 2.1 yields that $z = Jy' - \alpha y \in \mathbf{Z}$.

On the other hand, suppose that $z \in \mathbf{Z}$ and its Fourier series is

$$z(t) = \sum_{\substack{j\in\mathbb{N}^*\\ j\neq k}} \left[a_j \cos\left(\frac{\pi}{2}(2j-1)t\right) + b_j \sin\left(\frac{\pi}{2}(2j-1)t\right) \right],$$

where $b_j = (-1)^j J a_j$ for $j \in \mathbb{N}^* \setminus \{k\}$. According to (2.4), for any $y \in L^2(0, 4, \mathbb{R})$, its Fourier series is

$$y(t) = A_0 + \sum_{l=1}^{\infty} \left[A_l \cos\left(\frac{\pi}{2} lt\right) + B_l \sin\left(\frac{\pi}{2} lt\right) \right].$$

Direct computation yields

$$Jy' - \alpha y = -\alpha A_0 + \sum_{l=1}^{\infty} \left[\left(\frac{\pi}{2} l J B_l - \alpha A_l \right) \cos\left(\frac{\pi}{2} lt\right) + \left(-\frac{\pi}{2} l J A_l - \alpha B_l \right) \sin\left(\frac{\pi}{2} lt\right) \right].$$

If y satisfies $Jy' = z + \alpha y$, then

$$-\alpha A_0 = \frac{\pi}{2}(2l) J B_{2l} - \alpha A_{2l} = -\frac{\pi}{2}(2l) J A_{2l} - \alpha B_{2l} = 0$$

for $l \in \mathbb{N}^*$ and

$$\frac{\pi}{2}(2l-1) J B_{2l-1} - \alpha A_{2l-1} = a_l, \qquad -\frac{\pi}{2}(2l-1) J A_{2l-1} - \alpha B_{2l-1} = b_l$$

for $l \in \mathbb{N}^* \setminus \{k\}$, that is,

$$A_0 = A_{2l} = B_{2l} = 0$$

for $l \in \mathbb{N}^*$ and

$$A_{2l-1} = \gamma_l a_l, \qquad B_{2l-1} = \gamma_l b_l$$

for $l \in \mathbb{N}^* \setminus \{k\}$, where $\gamma_l = \frac{1}{(-1)^{l-1}(2l-1)\frac{\pi}{2} - (-1)^{k-1}(2k-1)\frac{\pi}{2}}$. It follows that

$$\begin{aligned} y(t) = &\sum_{\substack{l \in \mathbb{N}^* \\ l \neq k}} \gamma_l \left[a_l \cos\left(\frac{\pi}{2}(2l-1)t\right) + b_l \sin\left(\frac{\pi}{2}(2l-1)t\right) \right] \\ &+ \left[A_{2k-1} \cos\left(\frac{\pi}{2}(2k-1)t\right) + B_{2k-1} \sin\left(\frac{\pi}{2}(2k-1)t\right) \right]. \end{aligned}$$

Hence $y \in \mathbf{Z}$ if and only if $A_{2k-1} = B_{2k-1} = 0$. Therefore, (2.6) has a unique solution $Lz \in \mathbf{Z}$, where

$$(Lz)(t) = \sum_{\substack{l \in \mathbb{N}^* \\ l \neq k}} \gamma_l \left[a_l \cos\left(\frac{\pi}{2}(2l-1)t\right) + b_l \sin\left(\frac{\pi}{2}(2l-1)t\right) \right]. \qquad \square$$

In what follows, we give an explicit expression of Lz in terms of z. For any $z \in \mathbf{Z}$, the general solution of (2.6) is given by

$$\begin{aligned} y(t) &= \exp(-J\alpha t)c - \int_0^t \exp(-J\alpha(t-s)) J z(s) ds \\ &= c\cos(\alpha t) - Jc \sin(\alpha t) - \int_0^t [Jz(s)\cos\alpha(t-s) + z(s)\sin\alpha(t-s)] ds, \end{aligned} \tag{2.7}$$

where $c \in \mathbb{R}^2$.

Proposition 2.3. *For any $z \in \mathbf{Z}$,*

$$\begin{aligned} Lz(t) = &\frac{1}{4}\int_0^4 (4-s)[Jz(s)\cos(\alpha(t-s)) + z(s)\sin(\alpha(t-s))] ds \\ &- \int_0^t [Jz(s)\cos\alpha(t-s) + z(s)\sin\alpha(t-s)] ds. \end{aligned}$$

Proof. The proof is done by direct computation. In fact, since $Lz \in \mathbf{Z}$, we have $\int_0^4 Lz(t)\cos(\alpha t)dt = 0$. As Lz has the expression (2.7), multiplying both sides of (2.7) by $\cos(\alpha t)$ and integrating over $[0,4]$, we have

$$\begin{aligned}
&\int_0^4 [c\cos(\alpha t) - Jc\sin(\alpha t)]cos(\alpha t)]dt \\
&= \int_0^4 \int_0^t [Jz(s)\cos(\alpha(t-s)) + z(s)\sin(\alpha(t-s))]ds\cos(\alpha t)dt.
\end{aligned}$$

Note that

$$\int_0^4 [c\cos(\alpha t) - Jc\sin(\alpha t)]\cos(\alpha t)dt = \int_0^4 c\cos(\alpha t)\cos(\alpha t)dt = 2c$$

and

$$\begin{aligned}
&\int_0^4 \int_0^t [Jz(s)\cos(\alpha(t-s)) + z(s)\sin(\alpha(t-s))]ds\cos(\alpha t)dt \\
&= \int_0^4 \int_s^4 [Jz(s)\cos(\alpha(t-s))\cos(\alpha t) + z(s)\sin(\alpha(t-s))\cos(\alpha t)]dtds \\
&= \int_0^4 [Jz(s)\int_s^4 \cos(\alpha(t-s))\cos(\alpha t)dt + z(s)\int_s^4 \sin(\alpha(t-s))\cos(\alpha t)dt]ds \\
&= \frac{1}{2}\int_0^4 \Big\{Jz(s)\Big[\frac{1}{2\alpha}\sin\Big(\alpha(2t-s)\Big) + \cos\Big(\alpha s\Big)t\Big]\Big|_s^4 \\
&\quad + z(s)\Big[-\frac{1}{2\alpha}\cos\Big(\alpha(2t-s)\Big) + \sin\Big(-\alpha s\Big)t\Big]\Big|_s^4\Big]\Big\}ds \\
&= \frac{1}{2}\int_0^4 \Big\{Jz(s)\Big[-\frac{1}{\alpha}\sin(\alpha s) + (4-s)\cos(\alpha s)\Big] + z(s)[(4-s)\sin(-\alpha s)]\Big\}ds \\
&= \frac{1}{2}\int_0^4 (4-s)[Jz(s)\cos(\alpha s) - (4-s)z(s)\sin(\alpha s)]ds.
\end{aligned}$$

It follows that

$$c = \frac{1}{4}\int_0^4 (4-s)[Jz(s)\cos(\alpha s) - z(s)\sin(\alpha s)]ds.$$

Then we can easily get the required result. □

For given $z \in \mathbf{Z}$, let $y \in E$ be a solution to (2.6). Denote $\overline{y} = y - Lz$. Then

$$\overline{y} \in \mathbf{Y} = \mathbf{Z}^{\perp} = \{\omega \in E|\ \omega(t) = c\cos(\alpha t) - Jc\sin(\alpha t),\ c \in \mathbb{R}^2\}$$

and the general solution of (2.6) is given by

$$y(t) = \overline{y}(t) + Lz(t). \tag{2.8}$$

Suppose that y is a solution to (2.2) with the symmetric structure (2.3), that is, $y \in E$. Let $z(t) = Jy'(t) - \alpha y(t)$. Then $z(t) = \nabla H(t, y(t))$. By the reciprocal formula, y satisfies the equation

$$y(t) = \nabla H^*(t, z(t)) \qquad \text{a.e. on } [0,4], \tag{2.9}$$

where $H^*(t,\cdot)$ is the Fenchel transform of $H(t,\cdot)$. Substituting (2.8) into (2.9), we obtain

$$-Lz(t)+\nabla H^*(t,z(t))=\overline{y}(t) \qquad \text{a.e. on } [0,4] \tag{2.10}$$

or

$$-Lz(t)+\nabla H^*(t,z(t))\in \mathbf{Y} \qquad \text{a.e. on } [0,4]. \tag{2.11}$$

Conversely, if $z\in \mathbf{Z}$ satisfies (2.10) or (2.11), then defining y by (2.8), we see that the elimination of z implies (2.2). Also, y satisfies the symmetric structure (2.3). We note that (2.11) is the Euler equation for the critical points of χ on $\mathbf{Z}$, where

$$\chi(z)=\int_0^4\left[-\frac{1}{2}(Lz(t),z(t))+H^*(t,z(t))\right]dt. \tag{2.12}$$

Now we are ready to search critical points of (2.12) restricted on the subspace $\mathbf{Z}$.

For any $z\in\mathbf{Z}$, it follows from (2.5) that

$$z(t)=\sum_{\substack{m\in\mathbb{Z}\\ m\neq -k+1,k}} z_m e^{\frac{i(2m-1)\pi t}{2}}, \tag{2.13}$$

where $\overline{z}_{-m+1}=z_m=z_m^1+iz_m^2$ such that $z_m^2=(-1)^{m-1}Jz_m^1$ and $z_m^i\in\mathbb{R}^2$ for $i=1,2$. Substituting (2.13) into $Lz(t)$, we have

$$\begin{aligned}
Lz(t)=&\frac{1}{4}\int_0^4(4-s)[Jz(s)\cos\alpha(t-s)+z\sin\alpha(t-s)]ds\\
&-\int_0^t[Jz(s)\cos\alpha(t-s)+z(s)\sin\alpha(t-s)]ds\\
=&\frac{1}{4}\int_0^4(4-s)\Big[J\sum_{\substack{m\in\mathbb{Z}\\ m\neq -k+1,k}} z_m e^{\frac{i(2m-1)\pi s}{2}}\cos(\alpha(t-s))\\
&+\sum_{\substack{m\in\mathbb{Z}\\ m\neq -k+1,k}} z_m e^{\frac{i(2m-1)\pi s}{2}}\sin(\alpha(t-s))\Big]ds\\
&-\int_0^t\Big[J\sum_{\substack{m\in\mathbb{Z}\\ m\neq -k+1,k}} z_m e^{\frac{i(2m-1)\pi s}{2}}\cos\alpha(t-s)\\
&+\sum_{\substack{m\in\mathbb{Z}\\ m\neq -k+1,k}} z_m e^{\frac{i(2m-1)\pi s}{2}}\sin\alpha(t-s)\Big]ds\\
=&\sum_{\substack{m\in\mathbb{Z}\\ m\neq\pm\frac{2}{\pi}\alpha}} Jz_m\Big[\int_0^4\frac{4-s}{4}e^{\frac{i(2m-1)\pi s}{2}}\cos(\alpha(t-s))ds\\
&-\int_0^t e^{\frac{i(2m-1)\pi s}{2}}\cos\alpha(t-s)ds\Big]\\
&+\sum_{\substack{m\in\mathbb{Z}\\ m\neq -k+1,k}} z_m\Big[\int_0^4\frac{4-s}{4}e^{\frac{i(2m-1)\pi s}{2}}\sin(\alpha(t-s))ds\\
&-\int_0^t e^{\frac{i(2m-1)\pi s}{2}}\sin\alpha(t-s)ds\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\substack{m\in\mathbb{Z}\\ m\neq -k+1,k}} Jz_m \Big[\int_0^4 e^{\frac{i(2m-1)\pi s}{2}} \cos(\alpha(t-s))ds \\
&\quad -\frac{1}{4}\int_0^4 s e^{\frac{i(2m-1)\pi s}{2}} \cos(\alpha(t-s))ds \\
&\quad -\int_0^t e^{\frac{i(2m-1)\pi s}{2}} \cos\alpha(t-s)ds\Big] \\
&\quad + \sum_{\substack{m\in\mathbb{Z}\\ m\neq -k+1,k}} z_m \Big[\int_0^4 e^{\frac{i(2m-1)\pi s}{2}} \sin(\alpha(t-s))ds \\
&\quad -\frac{1}{4}\int_0^4 s e^{\frac{i(2m-1)\pi s}{2}} \sin(\alpha(t-s))ds \\
&\quad -\int_0^t e^{\frac{i(2m-1)\pi s}{2}} \sin\alpha(t-s)ds\Big] \\
&= \sum_{\substack{m\in\mathbb{Z}\\ m\neq -k+1,k}} Jz_m \frac{(2m-1)^2}{(2k-1)^2-(2m-1)^2}\frac{1}{i(2m-1)\frac{\pi}{2}} e^{i(2m-1)\frac{\pi}{2}t} \\
&\quad + \sum_{\substack{m\in\mathbb{Z}\\ m\neq -k+1,k}} z_m \frac{(2m-1)^2}{(2k-1)^2-(2m-1)^2}\frac{(-1)^{k-1}(2k-1)\frac{\pi}{2}}{(i(2m-1)\frac{\pi}{2})^2} e^{i(2m-1)\frac{\pi}{2}t} \\
&= \sum_{\substack{m\in\mathbb{Z}\\ m\neq -k+1,k}} \frac{(2m-1)^2}{(2k-1)^2-(2m-1)^2}\Big[Jz_m\frac{1}{i(2m-1)\frac{\pi}{2}} + z_m\frac{(-1)^{k-1}(2k-1)\frac{\pi}{2}}{(i(2m-1)\frac{\pi}{2})^2}\Big] e^{i(2m-1)\frac{\pi}{2}t}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\int_0^4 (Lz(t), z(t))dt \\
&= \int_0^4 \Bigg(\sum_{\substack{m\in\mathbb{Z}\\ m\neq -k+1,k}} \frac{(2m-1)^2}{(2k-1)^2-(2m-1)^2}\Big[\frac{Jz_m}{i(2m-1)\frac{\pi}{2}} \\
&\quad + \frac{z_m(-1)^{k-1}(2k-1)\frac{\pi}{2}}{(i(2m-1)\frac{\pi}{2})^2}\Big] e^{i(2m-1)\frac{\pi}{2}t}, \sum_{\substack{m\in\mathbb{Z}\\ m\neq -k+1,k}} z_m e^{\frac{i(2m-1)\pi t}{2}} \Bigg) dt \\
&= \sum_{\substack{m\in\mathbb{Z}\\ m\neq -k+1,k}} \frac{(2m-1)^2}{(2m-1)^2-(2k-1)^2}\Big[\frac{4}{(2m-1)\frac{\pi}{2}} i(Jz_m, z_{-m+1}) + \frac{(-1)^{k-1}(2k-1)2\pi}{((2m-1)\frac{\pi}{2})^2}|z_m|^2\Big].
\end{aligned} \tag{2.14}
$$

If $z_m = (z_m^1, z_m^2)$ with $z_m^j = a_j + ib_j$, $j = 1, 2$, then

$$
\begin{aligned}
i(Jz_m, z_{-m+1}) &= i\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a_1+ib_1 \\ a_2+ib_2 \end{pmatrix}, \begin{pmatrix} a_1-ib_1 \\ a_2-ib_2 \end{pmatrix}\right) \\
&= i[(a_2+ib_2)(a_1-ib_1) + (-a_1-ib_1)(a_2-ib_2)]
\end{aligned}
$$

$$
\begin{aligned}
&= 2a_2b_1 - 2a_1b_2 = 2(a_2, -a_1)\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = 2(a_2, -a_1)(-1)^{m-1}J\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \\
&= (-1)^{m-1}2(a_1^2 + a_2^2) = (-1)^{m-1}(a_1^2 + a_2^2 + b_1^2 + b_2^2) \\
&= (-1)^{m-1}|z_m|^2. \qquad (2.15)
\end{aligned}
$$

Substituting (2.15) into (2.14), one obtains

$$
\begin{aligned}
&\int_0^4 (Lz(t), z(t))dt \\
&= \sum_{\substack{m\in\mathbb{Z} \\ m\neq -k+1,k}} \frac{(2m-1)^2}{(2m-1)^2-(2k-1)^2}\left[\frac{4(-1)^{m-1}}{(2m-1)\frac{\pi}{2}} + \frac{(-1)^{k-1}(2k-1)2\pi}{((2m-1)\frac{\pi}{2})^2}\right]|z_m|^2 \\
&= \sum_{\substack{m\in\mathbb{Z} \\ m\neq -k+1,k}} \frac{8}{\pi}\frac{1}{(2m-1)^2-(2k-1)^2}\Big[(-1)^{m-1}(2m-1) + (-1)^{k-1}(2k-1)\Big]|z_m|^2 \\
&= \sum_{\substack{m\in\mathbb{Z} \\ m\neq -k+1,k}} \frac{8}{\pi}\frac{1}{(-1)^{m-1}(2m-1)-(-1)^{k-1}(2k-1)}|z_m|^2 \\
&\leq \sum_{\substack{m\in\mathbb{Z} \\ m\neq -k+1,k}} \frac{2}{\pi}|z_m|^2 \\
&= \frac{1}{2\pi}\|z\|_{L^2}^2.
\end{aligned}
$$

3. Main results and their proofs

Now we are in the position to present our main results.

Theorem 3.1. *Suppose that* $(\mathbf{f}_1)$ *and* $(\mathbf{f}_2)$ *hold and*

$$H(t, y) = H(t, x_1, x_2) = \int_0^{x_1} f(t, s)ds + \int_0^{x_2} f(t, s)ds$$

satisfies the following conditions

$(\mathbf{H}_1)$ *There exists* $l \in L^4(0, 4; \mathbb{R}^2)$ *such that*

$$H(t, y) \geq (l(t), y) \qquad \text{for all } y \in \mathbb{R}^2 \text{ and a.e. } t \in [0, 4];$$

$(\mathbf{H}_2)$ *There exist* $\beta \in (0, 2\pi)$ *and* $\gamma \in L^2(0, 4; \mathbb{R}^+)$ *such that*

$$H(t, y) \leq \frac{\beta}{2}|y|^2 + \gamma(t) \qquad \text{for } y \in \mathbb{R}^2 \text{ and a.e. } t \in [0, 4];$$

$(\mathbf{H}_3)$ $\int_0^4 H(t, b_1\cos(\alpha t) + b_2\sin(\alpha t))dt \to \infty$ *as* $|b_1| + |b_2| \to \infty$*, where* b_1*,* $b_2 \in \mathbb{R}^2$*.*

Then (2.2) *has at least one solution* y *such that* $z = Jy' - \alpha y$ *minimizes the dual functional* χ *on* $\mathbf{Z}$*.*

Remark 3.1. The assumptions ($\mathbf{H}_1$-$\mathbf{H}_3$) had been used to study the existence of periodic solutions to Hamiltonian systems. Here we refer the interested reader to [19, 20] and their references.

Let $\varepsilon_0 > 0$ such that $\beta + \varepsilon_0 < 2\pi$. For $\varepsilon \in (0, \varepsilon_0)$, define $H_\varepsilon : [0,4] \times \mathbb{R}^2 \to \mathbb{R}$ by

$$H_\varepsilon(t,y) = \frac{1}{2}\varepsilon|y|^2 + H(t,y).$$

In order to prove Theorem 3.1, we consider the disturbed Hamiltonian system,

$$\begin{cases} Jy'(t) = \alpha y + \nabla H_\varepsilon(t,y), \\ y(0) = y(4). \end{cases} \tag{3.1}$$

The dual variational functional corresponding to (3.1) is

$$\chi_\varepsilon(z) = \int_0^4 \Big[-\frac{1}{2}(Lz(t), z(t)) + H_\varepsilon^*(t, z(t)) \Big] dt.$$

Lemma 3.1. *Assume that the assumptions of Theorem 3.1 hold. Then the disturbed Hamiltonian system* (3.1) *possesses at least one solution y_ε such that $z_\varepsilon = Jy'_\varepsilon - \alpha y_\varepsilon$ minimizes the dual functional χ_ε on $\mathbf{Z}$.*

Proof. It is easy to check that $H_\varepsilon(t,\cdot)$ is strictly convex and continuously differentiable for a.e. $t \in [0,4]$ and $H_\varepsilon(\cdot, y)$ is measurable on $[0,4]$ for every $y \in \mathbb{R}^2$. By $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$, we know that H_ε satisfies

$$\frac{\varepsilon}{2}|y|^2 - |l(t)| \cdot |y| \le H_\varepsilon(t,y) \le (\beta + \varepsilon_0)\frac{|y|^2}{2} + \gamma(t).$$

Consequently,

$$\frac{\varepsilon}{4}|y|^2 - \frac{|l(t)|^2}{\varepsilon} \le H_\varepsilon(t,y) \le (\beta + \varepsilon_0)\frac{|y|^2}{2} + \gamma(t). \tag{3.2}$$

By Theorem 2.3 in [20], the functional

$$\varphi_\varepsilon(z) = \int_0^4 H_\varepsilon^*(t,z)dt$$

is continuously differentiable on $\mathbf{Z}$. Define $\phi : \mathbf{Z} \to \mathbb{R}$ by

$$\phi(z) = -\int_0^4 \frac{1}{2}(Lz(t), z(t))dt.$$

Then ϕ is continuously differentiable on $\mathbf{Z}$. Hence, $\chi_\varepsilon(z) = \phi(z) + \varphi_\varepsilon(z)$ is continuously differentiable on $\mathbf{Z}$.

Since (3.2) holds, arguing similarly as in the proof of Proposition 2.2 of [20], we have

$$H_\varepsilon^*(t,z) \ge \frac{1}{2(\beta + \varepsilon_0)}|z|^2 - \gamma(t).$$

Then

$$\chi_\varepsilon(z) \ge \frac{1}{2}\Big(\frac{1}{\beta + \varepsilon_0} - \frac{1}{2\pi}\Big)\|z\|_{L^2}^2 - \int_0^4 \gamma(t)dt = \delta_0\|z\|_{L^2}^2 - \gamma_0, \tag{3.3}$$

where $\delta_0 = \frac{1}{2}(\frac{1}{\beta+\varepsilon_0} - \frac{1}{2\pi}) > 0$ and $\gamma_0 = \int_0^4 \gamma(t)dt$. Therefore, χ_ε is bounded from below and every minimizing sequence for χ_ε is bounded. Since φ_ε is lower semi-continuous and convex on $\mathbf{Z}$, φ_ε is weakly lower semi-continuous on $\mathbf{Z}$. Since ϕ is weakly continuous, χ_ε is weakly lower semi-continuous on $\mathbf{Z}$. Hence, χ_ε attains its minimum at some $z_\varepsilon \in \mathbf{Z}$, for which

$$< \chi_\varepsilon'(z_\varepsilon), y >= \int_0^4 [-(Lz_\varepsilon(t), y(t)) + (\nabla H_\varepsilon^*(t, z_\varepsilon(t)), y(t))]dt = 0 \qquad \text{for all } y \in \mathbf{Z}.$$

It follows that

$$-Lz_\varepsilon + \nabla H_\varepsilon^*(\cdot, z_\varepsilon(\cdot)) \in \mathbf{Y}.$$

Set $\overline{y}_\varepsilon = -Lz_\varepsilon + \nabla H_\varepsilon^*(\cdot, z_\varepsilon(\cdot))$. Then $y_\varepsilon = \overline{y}_\varepsilon + Lz_\varepsilon$ is a solution of (3.1). □

Lemma 3.2. *Assume that the assumptions of Theorem 3.1 hold. Then $\|y_\varepsilon\|_{L^2}$ and $\|y_\varepsilon'\|_{L^2}$ are both bounded for $\varepsilon \in (0, \varepsilon_0)$, where y_ε is the solution of* (3.1).

Proof. For any $y \in \mathbb{R}^2$, let $z = \nabla H(t, y)$. Computing directly, by $\mathbf{(H_1)}$ and $\mathbf{(H_2)}$, we have

$$\frac{1}{2\beta}|z|^2 - \gamma(t) \le H^*(t, z) = (z, y) - H(t, y) \le (|z| + |l(t)|)|y|. \tag{3.4}$$

If $|z| \ge 1$, then (3.4) yields that

$$|\nabla H(t, y)| = |z| \le 2\beta[(1 + |l(t)|)|y| + |\gamma|].$$

Hence

$$|\nabla H(t, y)| \le 2\beta[(1 + |l(t)|)|y| + |\gamma|] + 1 \qquad \text{for } y \in \mathbb{R}^2. \tag{3.5}$$

Define a function $\overline{H} : \mathbb{R}^4 \to \mathbb{R}$ by

$$\overline{H}(a, b) = \int_0^4 H(t, a\cos(\alpha t) + b\sin(\alpha t))dt.$$

Because of $\mathbf{(H_2)}$ and (3.5), applying Theorem 1.4 of [20], we know that $\overline{H}$ is continuously differentiable on $\mathbb{R}^4$. Then $\mathbf{(H_3)}$ implies that $\overline{H}$ attains its minimum at some point, denoted by $(\overline{a}, \overline{b})$, and

$$\begin{aligned}
&\int_0^4 \nabla H(t, \overline{a}\cos(\alpha t) + \overline{b}\sin(\alpha t))\cos(\alpha t)dt \\
&= \int_0^4 \nabla H(t, \overline{a}\cos(\alpha t) + \overline{b}\sin(\alpha t))\sin(\alpha t)dt \\
&= 0.
\end{aligned}$$

Thus $\nabla H(t, \overline{a}\cos(\alpha t) + \overline{b}\sin(\alpha t)) \in \mathbf{Z}$. Let

$$z(t) = \nabla H(t, \overline{a}\cos(\alpha t) + \overline{b}\sin(\alpha t)).$$

By duality, we have

$$H^*(t, z(t)) = (z(t), \overline{a}\cos(\alpha t) + \overline{b}\sin(\alpha t)) - H(t, \overline{a}\cos(\alpha t) + \overline{b}\sin(\alpha t))$$

for a.e. $t \in [0,4]$. Since $F(\cdot, y)$ is measurable for every $y \in \mathbb{R}^2$, $F(\cdot, y)$ and hence $H(\cdot, y)$ are integrable. Moreover, $H^*(t, z(t))$ is integrable since $(z(t), \overline{a}\cos(\alpha t) + \overline{b}\sin(\alpha t))$ is continuous. It follows from (3.3) that

$$\delta_0 \|z_\varepsilon\|_{L^2}^2 - \gamma_0 \le \chi_\varepsilon(z_\varepsilon) \le \chi_\varepsilon(z) \le \chi(z) = c_1 < \infty,$$

where c_1 is a constant. Thus $\|z_\varepsilon\|_{L^2} \le c_2$, where c_2 is a constant independent of ε. Consequently,

$$\|Jy_\varepsilon' - \alpha y_\varepsilon\|_{L^2} = \|z_\varepsilon\|_{L^2} \le c_2.$$

We note from Proposition 2.3 that the operator $L : \mathbf{Z} \to \mathbf{Z}$ is bounded. Let $\widetilde{y}_\varepsilon = y_\varepsilon - \overline{y}_\varepsilon$. Then $\|\widetilde{y}_\varepsilon\|_{L^2} = \|Lz_\varepsilon\|_{L^2} \le c_3$ for some suitable constant $c_3 > 0$.

Since $H(t, \cdot)$ is convex, $(\mathbf{H}_2)$ implies that

$$\begin{aligned} H\Big(t, \frac{\overline{y}_\varepsilon(t)}{2}\Big) &= H\Big(t, \frac{1}{2}y_\varepsilon(t) - \frac{1}{2}\widetilde{y}_\varepsilon(t)\Big) \\ &\le \frac{1}{2}H(t, y_\varepsilon(t)) + \frac{1}{2}H(t, -\widetilde{y}_\varepsilon(t)) \\ &\le \frac{1}{2}(\nabla H(t, y_\varepsilon(t)), y_\varepsilon(t)) + \frac{1}{2}H(t,0) + \frac{\beta}{4}|\widetilde{y}_\varepsilon(t)|^2 + \frac{\gamma(t)}{2} \\ &\le \frac{1}{2}(Jy_\varepsilon'(t) - ay_\varepsilon(t), y_\varepsilon(t)) + \gamma(t) + \frac{\beta}{4}|\widetilde{y}_\varepsilon(t)|^2. \end{aligned}$$

It follows from $z_\varepsilon = Jy_\varepsilon' - \alpha y_\varepsilon \in \mathbf{Z}$ and $\overline{y}_\varepsilon \in \mathbf{Y}$ that $\int_0^4 (Jy_\varepsilon' - \alpha y_\varepsilon, \overline{y}_\varepsilon)dt = 0$. Then

$$\begin{aligned} \int_0^4 H\Big(t, \frac{\overline{y}_\varepsilon(t)}{2}\Big)dt &\le \frac{1}{2}\int_0^4 (Jy_\varepsilon'(t) - ay_\varepsilon(t), y_\varepsilon(t))dt + \int_0^4 \gamma(t)dt + \frac{\beta}{4}\int_0^4 |\widetilde{y}_\varepsilon(t)|^2 dt \\ &= \frac{1}{2}\int_0^4 (Jy_\varepsilon'(t) - ay_\varepsilon(t), \widetilde{y}_\varepsilon(t))dt + \int_0^4 \gamma(t)dt + \frac{\beta}{4}\int_0^4 |\widetilde{y}_\varepsilon(t)|^2 dt \\ &\le \frac{1}{2}\|Jy_\varepsilon' - ay_\varepsilon\|_{L^2} \cdot \|\widetilde{y}_\varepsilon\|_{L^2} + \int_0^4 \gamma(t)dt + \frac{\beta}{4}\|\widetilde{y}_\varepsilon\|_{L^2}^2 \\ &\le c_4, \end{aligned}$$

where c_4 is a suitable positive constant independent of ε. By hypothesis $(\mathbf{H}_3)$, there exist positive constants c_5, c_6 and c_7 such that $\|\overline{y}_\varepsilon\|_{L^2} \le c_5$ and hence $\|y_\varepsilon\|_{L^2} \le c_6$, $\|Jy_\varepsilon'\|_{L^2} \le \|z_\varepsilon\|_{L^2} + \|\alpha y_\varepsilon\|_{L^2} \le c_2 + |\alpha|c_6 = c_7$. Consequently, $\|y_\varepsilon'\|_{L^2} \le c_7$. □

Proof of Theorem 3.1. For any $\varepsilon \in (0, \varepsilon_0)$, Lemma 3.1 implies that the disturbed Hamiltonian system (3.1) possesses a solutions y_ε. By Lemma 3.2, both $\|y_\varepsilon\|_{L^2}$ and $\|y_\varepsilon'\|_{L^2}$ are bounded. Since $y_\varepsilon \in E$, the mean value theorem implies that there exists $\eta \in [0,4]$ depending on ε such that

$$y_\varepsilon(\eta) = \frac{1}{4}\int_0^4 y_\varepsilon(t)dt = 0.$$

For any $t \in [0,4]$, we have

$$\begin{aligned} |y_\varepsilon(t)| &= |y_\varepsilon(t) - y_\varepsilon(\eta)| \\ &= |\int_\eta^t y_\varepsilon'(\tau)d\tau| \end{aligned}$$

$$\begin{aligned}&\le \int_0^4 |y'_\varepsilon(t)|dt\\&\le \Big(\int_0^4 |y'_\varepsilon(t)|^2dt\Big)^{\frac12}\Big(\int_0^4 dt\Big)^{\frac12}\\&= 2\|y'_\varepsilon\|_{L^2}.\end{aligned}$$

Thus $\{\|y_\varepsilon\|_{L^\infty}\}$ is bounded. For $0 \le s \le t \le 4$, we have

$$|y_\varepsilon(t) - y_\varepsilon(s)| \le \int_s^t |y'_\varepsilon(\tau)|d\tau \le (t-s)^{\frac12}\Big(\int_s^t |y'_\varepsilon(\tau)|^2d\tau\Big)^{\frac12} \le \|y'\|_{L^2}(t-s)^{\frac12}.$$

Hence $\{y_\varepsilon\}$ is equi-continuous. By Ascoli-Arzela's theorem, there is a sequence $\{\varepsilon_n\} \subset (0,\varepsilon_0)$ tending to zero and some $y \in C(0,4;\mathbb{R}^2)$ such that $y_{\varepsilon_n} \to y$ as $n \to \infty$. Obviously, $\{y_{\varepsilon_n}\}$ possesses the symmetric structure (2.3). Since the set of functions satisfying the symmetry property is closed, y possesses the symmetric structure (2.3). Integrating (3.1), one obtains

$$Jy_{\varepsilon_n}(t) - Jy_{\varepsilon_n}(0) = \int_0^t [\alpha y_{\varepsilon_n} + \nabla H_{\varepsilon_n}(t, y_{\varepsilon_n})]dt.$$

Since $\{y_{\varepsilon_n}\}$ converges to y,

$$Jy(t) - Jy(0) - \int_0^t [\alpha y(s) + \nabla H(s, y(s))]ds = 0,$$

which implies that y is a solution of (2.2).

Since $H_\varepsilon(t,y) = \varepsilon/2|y|^2 + H(t,y) \ge H(t,y)$, we have $H^*_\varepsilon(t,z) \le H^*(t,z)$. It follows that $\chi_{\varepsilon_n}(z_{\varepsilon_n}) \le \chi_{\varepsilon_n}(h) \le \chi(h)$ for any $h \in \mathbf{Z}$. Then

$$\begin{aligned}\chi_{\varepsilon_n}(z_{\varepsilon_n}) &= \int_0^4 \Big[-\frac12(Lz_{\varepsilon_n}(t), z_{\varepsilon_n}(t)) + H^*_{\varepsilon_n}(t, z_{\varepsilon_n}(t))\Big]dt\\&= \int_0^4 \Big[-\frac12(Lz_{\varepsilon_n}(t), z_{\varepsilon_n}(t)) + (y_{\varepsilon_n}, z_{\varepsilon_n}) - H_{\varepsilon_n}(t, y_{\varepsilon_n}(t))\Big]dt\\&= \int_0^4 \Big[-\frac12(Lz_{\varepsilon_n}(t), z_{\varepsilon_n}(t)) + (y_{\varepsilon_n}, z_{\varepsilon_n}) - H(t, y_{\varepsilon_n}(t)) - \frac12\varepsilon_n|y_{\varepsilon_n}|^2\Big]dt.\end{aligned}$$

Since $z_\varepsilon = \nabla H(t, y_\varepsilon)$, we have

$$\begin{aligned}\lim_{n\to\infty}\chi_{\varepsilon_n}(z_{\varepsilon_n}) &= \int_0^4 \Big[-\frac12(Lz,z) + (y, \nabla H(t,y)) - H(t,y)\Big]dt\\&= \int_0^4 \Big[-\frac12(Lz,z) + H^*(t,z)\Big]dt\\&= \chi(z).\end{aligned}$$

It follows that $\chi(z) \le \chi(h)$ for all $h \in H^1_4 \cap \mathbf{Z}$, where H^1_4 denote the Sobolev space which contains all 4-periodic functions with the property that themselves and their weak derivatives are squarely integrable. This completes the proof of Theorem 3.1. $\square$

Theorem 3.2. *Assume that f satisfies* $(\mathbf{f}_1)$ *and* $(\mathbf{f}_2)$. *Moreover, there exist positive numbers α, β with $0 < \alpha \leq \beta < 2\pi$ and a function $\gamma \in L^2(0, 4; \mathbb{R}^+)$ such that, for every $y \in \mathbb{R}^2$ and a.e. $t \in [0, 4]$,*

$$\frac{\alpha}{2}|y|^2 - \gamma(t) \leq H(t, y) \leq \frac{\beta}{2}|y|^2 + \gamma(t).$$

Then (2.2) *has at least one solution y such that $z = Jy' - \alpha y$ minimizes the dual functional χ on* $\mathbf{Z}$.

Proof. One can easily verify that $(\mathbf{H}_2)$ and $(\mathbf{H}_3)$ hold. As for the assumption $(\mathbf{H}_1)$, we have

$$H(t, y) \geq \frac{\alpha}{2}|y|^2 - \gamma(t) \geq \alpha y - \frac{\alpha}{2} - \gamma(t).$$

The rest of the proof is similar to that of Theorem 3.1 and hence is omitted. □

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