Journal of Applied Analysis and Computation
Volume 9, Number 3, June 2019, 1165–1182
Website:http://jaac.ijournal.cn
DOI:10.11948/2156-907X.20190022

INFINITELY MANY SOLUTIONS FOR FRACTIONAL SCHRÖDINGER-MAXWELL EQUATIONS*

Jiafa Xu[1,†], Zhongli Wei[2], Donal O'Regan[3] and Yujun Cui[4]

Abstract In this paper using fountain theorems we study the existence of infinitely many solutions for fractional Schrödinger-Maxwell equations

$$\begin{cases}(-\Delta)^\alpha u+\lambda V(x)u+\phi u=f(x,u)-\mu g(x)|u|^{q-2}u, \text{ in } \mathbb{R}^3,\\ (-\Delta)^\alpha \phi=K_\alpha u^2, \text{ in } \mathbb{R}^3,\end{cases}$$

where $\lambda,\mu>0$ are two parameters, $\alpha\in(0,1]$, $K_\alpha=\frac{\pi^{-\alpha}\Gamma(\alpha)}{\pi^{-(3-2\alpha)/2}\Gamma((3-2\alpha)/2)}$ and $(-\Delta)^\alpha$ is the fractional Laplacian. Under appropriate assumptions on f and g we obtain an existence theorem for this system.

Keywords Fractional Laplacian, Schrödinger-Maxwell equations, infinitely many solutions.

MSC(2010) 35J20, 35J60.

1. Introduction

In this paper we study the fractional Schrödinger-Maxwell equations

$$\begin{cases}(-\Delta)^\alpha u+\lambda V(x)u+\phi u=f(x,u)-\mu g(x)|u|^{q-2}u, \text{ in } \mathbb{R}^3,\\ (-\Delta)^\alpha \phi=K_\alpha u^2, \text{ in } \mathbb{R}^3,\end{cases} \tag{1.1}$$

where $\lambda,\mu>0$ are two parameters, $\alpha\in(0,1]$, $K_\alpha=\frac{\pi^{-\alpha}\Gamma(\alpha)}{\pi^{-(3-2\alpha)/2}\Gamma((3-2\alpha)/2)}$ and $(-\Delta)^\alpha$ is the fractional Laplacian. Here the fractional Laplacian $(-\Delta)^\alpha$ with $\alpha\in(0,1]$ of a function $\varphi:\mathbb{R}^3\to\mathbb{R}$ is defined by $\mathcal{F}((-\Delta)^\alpha\varphi)(\xi)=|\xi|^{2\alpha}\mathcal{F}(\varphi)(\xi)$, $\forall\alpha\in(0,1]$, where $\mathcal{F}$ is the Fourier transform, i.e., $\mathcal{F}(\varphi)(\xi)=\frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}\exp\{-2\pi i\xi\cdot x\}dx$.

[†]the corresponding author.
Email address: xujiafa292@sina.com(J. Xu), jnwzl32@163.com(Z. Wei), donal.oregan@nuigalway.ie(D. O'Regan), cyj720201@163.com(Y. Cui)
[1]School of Mathematical Sciences, Chongqing Normal University, Chongqing 401331, China
[2]Department of Mathematics, Shandong Jianzhu University, Jinan 250101, China
[3]School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, Ireland
[4]State Key Laboratory of Mining Disaster Prevention and Control Co-founded by Shandong Province and the Ministry of Science and Technology, Shandong University of Science and Technology, Qingdao 266590, China
*The authors were supported by Talent Project of Chongqing Normal University (No. 02030307-0040), the National Natural Science Foundation of China(No. 11601048), Natural Science Foundation of Chongqing (No. cstc2016jcyjA0181), Natural Science Foundation of Chongqing Normal University (No. 16XYY24).

If φ is smooth enough then $(-\Delta)^\alpha$ can also be computed by the singular integral $(-\Delta)^\alpha\varphi(x) = c_{3,\alpha}\text{P.V.}\int_{\mathbb{R}^3}\frac{\varphi(x)-\varphi(y)}{|x-y|^{3+2\alpha}}dy$, where P.V. is the principal value and $c_{3,\alpha}$ is a normalization constant.

Fractional models are widely used in various fields, such as physics, signal processing, fluid mechanics, viscoelasticity, mathematical biology, and electrochemistry. For example, Bagley and Torvik [1] used fractional calculus to construct stress-strain relationships for viscoelastic materials, and they proposed a five-parameter model in the form

$$\sigma(t) + bD^\beta\sigma(t) = E_0\varepsilon(t) + E_1D^\alpha\varepsilon(t),$$

where D^α, D^β are fractional derivatives, and $\alpha, \beta, b, E_0, E_1$ are parameters. For more applications in this direction, we refer the reader to [2, 3, 5, 8, 11–14, 33, 34, 39–46] and the references therein. Fractional Schrödinger-Maxwell equations or Schrödinger-Poisson equations arise from standing waves for fractional nonlinear Schrödinger equations; for the physical background we refer the reader to [4, 9] and the references therein. For results on existence and multiplicity of solutions for Schrödinger-Poisson systems we refer the reader to [6, 7, 10, 15, 17, 19–26, 28–32, 36–38, 47–52] and the references therein. Li [19] adopted the (AR) condition to obtain an existence theorem for (1.1) when $\lambda = V = K_\alpha = 1, \mu = 0$. Teng [32] used the method of Pohozaev-Nehari manifolds, the arguments of Brezis-Nirenberg, a monotonic trick and a global compactness lemma to establish the existence of a nontrivial ground state solution when $f = \mu|u|^{q-1}u + |u|^{2^*_\alpha-2}u, 2^*_\alpha = \frac{6}{3-2\alpha}$ $(\lambda = K_\alpha = 1, \mu = 0)$. However, there are only a few papers in the literature which consider the effect of the parameter λ, μ and the perturbation term g on the existence of solutions of (1.1); see [17, 25, 28, 36, 37]. In [28], S. Secchi studied nonlinear fractional equations involving the Bessel operator

$$(I-\Delta)^\alpha u + \lambda V(x)u = f(x,u) + \mu\xi(x)|u|^{p-2}u, x \in \mathbb{R}^N,$$

where f satisfies the (AR) condition, and $\xi(x)|u|^{p-2}u$ is a sublinear perturbation term.

In our paper, in system (1.1), the functions $u, g, V : \mathbb{R}^3 \to \mathbb{R}$, $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ satisfy the following assumptions

(V) $V \in C(\mathbb{R}^3, \mathbb{R})$, and there is a positive constant $V_0 > 0$ such that $\inf_{x\in\mathbb{R}^3}\widetilde{V}(x) > 0$, $\lim_{|x|\to\infty}\widetilde{V}(x) = +\infty$, where $\widetilde{V}(x) = V(x) + V_0$, for $x \in \mathbb{R}^3$.

(H1) $\widetilde{f} \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$, and $\widetilde{f}(x,u) = o(u)$ uniformly in $x \in \mathbb{R}^3$ as $u \to 0$, where $\widetilde{f}(x,u) = f(x,u) + \lambda V_0 u$, for $(x,u) \in \mathbb{R}^3 \times \mathbb{R}$.

(H2) $\widetilde{F}(x,u) = \int_0^u \widetilde{f}(x,s)ds \geq 0$ and $\widetilde{\mathscr{F}}(x,u) = \frac{1}{4}\widetilde{f}(x,u)u - \widetilde{F}(x,u) \geq 0$, for $(x,u) \in \mathbb{R}^3 \times \mathbb{R}$.

(H3) $\lim_{|u|\to\infty}\frac{\widetilde{f}(x,u)u}{u^4} = +\infty$ uniformly in $x \in \mathbb{R}^3$.

(H4) There exist $d_1, L_1 > 0$ and $\tau \in (\frac{3}{2\alpha}, 2), \alpha > \frac{3}{4}$ such that

$$|\widetilde{f}(x,u)|^\tau \leq d_1\widetilde{\mathscr{F}}(x,u)|u|^\tau, \text{ for all } x \in \mathbb{R}^3, \text{ and } |u| \geq L_1.$$

(H5) $\widetilde{f}(x,-u) = -\widetilde{f}(x,u)$, for $(x,u) \in \mathbb{R}^3 \times \mathbb{R}$.

(g) $g \in L^{q'}(\mathbb{R}^3)$, and $g(x) \geq 0$ $(\not\equiv 0)$, for $x \in \mathbb{R}^3$, where $q' \in \left(\frac{2^*_\alpha}{2^*_\alpha - q}, \frac{2}{2-q}\right], q \in (1,2)$.

Now, we state the main result of our paper.

Theorem 1.1. *Suppose that (V), (H1)-(H5) and (g) hold. Then for any* $\mu > 0$, *there exists* $\Lambda > 0$ *such that system* (1.1) *possesses infinitely many solutions when* $\lambda \geq \Lambda$.

Remark 1.1. Condition (H4) (see [6,20,38]) is weaker than the (AR) condition (H4)$'$ there exists $\vartheta > 4$ such that $0 < \vartheta \widetilde{F}(x, u) \leq \widetilde{f}(x, u)u$ for all $(x, u) \in \mathbb{R}^3 \times \mathbb{R}$ with $u \neq 0$.

Note if $\widetilde{f}(x, t) = e^{|x|}t^3[2\ln(1 + t^2) + \frac{t^2}{1+t^2}]$ for all $(x, t) \in \mathbb{R}^3 \times \mathbb{R}$ then (H4) is satisfied but (H4)$'$ is not. Moreover, from the proof of [28, Lemma 2.3] note (H1) and (H4)$'$ imply (H4).

Finally we note that in [17,25,28,36,37] the authors used the (AR) condition (not (H4)) to discuss the effect of parameters and perturbation terms on the existence of solutions for their problem.

Remark 1.2. If the potential function V satisfies condition (V), then the following automatically holds:
(V1) $V \in C(\mathbb{R}^3, \mathbb{R})$, and V is bounded from below, i.e., there exists a positive constant $V_0 > 0$ such that $V(x) + V_0 > 0$ for all $x \in \mathbb{R}^3$.
(V2) There exists $b > 0$ such that $\text{meas}\{x \in \mathbb{R}^3 : \widetilde{V}(x) \leq b\}$ is finite; here meas denotes the Lebesgue measure.

2. Variational settings and preliminary results

For any $1 \leq r < \infty, L^r(\mathbb{R}^3)$ is the usual Lebesgue space with the norm

$$\|u\|_r = \left(\int_{\mathbb{R}^3} |u(x)|^r dx\right)^{\frac{1}{r}}.$$

The fractional order Sobolev space

$$H^\alpha(\mathbb{R}^3) = \left\{u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} (|\xi|^{2\alpha}\hat{u}^2 + \hat{u}^2)d\xi < \infty\right\},$$

where $\hat{u} = \mathcal{F}(u)$, and the norm is defined by

$$\|u\|_{H^\alpha(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} (|\xi|^{2\alpha}\hat{u}^2 + \hat{u})d\xi\right)^{\frac{1}{2}}.$$

The space $D^\alpha(\mathbb{R}^3)$ is defined as the completion of $C_0^\infty(\mathbb{R}^3)$ under the norms

$$\|u\|_{D^\alpha(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} (|\xi|^{2\alpha}\hat{u}^2)d\xi\right)^{\frac{1}{2}} = \left(\int_{\mathbb{R}^3} |(-\Delta)^{\alpha/2}u(x)|^2 dx\right)^{\frac{1}{2}}.$$

Note that, from Plancherel's theorem we have $\|u\|_2 = \|\hat{u}\|_2$, and

$$\begin{aligned}\int_{\mathbb{R}^3} |(-\Delta)^{\alpha/2}u(x)|^2 dx &= \int_{\mathbb{R}^3} ((\widehat{-\Delta)^{\alpha/2}u}(\xi))^2 d\xi = \int_{\mathbb{R}^3} (|\xi|^\alpha \hat{u}(\xi))^2 d\xi \\ &= \int_{\mathbb{R}^3} |\xi|^{2\alpha}\hat{u}^2 d\xi < \infty, \forall u \in H^\alpha(\mathbb{R}^3).\end{aligned}$$

It follows that

$$\|u\|_{H^\alpha(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} (|(-\Delta)^{\frac{\alpha}{2}}u(x)|^2 + u^2)dx\right)^{\frac{1}{2}}.$$

In our problem we work with the space

$$E := \left\{ u \in H^{\alpha}(\mathbb{R}^3) : \left(\int_{\mathbb{R}^3} (|(-\Delta)^{\frac{\alpha}{2}} u(x)|^2 + \lambda \widetilde{V}(x) u^2) dx \right)^{\frac{1}{2}} < \infty \right\}. \tag{2.1}$$

Now E is a Hilbert space with the inner product

$$(u, v) := \int_{\mathbb{R}^3} ((-\Delta)^{\frac{\alpha}{2}} u(x) \cdot (-\Delta)^{\frac{\alpha}{2}} v(x) + \lambda \widetilde{V}(x) uv) dx.$$

and its norm is $\|u\| = \sqrt{(u, u)}$.

Lemma 2.1 (see [7, 10]). *$H^{\alpha}(\mathbb{R}^3)$ is continuously embedded into $L^p(\mathbb{R}^3)$ for $p \in [2, 2^*_\alpha]$; and compactly embedded into $L^p_{loc}(\mathbb{R}^3)$ for $p \in [2, 2^*_\alpha)$ where $2^*_\alpha = \frac{6}{3-2\alpha}$. Therefore, there exists a positive constant C_P such that*

$$\|u\|_p \le C_p \|u\|_{H^{\alpha}(\mathbb{R}^3)}. \tag{2.2}$$

Lemma 2.2 (see [29]). *Under the assumptions (V), the space E is compactly embedded into $L^p(\mathbb{R}^3)$ for $p \in [2, 2^*_\alpha)$.*

Lemma 2.3 (see [16]). *For $1 < p < \infty$ and $0 < \alpha < N/p$, we have*

$$\|u\|_{L^{\frac{pN}{N-p\alpha}}(\mathbb{R}^N)} \le B \|(-\Delta)^{\alpha/2} u\|_{L^p(\mathbb{R}^N)}, \tag{2.3}$$

with best constant

$$B = 2^{-\alpha} \pi^{-\alpha/2} \frac{\Gamma((N-\alpha)/2)}{\Gamma((N+\alpha)/2)} \left(\frac{\Gamma(N)}{\Gamma(N/2)} \right)^{\alpha/N}.$$

Lemma 2.4. *For any $u \in H^{\alpha}(\mathbb{R}^N)$ and for any $h \in D^{-\alpha}(\mathbb{R}^N)$, there exists a unique solution $\phi = ((-\Delta)^{\alpha} + u^2)^{-1} h \in D^{\alpha}(\mathbb{R}^N)$ of the equation*

$$(-\Delta)^{\alpha} \phi + u^2 \phi = h,$$

(here $D^{-\alpha}(\mathbb{R}^N)$ is the dual space of $D^{\alpha}(\mathbb{R}^N)$). Moreover, for every $u \in H^{\alpha}(\mathbb{R}^N)$ and for every $h, g \in D^{-\alpha}(\mathbb{R}^N)$, we have

$$\langle h, ((-\Delta)^{\alpha} + u^2)^{-1} g \rangle = \langle g, ((-\Delta)^{\alpha} + u^2)^{-1} h \rangle, \tag{2.4}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $D^{-\alpha}(\mathbb{R}^N)$ and $D^{\alpha}(\mathbb{R}^N)$.

Proof. If $u \in H^{\alpha}(\mathbb{R}^N)$, then by the Hölder inequality and (2.3), we have

$$\int_{\mathbb{R}^N} u^2 \phi^2 dx \le \|u\|^2_{2p} \|\phi\|^2_{2q} \le B^2 \|u\|^2_{2p} \|\phi\|^2_{D^{\alpha}}, \tag{2.5}$$

where $\frac{1}{p} + \frac{1}{q} = 1, q = \frac{N}{N-2\alpha}, 2q = 2^*_\alpha$. Thus $(\int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2} \phi|^2 dx + \int_{\mathbb{R}^N} u^2 \phi^2 dx)^{1/2}$ is a norm in $D^{\alpha}(\mathbb{R}^N)$ equivalent to $\|\phi\|_{D^{\alpha}}$. Hence, by applying the Lax-Milgram Lemma, we establish the existence part. For every $u \in H^{\alpha}(\mathbb{R}^N)$ and for every

$h, g \in D^{-\alpha}(\mathbb{R}^N)$, we have $\phi_g = ((-\Delta)^\alpha + u^2)^{-1}g, \phi_h = ((-\Delta)^\alpha + u^2)^{-1}h$. Hence,

$$\begin{aligned}\langle h, ((-\Delta)^\alpha + u^2)^{-1}g\rangle &= \int_{\mathbb{R}^N} h((-\Delta)^\alpha + u^2)^{-1}g dx = \int_{\mathbb{R}^N} h\phi_g dx \\ &= \int_{\mathbb{R}^N} ((-\Delta)^\alpha + u^2)\phi_h\phi_g dx = \int_{\mathbb{R}^N} ((-\Delta)^\alpha\phi_h + u^2\phi_h)\phi_g dx \\ &= \int_{\mathbb{R}^N} ((-\Delta)^\alpha\phi_g + u^2\phi_g)\phi_h dx = \int_{\mathbb{R}^N} g\phi_h dx \\ &= \int_{\mathbb{R}^N} g((-\Delta)^\alpha + u^2)^{-1}h dx = \langle g, ((-\Delta)^\alpha + u^2)^{-1}h\rangle,\end{aligned}$$

so we have (2.4). □

Lemma 2.5 (see [18])**.** *Let f be a function in $C_0^\infty(\mathbb{R}^N)$. Then for $\alpha \in (0, n)$, we have*

$$\mathcal{C}_\alpha \doteq \pi^{-\alpha/2}\Gamma(-\alpha/2), \tag{2.6}$$

$$\mathcal{C}_\alpha(\xi^{-\alpha}\hat{f}(\xi))^\vee(x) = \mathcal{C}_{n-\alpha}\int_{\mathbb{R}^n} |x-y|^{\alpha-n} f(y)dy. \tag{2.7}$$

Lemma 2.6. *For every $u \in H^\alpha$ there exists a unique $\phi = \phi(u) \in D^\alpha$ which solves the second equation in* (1.1)*. Furthermore, $\phi(u)$ is given by*

$$\phi(u)(x) = \int_{\mathbb{R}^3} |x-y|^{2\alpha-3}u^2(y)dy. \tag{2.8}$$

As a consequence, the map $\Phi : u \in H^\alpha \longmapsto \phi(u) \in D^\alpha$ is of class C^1 and

$$[\Phi(u)]'(v)(x) = 2\int_{\mathbb{R}^3} |x-y|^{2\alpha-3}u(y)v(y)dy, \forall u, v \in H^\alpha. \tag{2.9}$$

Proof. The existence and uniqueness part follows from Lemma 2.4. From Lemma 2.5 and the Fourier transform of the second equation in (1.1), the representation formula (2.8) holds for $u \in C_0^\infty(\mathbb{R}^3)$; by density it can be extended for any $u \in H^\alpha$. The representation formula (2.9) is clear. □

System (1.1) is the Euler-Lagrange equations corresponding to the functional $J : H^\alpha(\mathbb{R}^3) \times D^\alpha(\mathbb{R}^3) \to \mathbb{R}$:

$$\begin{aligned}J(u,\phi) = &\frac{1}{2}\int_{\mathbb{R}^3}\left(|(-\Delta)^{\frac{\alpha}{2}}u(x)|^2 + \lambda\widetilde{V}(x)u^2 - \frac{1}{2}|(-\Delta)^{\frac{\alpha}{2}}\phi(x)|^2 + K_\alpha\phi u^2\right)dx \\ &- \int_{\mathbb{R}^3}\widetilde{F}(x,u)dx + \frac{\mu}{q}\int_{\mathbb{R}^3} g(x)|u|^q dx,\end{aligned}$$

where $\widetilde{F}(x,t) = \int_0^t \widetilde{f}(x,s)ds, t \in \mathbb{R}$.

Evidently, the action functional J belongs to $C^1(H^\alpha(\mathbb{R}^3) \times D^\alpha(\mathbb{R}^3), \mathbb{R})$ and the partial derivatives in (u,ϕ) are given, for $\xi \in H^\alpha(\mathbb{R}^3)$ and $\eta \in D^\alpha(\mathbb{R}^3)$, by

$$\begin{aligned}\left\langle \frac{\partial J}{\partial u}(u,\phi), \xi\right\rangle = &\int_{\mathbb{R}^3}((-\Delta)^{\frac{\alpha}{2}}u(x)(-\Delta)^{\frac{\alpha}{2}}\xi(x) + \lambda\widetilde{V}(x)u\xi + K_\alpha\phi u\xi)dx \\ &- \int_{\mathbb{R}^3}\widetilde{f}(x,u)\xi(x)dx + \mu\int_{\mathbb{R}^3} g(x)|u|^{q-2}u\xi(x)dx,\end{aligned}$$

$$\left\langle \frac{\partial J}{\partial \phi}(u,\phi), \eta\right\rangle = \frac{1}{2}\int_{\mathbb{R}^3}(-(-\Delta)^{\frac{\alpha}{2}}\phi(x)(-\Delta)^{\frac{\alpha}{2}}\eta(x) + K_\alpha u^2\eta)dx.$$

Thus, we have the following result:

Proposition 2.1. *The pair* (u, ϕ) *is a weak solution of system* (1.1) *if and only if it is a critical point of* J *in* $H^\alpha(\mathbb{R}^3) \times D^\alpha(\mathbb{R}^3)$.

We can consider the functional $J : H^\alpha(\mathbb{R}^3) \to \mathbb{R}$ defined by $J(u) = J(u, \phi(u))$. After multiplying the second equation in (1.1) by $\phi(u)$ and integrating by parts, we obtain

$$\int_{\mathbb{R}^3} |(-\Delta)^{\alpha/2}\phi(u)|^2 dx = K_\alpha \int_{\mathbb{R}^3} \phi(x)u^2 dx.$$

Therefore, the reduced functional takes the form

$$\begin{aligned} J(u) = & \frac{1}{2}\int_{\mathbb{R}^3} (|(-\Delta)^{\frac{\alpha}{2}} u(x)|^2 + \lambda \widetilde{V}(x)u^2)dx + \frac{1}{4}K_\alpha \int_{\mathbb{R}^3} u^2\phi(u)dx \\ & - \int_{\mathbb{R}^3} \widetilde{F}(x,u)dx + \frac{\mu}{q}\int_{\mathbb{R}^3} g(x)|u|^q dx. \end{aligned} \tag{2.10}$$

Lemma 2.7. *Assume that there exist* $c_1, c_2 > 0$ *and* $p \in (4, 2^*_\alpha)$ *such that*

$$|\widetilde{f}(x,s)| \le c_1|s| + c_2|s|^{p-1}, \forall x \in \mathbb{R}^3, s \in \mathbb{R}. \tag{2.11}$$

Then the following statements are equivalent:

(i) $(u, \phi) \in (H^\alpha \cap L^p) \times D^\alpha$ *is a solution of the system (1.1)-(1.2);*

(ii) $u \in H^\alpha \cap L^p$ *is a critical point of* J *and* $\phi = \phi(u)$.

Proof. From assumption (2.11), the Nemitsky operator $u \in H^\alpha \cap L^p \mapsto F(x,u) \in L^1$ is of class C^1. Hence, from Lemma 2.6, for every $u, v \in H^\alpha$, we have

$$\begin{aligned} < J'(u), v > = & \int_{\mathbb{R}^3} (-\Delta)^{\frac{\alpha}{2}} u(x)(-\Delta)^{\frac{\alpha}{2}} v(x)dx + \int_{\mathbb{R}^3} \lambda\widetilde{V}(x)uvdx \\ & + \frac{1}{2}K_\alpha \int_{\mathbb{R}^3} uv \int_{\mathbb{R}^3} |x-y|^{2\alpha-3}u^2(y)dydx \\ & + \frac{1}{2}K_\alpha \int_{\mathbb{R}^3} u^2 \int_{\mathbb{R}^3} |x-y|^{2\alpha-3}u(y)v(y)dydx \\ & - \int_{\mathbb{R}^3} \widetilde{f}(x,u)vdx + \mu\int_{\mathbb{R}^3} g(x)|u|^{q-2}uvdx \\ = & \int_{\mathbb{R}^3} (-\Delta)^{\frac{\alpha}{2}} u(x)(-\Delta)^{\frac{\alpha}{2}} v(x)dx + \int_{\mathbb{R}^3} \lambda\widetilde{V}(x)uvdx \\ & + K_\alpha \int_{\mathbb{R}^3} uv\phi(u)dx - \int_{\mathbb{R}^3} \widetilde{f}(x,u)vdx + \mu\int_{\mathbb{R}^3} g(x)|u|^{q-2}uvdx. \end{aligned}$$

From the Fubini-Tonelli Theorem we obtain the conclusion. □

Remark 2.1. Conditions (H1), (H2), (H4) imply (2.11). From (H2) and (H4), we have

$$|\widetilde{f}(x,u)|^\tau \le d_1\left(\frac{1}{4}\widetilde{f}(x,u)u - \widetilde{F}(x,u)\right)|u|^\tau \le \frac{d_1}{4}|\widetilde{f}(x,u)||u|^{\tau+1},$$

i.e., $|\widetilde{f}(x,u)|^{\tau-1} \le \frac{d_1}{4}|u|^{\tau+1}$ for large u with $\frac{2\tau}{\tau-1} \in (4, 2^*_\alpha)$. Combining this with (H1), there exist $c_1, c_2 > 0$ such that (2.11) holds. This also implies that

$$|\widetilde{F}(x,s)| \le \frac{c_1}{2}|s|^2 + \frac{c_2}{p}|s|^p, \forall x \in \mathbb{R}^3, s \in \mathbb{R}. \tag{2.12}$$

If $1 \le p < \infty$ and $a, b \ge 0$, then

$$(a+b)^p \le 2^{p-1}(a^p + b^p). \tag{2.13}$$

From (1.1), for any $u \in E$ using the Hölder inequality we have

$$\|\phi(u)\|_{D^\alpha}^2 = K_\alpha \int_{\mathbb{R}^3} \phi(u)u^2 dx \le K_\alpha \|\phi(u)\|_q \|u\|_{2p}^2 \le C\|\phi(u)\|_{D^\alpha}\|u\|_{2p}^2,$$

where $\frac{1}{p} + \frac{1}{q} = 1, q = 2_\alpha^* = \frac{6}{3-2\alpha}, \alpha > \frac{3}{4}$. Here and subsequently, C denotes an universal positive constant. This and Lemma 2.2 imply that

$$\|\phi(u)\|_{D^\alpha} \le C\|u\|_{2p}^2 \le C\|u\|_E^2, \tag{2.14}$$

$$\int_{\mathbb{R}^3} \phi(u)u^2 dx \le C\|u\|_{2p}^4 \le C\|u\|_E^4. \tag{2.15}$$

Lemma 2.8. *Suppose that $u_n \rightharpoonup u$ in E, $u_n(x) \to u(x)$ for a.e. $x \in \mathbb{R}^3$. Then we have*

$$\int_{\mathbb{R}^3} \phi(u_n)u_n^2 - \phi(u)u^2 dx = o(1), \ \text{as } n \to \infty, \tag{2.16}$$

and

$$\int_{\mathbb{R}^3} (\phi(u_n)u_n - \phi(u)u)v dx = o(1), \ \text{as } n \to \infty, \forall v \in E. \tag{2.17}$$

Proof. Now $u_n \to u$ in $L^r(\mathbb{R}^3)$ with $r \in [2, 2_\alpha^*)$ after passing to a subsequence. From [32, Lemma 2.3] and [47, Lemma 2.4], we have

(i) $T(u_n) = T(u) + T(u_n - u) + o(1)$ as $n \to \infty$, where $T(u) = \int_{\mathbb{R}^3} \phi(u)u^2 dx$.

(ii) if $u_n \rightharpoonup u$ in E, then $\phi(u_n) \rightharpoonup \phi(u)$ in $D^\alpha(\mathbb{R}^3)$.

From (i), by (2.15) for $p = \frac{6}{3+2\alpha}$ we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} \phi(u_n)u_n^2 - \phi(u)u^2 dx = \lim_{n\to\infty} \int_{\mathbb{R}^3} \phi(u_n - u)(u_n - u)^2 dx \le C\|u_n - u\|_{2p}^4 \to 0. \tag{2.18}$$

From (ii) we have $\phi(u_n) \to \phi(u)$ in $L^{\frac{12}{3+2\alpha}}(\mathbb{R}^3)$. Therefore, from (2.2) we have

$$\begin{aligned} &\int_{\mathbb{R}^3} u_n v(\phi(u_n) - \phi(u))dx \\ &\le \left(\int_{\mathbb{R}^3} |u_n|^{\frac{12}{3+2\alpha}} dx\right)^{\frac{3+2\alpha}{12}} \left(\int_{\mathbb{R}^3} |v|^{\frac{6}{3-2\alpha}} dx\right)^{\frac{3-2\alpha}{6}} \left(\int_{\mathbb{R}^3} |\phi(u_n) - \phi(u)|^{\frac{12}{3+2\alpha}} dx\right)^{\frac{3+2\alpha}{12}} \\ &\le C_{\frac{12}{3+2\alpha}} C_{\frac{6}{3-2\alpha}} \|u_n\|\|v\| \left(\int_{\mathbb{R}^3} |\phi(u_n) - \phi(u)|^{\frac{12}{3+2\alpha}} dx\right)^{\frac{3+2\alpha}{12}} \\ &\to 0, \ \text{as } n \to \infty, \ \text{for } u_n, u, v \in E. \end{aligned} \tag{2.19}$$

Note (2.5) and we obtain

$$\begin{aligned} \int_{\mathbb{R}^3} (u_n - u)v\phi(u)dx &\le \left(\int_{\mathbb{R}^3} |u_n - u|^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} v^2\phi^2(u)dx\right)^{\frac{1}{2}} \\ &\le B\|v\|_{\frac{12}{3+2\alpha}} \|\phi\|_{D^\alpha} \left(\int_{\mathbb{R}^3} |u_n - u|^2 dx\right)^{\frac{1}{2}} \\ &\to 0, \ \text{as } n \to \infty, \ \text{for } u_n, u, v \in E. \end{aligned}$$

Therefore, when $n \to \infty$ we have

$$\int_{\mathbb{R}^3} (\phi(u_n)u_n - \phi(u)u)v dx = \int_{\mathbb{R}^3} (\phi(u_n)u_n - \phi(u)u_n + \phi(u)u_n - \phi(u)u)v dx = o(1).$$

□

Next we introduce the Fountain theorem under the condition (C), which is weaker than the (PS) condition.

Definition 2.1 (see [27]). Assume that X is a Banach space. We say that $J \in C^1(X, \mathbb{R})$ satisfies the Cerami condition (C), if for all $c \in \mathbb{R}$:

(i) any bounded sequence $\{u_n\} \subset X$ satisfying $J(u_n) \to c, J'(u_n) \to 0$ possesses a convergent subsequence;

(ii) there exist $\sigma, R, \beta > 0$ such that for any $u \in J^{-1}([c-\sigma, c+\sigma])$ with $\|u\| \geq R, \|J'(u)\|\|u\| \geq \beta$.

Lemma 2.9 (see [27]). *Assume that* $X = \overline{\bigoplus_{j=1}^{\infty} X_j}$, *where* X_j *are finite dimensional subspaces of* X. *For each* $k \in \mathbb{N}$, *let* $Y_k = \bigoplus_{j=1}^{k} X_j$, $Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}$. *Suppose that* $J \in C^1(X, \mathbb{R})$ *satisfies condition (C), and* $J(-u) = J(u)$. *Assume for each* $k \in \mathbb{N}$, *there exist* $\rho_k > r_k > 0$ *such that*

(i) $b_k = \inf_{u \in Z_k \cap S_{r_k}} J(u) \to +\infty, k \to \infty$,

(ii) $a_k = \max_{u \in Y_k \cap S_{\rho_k}} J(u) \leq 0$, *where* $S_\rho = \{u \in X : \|u\| = \rho\}$.

Then J *has a sequence of critical points* u_n, *such that* $J(u_n) \to +\infty$ *as* $n \to \infty$.

3. Proof of Theorem 1.1

We first prove that the energy functional J satisfies condition (C) in Definition 2.1.

Lemma 3.1. *Suppose the assumptions in Theorem 1.1 hold (with* $\Lambda > 0$ *chosen appropriately). Then* J *satisfies condition (C).*

Proof. For every $c \in \mathbb{R}$, we assume that $\{u_n\}_{n \in \mathbb{N}} \subset E$ is bounded and

$$J(u_n) \to c, \ J'(u_n) \to 0, \text{ as } n \to \infty.$$

Therefore, passing to a subsequence if necessary, there exists $u \in E$ such that

$$\begin{cases} u_n \rightharpoonup u \text{ weakly in } E, \\ u_n \to u \text{ strongly in } L^p(\mathbb{R}^3) \text{ for } p \in [2, 2_\alpha^*), \\ u_n \to u \text{ for a.e. } x \in \mathbb{R}^3. \end{cases} \tag{3.1}$$

We now show that

$$J(u_n - u) = c - J(u) + o(1), \quad < J'(u_n - u), v >= o(1), \text{ as } n \to \infty, \forall v \in E. \tag{3.2}$$

Let $w_n = u_n - u$. Then $w_n \rightharpoonup 0$ in E, $w_n \to 0$ in $L^r(\mathbb{R}^3)$ with $r \in [2, 2_\alpha^*)$ and $w_n \to 0$ for a.e. $x \in \mathbb{R}^3$ after passing to a subsequence. Since $u_n \rightharpoonup u$ in E, we have $(u_n - u, u) \to 0$ as $n \to \infty$, which implies

$$\|u_n\|^2 = (w_n + u, w_n + u) = \|w_n\|^2 + \|u\|^2 + o(1), \text{ as } n \to \infty.$$

Note (2.18) and (2.19), and we easily see that

$$\int_{\mathbb{R}^3} \phi(u_n - u)(u_n - u)^2 dx \to 0, \text{ as } n \to \infty,$$

and

$$\int_{\mathbb{R}^3} \phi(u_n - u)(u_n - u) v dx \to 0, \text{ as } n \to \infty, \forall v \in E.$$

To prove (3.2), it will be enough to show that as $n \to \infty$,

$$\int_{\mathbb{R}^3} (\widetilde{F}(x, u_n) - \widetilde{F}(x, u_n - u) - \widetilde{F}(x, u)) dx = o(1), \tag{3.3}$$

$$\int_{\mathbb{R}^3} g(x)(|u_n|^q - |u_n - u|^q - |u|^q) dx = o(1), \tag{3.4}$$

$$\int_{\mathbb{R}^3} (\widetilde{f}(x, u_n) - \widetilde{f}(x, u_n - u) - \widetilde{f}(x, u)) v dx = o(1), \tag{3.5}$$

and

$$\int_{\mathbb{R}^3} g(x)(|u_n|^{q-2} u_n - |u_n - u|^{q-2}(u_n - u) - |u|^{q-2} u) v dx = o(1), \tag{3.6}$$

for all $v \in E$.

We only prove (3.3) and (3.4) (the proofs of (3.5) and (3.6) are similar). Note that $u_n \rightharpoonup u$ in E, and from Lemma A.1 of [35], there exists $\sigma(x) \in L^r(\mathbb{R}^3)$ with $r \in [2, 2^*_\alpha)$ such that

$$|u_n(x)| \leq \sigma(x), \quad |u(x)| \leq \sigma(x), \text{ for } x \in \mathbb{R}^3, n \in \mathbb{N}. \tag{3.7}$$

From this and (2.12), for $\sigma_1 \in L^2(\mathbb{R}^3), \sigma_2 \in L^p(\mathbb{R}^3)$ with $p \in (4, 2^*_\alpha)$, we have

$$\begin{aligned} |\widetilde{F}(x, u_n) - \widetilde{F}(x, u)| &\leq \frac{c_1}{2}(|u_n|^2 + |u|^2) + \frac{c_2}{p}(|u_n|^p + |u|^p) \\ &\leq c_1 \sigma_1^2(x) + \frac{2c_2}{p} \sigma_2^p(x) \in L^1(\mathbb{R}^3). \end{aligned}$$

Hence, the Lebesgue dominated convergence theorem yields

$$\begin{aligned} &\lim_{n\to\infty} \left| \int_{\mathbb{R}^3} (\widetilde{F}(x, u_n) - \widetilde{F}(x, u)) dx \right| \leq \lim_{n\to\infty} \int_{\mathbb{R}^3} \left| (\widetilde{F}(x, u_n) - \widetilde{F}(x, u)) \right| dx \\ &= \int_{\mathbb{R}^3} \lim_{n\to\infty} \left| (\widetilde{F}(x, u_n) - \widetilde{F}(x, u)) \right| dx \to 0. \end{aligned}$$

On the other hand, from (2.12) and the Hölder inequality, for $p \in (4, 2^*_\alpha)$ we have

$$\int_{\mathbb{R}^3} \widetilde{F}(x, u_n - u) dx \leq \int_{\mathbb{R}^3} (\frac{c_1}{2}|w_n|^2 + \frac{c_2}{p}|w_n|^p) dx \to 0, \text{ as } n \to \infty.$$

This proves (3.3). From (g) and the Hölder inequality, for $\frac{qq'}{q'-1} \in [2, 2^*_\alpha)$ we have

$$\int_{\mathbb{R}^3} g(x)|u_n - u|^q dx \leq \|g\|_{q'} \left(\int_{\mathbb{R}^3} |u_n - u|^{\frac{qq'}{q'-1}} dx \right)^{\frac{q'-1}{q'}} \to 0, \text{ as } n \to \infty.$$

Since

$$\left|\int_{\mathbb{R}^3} g(x)(|u_n|^q - |u|^q)dx\right| \le \int_{\mathbb{R}^3} g(x)|u_n - u|^q dx,$$

the proof of (3.4) is complete.

Recall $w_n = u_n - u$. From (2.11), (2.12) and (3.7) we have

$$\begin{aligned}\int_{\mathbb{R}^3} \widetilde{\mathscr{F}}(x,w_n)dx &= \int_{\mathbb{R}^3}\left(\frac{1}{4}\widetilde{f}(x,w_n)w_n - \widetilde{F}(x,w_n)\right)dx\\ &\le \int_{\mathbb{R}^3}(\frac{3}{4}c_1|w_n|^2 + \frac{p+4}{4p}c_2|w_n|^p)dx\\ &\le \int_{\mathbb{R}^3}(3c_1\sigma_1^2(x) + \frac{p+4}{p}2^{p-2}c_2\sigma_2^p(x))dx\\ &\le \widetilde{M},\end{aligned}$$

where $\widetilde{M} > 0$.

As $\widetilde{V}(x) < b$ on a set of finite measure (see Remark 1.2) and $w_n \rightharpoonup 0$ in E, from (2.2) we have

$$\|w_n\|_2^2 = \int_{\mathbb{R}^3}|w_n|^2dx \le \frac{1}{\lambda b}\int_{\widetilde{V}\ge b}\lambda\widetilde{V}(x)|w_n|^2dx + \int_{\widetilde{V}<b}|w_n|^2dx \le \frac{1}{\lambda b}\|w_n\|^2 + o(1).$$

From this and the Hölder inequality, for $s = \frac{2\tau}{\tau-1} \in (4, 2_\alpha^*)$, fixed $\nu \in (s, 2_\alpha^*)$ we have

$$\begin{aligned}\|w_n\|_s^s &= \int_{\mathbb{R}^3}|w_n|^s dx = \int_{\mathbb{R}^3}|w_n|^{\frac{2(\nu-s)}{\nu-2}}|w_n|^{s-\frac{2(\nu-s)}{\nu-2}}dx\\ &\le \left(\int_{\mathbb{R}^3}|w_n|^{\frac{2(\nu-s)}{\nu-2}\frac{\nu-2}{\nu-s}}dx\right)^{\frac{\nu-s}{\nu-2}}\left(\int_{\mathbb{R}^3}|w_n|^{(s-\frac{2(\nu-s)}{\nu-2})\frac{\nu-2}{s-2}}dx\right)^{\frac{s-2}{\nu-2}}\\ &= \left(\int_{\mathbb{R}^3}|w_n|^2dx\right)^{\frac{\nu-s}{\nu-2}}\left(\int_{\mathbb{R}^3}|w_n|^\nu dx\right)^{\frac{s-2}{\nu-2}}\\ &\le \left(\frac{1}{\lambda b}\right)^{\frac{\nu-s}{\nu-2}} C_\nu^{\frac{\nu(s-2)}{\nu-2}}\|w_n\|^{\frac{2(\nu-s)}{\nu-2}}\|w_n\|^{\frac{\nu(s-2)}{\nu-2}}\\ &= \left(\frac{1}{\lambda b}\right)^{\frac{\nu-s}{\nu-2}} C_\nu^{\frac{\nu(s-2)}{\nu-2}}\|w_n\|^s, \text{ for } C_\nu > 0.\end{aligned}$$

From (H1), for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $|\widetilde{f}(x,u)| \le \varepsilon|u|$ for $x \in \mathbb{R}^3$ and $|u| \le \delta$. Moreover, (H4) is also satisfied for $|u| \ge \delta$. Therefore, we have

$$\int_{|w_n|\le\delta}\widetilde{f}(x,w_n)w_ndx \le \varepsilon\int_{|w_n|\le\delta}|w_n|^2dx \le \frac{\varepsilon}{\lambda b}\|w_n\|^2 + o(1),$$

and

$$\begin{aligned}\int_{|w_n|\ge\delta}\widetilde{f}(x,w_n)w_ndx &= \int_{|w_n|\ge\delta}\frac{\widetilde{f}(x,w_n)}{w_n}w_n^2dx\\ &\le \left(\int_{|w_n|\ge\delta}\left|\frac{\widetilde{f}(x,w_n)}{w_n}\right|^\tau dx\right)^{1/\tau}\left(\int_{|w_n|\ge\delta}|w_n|^{\frac{2\tau}{\tau-1}}dx\right)^{(\tau-1)/\tau}\end{aligned}$$

$$\leq \left(\int_{|w_n|\geq \delta} d_1 \widetilde{\mathscr{F}}(x,u)dx \right)^{1/\tau} \|w_n\|_s^2$$
$$\leq (d_1 \widetilde{M})^{1/\tau} \left(\frac{1}{\lambda b}\right)^{\frac{2(\nu-s)}{s(\nu-2)}} C_\nu^{\frac{2\nu(s-2)}{s(\nu-2)}} \|w_n\|^2 + o(1).$$

Consequently, from (3.2) we obtain

$$\begin{aligned} o(1) &=< J'(w_n), w_n > \\ &= \|w_n\|^2 + K_\alpha \int_{\mathbb{R}^3} w_n^2 \phi(w_n)dx - \int_{\mathbb{R}^3} \widetilde{f}(x,w_n)w_n dx + \mu \int_{\mathbb{R}^3} g(x)|w_n|^q dx \\ &\geq \left[1 - \frac{\varepsilon}{\lambda b} - (d_1\widetilde{M})^{1/\tau} \left(\frac{1}{\lambda b}\right)^{\frac{2(\nu-s)}{s(\nu-2)}} C_\nu^{\frac{2\nu(s-2)}{s(\nu-2)}} \right] \|w_n\|^2 + o(1). \end{aligned}$$

Thus there exists $\Lambda > 0$ such that $w_n \to 0$ in E when $\lambda > \Lambda$. This implies that $u_n \to u$ in E, and Definition 2.1 (i) holds.

Next, we prove condition in Definition 2.1 (ii) holds. If not, there exist $c \in \mathbb{R}$ and $\{u_n\}_{n\in\mathbb{N}} \subset E$ satisfying

$$J(u_n) \to c,\ \|u_n\| \to \infty,\ \|J'(u_n)\|\|u_n\| \to 0,\ \text{as } n \to \infty. \tag{3.8}$$

Then we have

$$\begin{aligned} c + o(1) &= J(u_n) - \frac{1}{4} < J'(u_n), u_n > \\ &= \frac{1}{2}\|u_n\|^2 + \int_{\mathbb{R}^3} \widetilde{\mathscr{F}}(x,u_n)dx + \left(\frac{\mu}{q} - \frac{\mu}{4}\right) \int_{\mathbb{R}^3} g(x)|u_n|^q dx \\ &\geq \int_{\mathbb{R}^3} \widetilde{\mathscr{F}}(x,u_n)dx. \end{aligned} \tag{3.9}$$

In view of the definition of J', (3.8), (2.2) and (g) we obtain

$$\begin{aligned} 1 &= \frac{\|u_n\|^2}{\|u_n\|^2} \\ &= \frac{< J'(u_n), u_n >}{\|u_n\|^2} - \frac{K_\alpha \int_{\mathbb{R}^3} u_n^2 \phi(u_n)dx}{\|u_n\|^2} + \frac{\int_{\mathbb{R}^3} \widetilde{f}(x,u_n)u_n dx}{\|u_n\|^2} - \mu \frac{\int_{\mathbb{R}^3} g(x)|u_n|^q dx}{\|u_n\|^2} \\ &\leq \limsup_{n\to\infty} \left[\frac{< J'(u_n), u_n >}{\|u_n\|^2} + \frac{\int_{\mathbb{R}^3} \widetilde{f}(x,u_n)u_n dx}{\|u_n\|^2} + \mu \frac{\|g\|_{q'} C^q_{\frac{qq'}{q'-1}} \|u_n\|^q}{\|u_n\|^2} \right] \\ &\leq \limsup_{n\to\infty} \frac{\int_{\mathbb{R}^3} \widetilde{f}(x,u_n)u_n dx}{\|u_n\|^2}. \end{aligned} \tag{3.10}$$

Define $v_n = \frac{u_n}{\|u_n\|}$, and note $\|v_n\| = 1$. Passing to a subsequence, there exists a $v \in E$ such that $v_n \rightharpoonup v$ weakly in E, $v_n \to v$ strongly in $L^r(\mathbb{R}^3)$ with $r \in [2, 2^*_\alpha)$, $v_n(x) \to v(x)$ for a.e. $x \in \mathbb{R}^3$. For $0 \leq a < b$, let $\Omega_n(a,b) = \{x \in \mathbb{R}^3 : a \leq |u_n(x)| < b\}$. Now we consider the following two cases.

Case 1: Suppose $v = 0$.

Then $v_n \to 0$ in $L^r(\mathbb{R}^3)$ with $r \in [2, 2^*_\alpha)$, $v_n(x) \to 0$ for a.e. $x \in \mathbb{R}^3$. Let L_1 be as in (H4), and from (2.11) we have

$$\begin{aligned}\int_{\Omega_n(0,L_1)} \frac{\widetilde{f}(x,u_n)u_n}{\|u_n\|^2}\mathrm{d}x &= \int_{\Omega_n(0,L_1)} \frac{\widetilde{f}(x,u_n)u_n}{|u_n|^2}|v_n|^2\mathrm{d}x \\ &\le (c_1 + c_2L_1^{p-2})\int_{\Omega_n(0,L_1)} |v_n|^2\mathrm{d}x \qquad (3.11)\\ &\le (c_1 + c_2L_1^{p-2})\int_{\mathbb{R}^3} |v_n|^2\mathrm{d}x \to 0.\end{aligned}$$

On the other hand, if we set $\tau' = \tau/(\tau - 1)$, then $2\tau' \in (4, 2^*_\alpha)$. Frpm the Hölder inequality, (3.9) and (H4) we obtain

$$\begin{aligned}\int_{\Omega_n(L_1,\infty)} \frac{\widetilde{f}(x,u_n)u_n}{\|u_n\|^2}\mathrm{d}x &= \int_{\Omega_n(L_1,\infty)} \frac{\widetilde{f}(x,u_n)u_n}{|u_n|^2}|v_n|^2\mathrm{d}x \\ &\le \left(\int_{\Omega_n(L_1,\infty)} \left(\frac{\widetilde{f}(x,u_n)u_n}{|u_n|^2}\right)^\tau \mathrm{d}x\right)^{\frac{1}{\tau}} \left(\int_{\Omega_n(L_1,\infty)} |v_n|^{2\tau'}\mathrm{d}x\right)^{\frac{1}{\tau'}} \\ &\le \left(\int_{\Omega_n(L_1,\infty)} \left|\frac{\widetilde{f}(x,u_n)}{u_n}\right|^\tau \mathrm{d}x\right)^{\frac{1}{\tau}} \left(\int_{\Omega_n(L_1,\infty)} |v_n|^{2\tau'}\mathrm{d}x\right)^{\frac{1}{\tau'}} \\ &\le \left(\int_{\Omega_n(L_1,\infty)} d_1\widetilde{\mathscr{F}}(x,u)\mathrm{d}x\right)^{\frac{1}{\tau}} \left(\int_{\Omega_n(L_1,\infty)} |v_n|^{2\tau'}\mathrm{d}x\right)^{\frac{1}{\tau'}} \\ &\le [d_1(c+1)]^{\frac{1}{\tau}} \left(\int_{\mathbb{R}^3} |v_n|^{2\tau'}\mathrm{d}x\right)^{\frac{1}{\tau'}} \to 0.\end{aligned} \qquad (3.12)$$

Combining (3.11) and (3.12), we have

$$\int_{\mathbb{R}^3} \frac{\widetilde{f}(x,u_n)u_n}{\|u_n\|^2}\mathrm{d}x = \int_{\Omega_n(0,L_1)} \frac{\widetilde{f}(x,u_n)u_n}{\|u_n\|^2}\mathrm{d}x + \int_{\Omega_n(L_1,\infty)} \frac{\widetilde{f}(x,u_n)u_n}{\|u_n\|^2}\mathrm{d}x \to 0, \qquad (3.13)$$

which contradicts (3.10).

Case 2: Suppose $v \neq 0$.

Then we set $A = \{x \in \mathbb{R}^3 : v(x) \neq 0\}$ and $\mathrm{meas}(A) > 0$. For $x \in A$, we have $\lim_{n\to\infty} |u_n(x)| = \infty$, and hence $A \subset \Omega_n(L_1, \infty)$ for large n. From (2.11), (2.15), (2.2) and (H3), note the nonnegativity of $f(x, u)u$, Fatou's Lemma enables us to obtain

$$\begin{aligned}0 &= \lim_{n\to\infty} \frac{< J'(u_n), u_n >}{\|u_n\|^4} \\ &= \lim_{n\to\infty} \left[\frac{\|u_n\|^2}{\|u_n\|^4} + K_\alpha \frac{\int_{\mathbb{R}^3} u_n^2\phi(u_n)dx}{\|u_n\|^4} - \frac{\int_{\mathbb{R}^3} \widetilde{f}(x,u_n)u_n dx}{\|u_n\|^4} + \mu\frac{\int_{\mathbb{R}^3} g(x)|u_n|^q dx}{\|u_n\|^4}\right] \\ &\le \lim_{n\to\infty} \left[\frac{\|u_n\|^q}{\|u_n\|^4}\mu\|g\|_{q'}C^q_{\frac{qq'}{q'-1}} + K_\alpha C\frac{\|u_n\|^4}{\|u_n\|^4} - \int_{\Omega_n(0,L_1)} \frac{\widetilde{f}(x,u_n)u_n}{\|u_n\|^4}\mathrm{d}x \right.\\ &\qquad \left. - \int_{\Omega_n(L_1,\infty)} \frac{\widetilde{f}(x,u_n)u_n}{|u_n|^4}|v_n|^4\mathrm{d}x\right]\end{aligned}$$

$$\begin{aligned}
&\leq K_\alpha C + \limsup_{n\to\infty}\int_{\Omega_n(0,L_1)}\frac{\widetilde{f}(x,u_n)u_n}{\|u_n\|^4}\mathrm{d}x - \liminf_{n\to\infty}\int_{\Omega_n(L_1,\infty)}\frac{\widetilde{f}(x,u_n)u_n}{|u_n|^4}|v_n|^4\mathrm{d}x\\
&\leq K_\alpha C + \limsup_{n\to\infty}\frac{c_1L_1^2+c_2L_1^p}{\|u_n\|^4}\cdot \mathrm{meas}(\Omega_n(0,L_1))\\
&\quad -\liminf_{n\to\infty}\int_{\Omega_n(L_1,\infty)}\frac{\widetilde{f}(x,u_n)u_n}{|u_n|^4}[\chi_{\Omega_n(L_1,\infty)}(x)]|v_n|^4\mathrm{d}x\\
&\leq K_\alpha C - \int_{\Omega_n(L_1,\infty)}\liminf_{n\to\infty}\frac{\widetilde{f}(x,u_n)u_n}{|u_n|^4}[\chi_{\Omega_n(L_1,\infty)}(x)]|v_n|^4\mathrm{d}x\\
&\to -\infty.
\end{aligned}$$

This is also a contradiction.

Combining the above two cases we have that Definition 2.1 (ii) holds. □

Lemma 3.2. *Suppose the assumptions in Theorem 1.1 hold. Then there exist constants $\rho,\beta>0$ such that $J(u)\geq\beta$ when $\|u\|=\rho$.*

Proof. Note that $q\in\left(\frac{6}{3+2\alpha},2\right]$. From (2.12) and (2.2), we have

$$\begin{aligned}
J(u) &= \frac{1}{2}\parallel u\parallel^2+\frac{1}{4}K_a\int_{R^3}\phi(u)u^2dx-\int_{\mathbb{R}^3}\widetilde{F}(x,u)dx-\mu\int_{\mathbb{R}^3}g(x)udx\\
&\geq \frac{1}{2}\parallel u\parallel^2-\int_{\mathbb{R}^3}\widetilde{F}(x,u)dx-\mu\int_{\mathbb{R}^3}g(x)udx\\
&\geq \frac{1}{2}\parallel u\parallel^2-\frac{c_1}{2}\parallel u\parallel_2^2-\frac{c_2}{p}\parallel u\parallel_p^p-\mu\|g\|_q\|u\|_{\frac{q}{q-1}}\\
&\geq \frac{1}{2}\parallel u\parallel^2-\frac{c_1}{2}C_2^2\parallel u\parallel^2-\frac{c_2}{p}C_p^p\parallel u\parallel^p-\mu C_{\frac{q}{q-1}}\|g\|_q\|u\|.
\end{aligned}$$

Note that c_1 can be arbitrarily small, and let $c_1=\frac{1}{2C_2^2}$, and then

$$\begin{aligned}
J(u) &\geq \frac{1}{4}\parallel u\parallel^2-\frac{c_2}{p}C_p^p\parallel u\parallel^p-\mu C_{\frac{q}{q-1}}\|g\|_q\|u\|\\
&\geq \|u\|\left(\frac{1}{4}\parallel u\parallel-\frac{c_2}{p}C_p^p\parallel u\parallel^{p-1}-\mu C_{\frac{q}{q-1}}\|g\|_q\right).
\end{aligned}$$

Note that $p\in(4,2_\alpha^*)$, and we obtain a $\rho>0$ such that $h(\rho)=\frac{1}{4}\rho-\frac{c_2}{p}C_p^p\rho^{p-1}>0$. Consequently, we choose a sufficiently small $\mu>0$ such that $h(\rho)-\mu C_{\frac{q}{q-1}}\|g\|_q>0$. □

Proof of Theorem 1.1. Now, we use Lemma 2.9 to prove Theorem 1.1. We first give the direct decomposition for the space E. Note that E is a Hilbert space, so let e_j be an orthonomormal basis of E and define $X_j=\mathbb{R}_{e_j}$, and we have

$$Y_k=\bigoplus_{j=1}^{k}X_j,\quad Z_k=\overline{\bigoplus_{j=k}^{\infty}X_j},\quad k\in\mathbb{N}. \tag{3.14}$$

In what follows, we show that, for each $k\in\mathbb{N}$, there exist $\rho_k>r_k>0$ such that

$$b_k=\inf_{u\in Z_k,\|u\|=r_k}J(u)\to+\infty,\ k\to\infty, \tag{3.15}$$

and

$$a_k = \max_{u\in Y_k, \|u\|=\rho_k} J(u) \leq 0. \tag{3.16}$$

From Lemma 2.2 and Lemma 3.8 in [35], for $r \in [2, 2_\alpha^*)$ we have

$$\beta_k(r) = \sup_{u\in Z_k, \|u\|=1} \| u \|_r \to 0, k \to \infty. \tag{3.17}$$

This, together with (2.12) and (2.2), implies that

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{4}K_\alpha \int_{\mathbb{R}^3} u^2\phi(u)dx - \int_{\mathbb{R}^3} \widetilde{F}(x,u)dx + \frac{\mu}{q}\int_{\mathbb{R}^3} g(x)|u|^q dx \\ &\geq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^3} \widetilde{F}(x,u)dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{c_1}{2}\|u\|_2^2 - \frac{c_2}{p}\|u\|_p^p \\ &\geq \frac{1}{2}\|u\|^2 - \frac{c_1}{2}C_2^2\|u\|^2 - \frac{c_2}{p}\|u\|_p^p. \end{aligned} \tag{3.18}$$

Note from (H1) we see that c_1 can be chosen arbitrarily small, so if we take $c_1 \leq \frac{1}{2}C_2^{-2}$ and $r_k = (c_2\beta_k^p)^{\frac{1}{2-p}}$, by (3.18), for $u \in Z_k$, and $\|u\| = r_k$, we find

$$J(u) \geq \frac{1}{4}\|u\|^2 - \frac{c_2}{p}\beta_k^p\|u\|^p \geq \left(\frac{1}{4} - \frac{1}{p}\right)(c_2\beta_k^p)^{\frac{2}{2-p}} \to +\infty, \text{ as } k \to +\infty, \text{ with } p > 4.$$

Therefore, (3.15) holds.

On the other hand, from L'Hospital rule and (H3) we have

$$\lim_{|u|\to\infty} \frac{\widetilde{F}(x,u)}{|u|^4} = +\infty \text{ uniformly in } x \in \mathbb{R}^3.$$

Hence, there exists sufficiently large $\vartheta_k > 0$ such that

$$\widetilde{F}(x,u) \geq \vartheta_k|u|^4, \text{ for } x \in \mathbb{R}^3, |u| > L_2, \text{ for some } L_2 > 0.$$

From (2.12) with $p \in (4, 2_\alpha^*)$, we have

$$\widetilde{F}(x,u) \leq |u|^2\left(\frac{c_1}{2} + \frac{c_2}{p}|u|^{p-2}\right) \leq \left(\frac{c_1}{2} + \frac{c_2}{p}L_2^{p-2}\right)|u|^2, \text{ for } x \in \mathbb{R}^3, |u| \leq L_2.$$

As a result, there exists $\mathcal{M} = \frac{c_1}{2} + \frac{c_2}{p}L_2^{p-2}$ such that

$$\widetilde{F}(x,u) \geq \vartheta_k|u|^4 - \mathcal{M}|u|^2, \text{ for } x \in \mathbb{R}^3, u \in \mathbb{R}. \tag{3.19}$$

Since $\dim Y_k < \infty$ and all norms are equivalent in the finite-dimensional space, from (3.19), (2.15) and (g) we have

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{4}K_\alpha \int_{\mathbb{R}^3} u^2\phi(u)dx - \int_{\mathbb{R}^3} \widetilde{F}(x,u)dx + \frac{\mu}{q}\int_{\mathbb{R}^3} g(x)|u|^q dx \\ &\leq \frac{1}{2}\|u\|^2 + \frac{1}{4}K_\alpha C\|u\|^4 - \vartheta_k\|u\|_4^4 + \mathcal{M}\|u\|_2^2 + \frac{\mu}{q}\|g\|_{g'}\|u\|^q_{\frac{qq'}{q'-1}} \\ &\leq \frac{1}{2}\|u\|^2 + \frac{1}{4}K_\alpha C\|u\|^4 - \vartheta_k C_k^4\|u\|^4 + \mathcal{M}\|u\|_2^2 + \frac{\mu}{q}\|g\|_{g'}\|u\|^q_{\frac{qq'}{q'-1}}, C_k > 0. \end{aligned} \tag{3.20}$$

Note that ϑ_k can be chosen large enough, so we take $u \in Y_k$ and large ρ_k ($\rho_k > r_k$) such that

$$J(u) \leq 0, \text{ for } u \in Y_k, \|u\| = \rho_k.$$

Thus (3.16) holds.

Finally, (H5) implies that J is an even functional on E. Thus J satisfies all conditions of Lemma 2.9. Then J has a sequence of critical points u_n, such that $J(u_n) \to +\infty$ as $n \to \infty$. This means that (1.1) has infinitely many high energy solutions $\{u_n\}_{n\in\mathbb{N}}$ such that

$$\frac{1}{2}\|u_n\|^2 + \frac{1}{4}K_\alpha \int_{\mathbb{R}^3} u_n^2\phi(u_n)dx - \int_{\mathbb{R}^3} \widetilde{F}(x, u_n)dx + \frac{\mu}{q}\int_{\mathbb{R}^3} g(x)|u_n|^q dx \to +\infty, \text{ as } n \to \infty.$$

□

Acknowledgements. We would like to express our thanks to the anonymous referees and the editor for their constructive comments and suggestions, which greatly improved this article.

References

[1] R. Bagley and P. Torvik, *A theoretical basis for the application of fractional calculus to viscoelasticity*, J. Rheol., 1983, 27(3), 201–210.

[2] Z. Bai and Y. Zhang, *Solvability of fractional three-point boundary value problems with nonlinear growth*, Appl. Math. Comput., 2011, 218(5), 1719–1725.

[3] Z. Bai, Y. Chen, H. Lian and S. Sun, *On the existence of blow up solutions for a class of fractional differential equations*, Fract. Calc. Appl. Anal., 2014, 17(4), 1175–1187.

[4] V. Benci and D. Fortunato, *An eigenvalue problem for the Schrödinger-Maxwell equations*, Topol. Methods Nonl. Anal., 1998, 11(2), 283–293.

[5] Y. Cui, *Uniqueness of solution for boundary value problems for fractional differential equations*, Appl. Math. Lett., 2016, 51, 48–54.

[6] B. Cheng and X. Tang, *New existence of solutions for the Fractional p-Laplacian equations with sign-changing potential and nonlinearity*, Mediterr. J. Math., 2016, 13(5), 3373–3387.

[7] X. Chang, *Ground state solutions of asymptotically linear fractional Schrödinger equations*, J. Math. Phys., 2013, 54, Article ID 061504.

[8] L. Debnath, *Recent applications of fractional calculus to science and engineering*, Int. J. Math. Math. Sci., 2003, 2003(54), 3413–3442.

[9] T. D'Aprile and D. Mugnai, *Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations*, Proc. Roy. Soc. Edinburgh Sect. A, 2004, 134(5), 893–906.

[10] P. Felmer, A. Quaas and J. Tan, *Positive solutions of the nonlinear Schrödinger equation with the fractional Laplacian*, Proc. Roy. Soc. Edinburgh Sect. A, 2012, 142(6), 1237–1262.

[11] Y. Guo, *Nontrivial solutions for boundary-value problems of nonlinear fractional differential equations*, Bull. Korean Math. Soc., 2010, 47(1), 81–87.

[12] Y. Guo, *Nontrivial periodic solutions of nonlinear functional differential systems with feedback control*, Turkish J. Math., 2010, 34(1), 35–44.

[13] X. Hao, H. Wang, L. Liu and Y. Cui, *Positive solutions for a system of nonlinear fractional nonlocal boundary value problems with parameters and p-Laplacian operator*, Bound. Value Probl., 2017, Article ID 182, 18.

[14] J. He, X. Zhang, L. Liu, Y. Wu and Y. Cui, *Existence and asymptotic analysis of positive solutions for a singular fractional differential equation with nonlocal boundary conditions*, Bound. Value Probl., 2018, Article ID 189, 17.

[15] X. He, A. Qian and W. Zou, *Existence and concentration of positive solutions for quasi-linear Schrödinger equations with critical growth*, Nonlinearity, 2013, 26(12), 3137–3168.

[16] H. Hajaiej, X. Yu and Z. Zhai, *Fractional Gagliardo-Nirenberg and Hardy inequalities under Lorentz norms*, J. Math. Anal. Appl., 2012, 396(2), 569–577.

[17] S. Khoutir and H. Chen, *Multiple nontrivial solutions for a nonhomogeneous Schrödinger-Poisson system in* $\mathbb{R}^3$, Electron. J. Qual. Theory Differ. Equ., 2017, 2017(28), 1–17.

[18] E. Lieb and M. Loss, *Analysis*, Vol. 14 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, second edition, 2001.

[19] K. Li, *Existence of nontrivial solutions for nonlinear fractional Schrödinger-Poisson equations*, Appl. Math. Lett., 2017, 72, 1–9.

[20] L. Li, A. Boucherif and N. Daoudi-Merzagui, *Multiple solutions for 4-superlinear Klein-Gordon-Maxwell system without odd nonlinearity*, Taiwanese J. Math., 2017, 21(1), 151–165.

[21] J. Liu and A. Qian, *Ground state solution for a Schrödinger-Poisson equation with critical growth*, Nonlinear Anal. Real World Appl., 2018, 40, 428–443.

[22] A. Mao and H. Chang, *Kirchhoff type problems in* R^N *with radial potentials and locally Lipschitz functional*, Appl. Math. Lett., 2016, 62, 49–54.

[23] A. Mao and W. Wang, *Nontrivial solutions of nonlocal fourth order elliptic equation of Kirchhoff type in* $\mathbb{R}^3$, J. Math. Anal. Appl., 2018, 459(1), 556–563.

[24] A. Mao, L. Yang, A. Qian and S. Luan, *Existence and concentration of solutions of Schrödinger-Poisson system*, Appl. Math. Lett., 2017, 68, 8–12.

[25] P. Pucci, M. Xiang and B. Zhang, *Multiple solutions for nonhomogeneous Schrödinger-Kirchhoff type equations involving the fractional p-Laplacian in* $\mathbb{R}^N$, Calc. Var., 2015, 54(3), 2785–2806.

[26] A. Qian, *Infinitely many sign-changing solutions for a Schrödinger equation*, Adv. Difference Equ., 2011, Article ID 39, 6.

[27] A. Qian and C. Li, *Infinitely many solutions for a Robin boundary value problem*, Int. J. Differ. Equ., 2010, Article ID 548702, 9.

[28] S. Secchi, *Concave-convex nonlinearities for some nonlinear fractional equations involving the Bessel operator*, Complex Var. Elliptic Equ., 2017, 62(5), 654–669.

[29] Z. Shen and F. Gao, *On the existence of solutions for the critical fractional Laplacian equation in* $\mathbb{R}^N$, Abstract Appl. Anal., 2014, Article ID 143741, 10.

[30] M. Shao and A. Mao, *Multiplicity of solutions to Schrödinger-Poisson system with concave-convex nonlinearities*, Appl. Math. Lett., 2018, 83, 212–218.

[31] Y. Sun, L. Liu and Y. Wu, *The existence and uniqueness of positive monotone solutions for a class of nonlinear Schrödinger equations on infinite domains*, J. Comput. Appl. Math., 2017, 321, 478–486.

[32] K. Teng, *Existence of ground state solutions for the nonlinear fractional Schrödinger-Poisson system with critical Sobolev exponent*, J. Differential Equations, 2016, 261(6), 3061–3106.

[33] J. Wu, X. Zhang, L. Liu, Y. Wu and Y. Cui, *The convergence analysis and error estimation for unique solution of a p-Laplacian fractional differential equation with singular decreasing nonlinearity*, Bound. Value Probl., 2018, Article ID 82, 15.

[34] Y. Wang and J. Jiang, *Existence and nonexistence of positive solutions for the fractional coupled system involving generalized p-Laplacian*, Adv. Difference Equ., 2017, Article ID 337, 19.

[35] M. Willem, *Minimax theorems*, Boston: Birkhäser, 1996.

[36] J. Xu, Z. Wei and W. Dong, *Weak solutions for a fractional p-Laplacian equation with sign-changing potential*, Complex Complex Var. Elliptic Equ., 2016, 61(2), 284–296.

[37] L. Yang, *Multiplicity of solutions for fractional Schrödinger equations with perturbation*, Bound. Value Probl., 2015, Article ID 56, 9.

[38] Y. Ye and C. Tang, *Existence and multiplicity of solutions for Schrödinger-Poisson equations with sign-changing potential*, Calc. Var., 2015, 53(1–2), 383–411.

[39] M. Zuo, X. Hao, L. Liu and Y. Cui, *Existence results for impulsive fractional integro-differential equation of mixed type with constant coefficient and antiperiodic boundary conditions*, Bound. Value Probl., 2017, Article ID 161, 15.

[40] Y. Zou and G. He, *On the uniqueness of solutions for a class of fractional differential equations*, Appl. Math. Lett., 2017, 74, 68–73.

[41] Z. Yue and Y. Zou, *New uniqueness results for fractional differential equation with dependence on the first order derivative*, Adv. Difference Equ., 2019, Article ID 38, 9.

[42] K. Zhang, *On a sign-changing solution for some fractional differential equations*, Bound. Value Probl., 2017, Article ID 59, 8.

[43] X. Zhang, J. Wu, L. Liu, Y. Wu and Y. Cui, *Convergence analysis of iterative scheme and error estimation of positive solution for a fractional differential equation*, Math. Model. Anal., 2018, 23(4), 611–626.

[44] Y. Zhang, *Existence results for a coupled system of nonlinear fractional multi-point boundary value problems at resonance*, J. Inequal. Appl., 2018, Article ID 198, 17.

[45] X. Zhang, L. Liu, Y. Wu and Y. Zou, *Existence and uniqueness of solutions for systems of fractional differential equations with Riemann-Stieltjes integral boundary condition*, Adv. Difference Equ., 2018, Article ID 204, 15.

[46] X. Zhang, L. Liu and Y. Zou, *Fixed-point theorems for systems of operator equations and their applications to the fractional differential equations*, J. Funct. Spaces, 2018, Article ID 7469868, 9.

[47] J. Zhang, João Marcos do Ó and M. Squassina, *Fractional Schrödinger-Poisson systems with a general subcritical or critical nonlinearity*, Adv. Nonlinear Stud., 2016, 16, 15–30.

[48] X. Zhang, L. Liu, Y. Wu and Y. Cui, *New result on the critical exponent for solution of an ordinary fractional differential problem*, J. Funct. Spaces, 2017, Article ID 3976469, 4.

[49] X. Zhang, J. Jiang, Y. Wu and Y. Cui, *Existence and asymptotic properties of solutions for a nonlinear Schrödinger elliptic equation from geophysical fluid flows*, Appl. Math. Lett., 2019, 90, 229–237.

[50] X. Zhang, L. Liu, Y. Wu and Y. Cui, *Entire blow-up solutions for a quasilinear p-Laplacian Schrödinger equation with a non-square diffusion term*, Appl. Math. Lett., 2017, 74, 85–93.

[51] X. Zhang, L. Liu, Y. Wu and Y. Cui, *Existence of infinitely solutions for a modified nonlinear Schrödinger equation via dual approach*, Electron. J. Differential Equ., 2018, 2018(147), 1–15.

[52] X. Zhang, L. Liu, Y. Wu and Y. Cui, *The existence and nonexistence of entire large solutions for a quasilinear Schrödinger elliptic system by dual approach*, J. Math. Anal. Appl., 2018, 464(2), 1089–1106.