Journal of Applied Analysis and Computation
Volume 9, Number 4, August 2019, 1571–1588
Website:http://jaac.ijournal.cn
DOI:10.11948/2156-907X.20190021

THE EXISTENCE OF RANDOM $\mathcal{D}$-PULLBACK ATTRACTORS FOR RANDOM DYNAMICAL SYSTEM AND ITS APPLICATIONS*

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Abstract In this paper, we establish a result on the existence of random $\mathcal{D}$-pullback attractors for norm-to-weak continuous non-autonomous random dynamical system. Then we give a method to prove the existence of random $\mathcal{D}$-pullback attractors. As an application, we prove that the non-autonomous stochastic reaction diffusion equation possesses a random $\mathcal{D}$-pullback attractor in H_0^1 with polynomial growth of the nonlinear term.

Keywords Random dynamical system, random pullback attractor, stochastic reaction diffusion equation.

MSC(2010) 35B40, 35B41, 60H15, 60H30, 60H40.

1. Introduction

The asymptotic behavior of autonomous dynamical system is captured by attractors, which are compact invariant set attracting all the orbits. For a non-autonomous dynamical system, the asymptotic behavior is captured by pullback attractors, which are families of compact invariant sets pullback attracting all the orbits. In either case, it is an important problem to prove the existence of attractors in dynamical systems. Many authors have paid much attention to these problems for a quite long time and have made a lot of progress (see references [3,8,9,12–14,16,22]).

Recently, the theory of random attractors has been well developed for random dynamical systems (see [2, 4, 7, 10, 11, 15, 23]), where a random attractor is a measurable random compact invariant set attracting any bounded random set, that is, for a random dynamical system (φ, θ) on a separable Banach space X, a set $\mathcal{A} = \{A(\omega) : \omega \in \Omega\}$ is called a random attractor if the following hold:

(i) $A(\omega)$ is a random compact set;

(ii) $\mathcal{A} = \{A(\omega)\}_{\omega\in\Omega}$ is φ-invariant; that is, for $\mathbb{P}$-a.s. $\omega \in \Omega$, $\varphi(t, \omega)A(\omega) = A(\theta_t\omega)$;

(iii) $\mathcal{A}$ attracts every set in X; i.e., for all bounded $B \subset X$,

$$\lim_{t\to\infty} d(\varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega), A(\omega)) = 0.$$

In fact, this definition is an extension of the notion of the attractors of autonomous dynamical system in the framework of random dynamical system, these

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*The authors are supported by National Natural Science Foundation of China (11761044, 11661048) and the key constructive discipline of Lanzhou City University(LZCU-ZDJSXK-201706).

can be found in [1, 17, 18, 24, 25]. Motivated by these problems and some ideas in [7, 9, 12], we consider the existence of attractors for non-autonomous random dynamical system, called the random $\mathcal{D}$-pullback attractors, see Definition 2.6.

Obviously, random $\mathcal{D}$-pullback attractors are also an extension of the $\mathcal{D}$-pullback attractors (see [3, 9, 14]) of non-autonomous dynamical system in the framework of random dynamical system. A few authors consider this problem.

For norm-to-weak continuous autonomous random dynamical system, the existence of random attractors has been proved in [11]; for continuous non-autonomous random dynamical system, the existence of random $\mathcal{D}$-pullback attractors has been proved in [26], here we consider the existence of random $\mathcal{D}$-pullback attractors for norm-to-weak continuous non-autonomous random dynamical system.

In this paper, we establish a result on the existence of random $\mathcal{D}$-pullback attractors for norm-to-weak continuous non-autonomous random dynamical system by using the theory of measure of non-compactness. We continue the works in [7, 9, 12] and give a method to verify the existence of random $\mathcal{D}$-pullback attractors.

As an application of our abstract results, we study the existence of random $\mathcal{D}$-pullback attractors for the following stochastic reaction diffusion equation:

$$\begin{cases} du - (\Delta u - f(u) + g(t))dt = bu(t) \circ dW(t), \quad x \in D, t \in \mathbb{R} \\ u(x,\tau) = u_\tau(x), \quad x \in D, \tau \leq t, \\ u(x,t)|_{\partial D} = 0, \end{cases} \tag{1.1}$$

where $g(\cdot) \in L^2_{loc}(\mathbb{R}, L^2(D))$, $W(t)$ is a two-sided real-valued Wiener process on a probability space.

Non-autonomous system instead of autonomous system is considered in reference [14], for autonomous stochastic system (1.1), many authors have studied the existence of random attractors (see [2, 4, 7, 10, 11, 18, 24, 25]); for non-autonomous case of (1.1), it has also been studied (see [18, 24, 25]) by many authors under some strict conditions, where the domain D is unbounded, $f(u)$ is replaced by $f(u) + \lambda u$, $\lambda > 0$. The term λu is required for the case of unbounded domain in order to obtain the dissipativity of the linear part. In fact, without this term, the pullback asymptotic compactness does not hold true. It is also difficult to get the pullback asymptotic compactness in H_0^1 even for the problem in bounded domains. However, a few authors consider the problem when the domain is bounded, especially with the exponential growth of the external force and polynomial growth of the nonlinear term. Here we will prove that the system (1.1) exists a random $\mathcal{D}$-pullback attractor under some dissipative conditions.

The rest of the paper is organized as follows. In section 2, we recall some basic concepts about attractors. In section 3, we present some sufficient conditons for the existence of random $\mathcal{D}$-pullback attractors for a norm-to-weak continuous random dynamical system. In section 4, we prove that the system (1.1) exists a random $\mathcal{D}$-pullback attractor.

2. Preliminaries

In this section, we introduce some concepts related to random $\mathcal{D}$-pullback attractors for random dynamical system, which are extensions of $\mathcal{D}$-pullback attractors for non-autonomous dynamical system. The reader is referred to [6-8,11-14,16,18] for more details.

Let $(X, \|\cdot\|_X)$ be a separable Banach space with Borel σ-algebra $\mathcal{B}(X)$ and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. In this paper, the term $\mathbb{P}$-a.s.(the abbreviation for $\mathbb{P}$ almost surely) denotes that an event happens with probability one. In other words, the set of possible exception may be non-empty, but it has probability zero.

Definition 2.1 ([2, 4, 7, 10, 11]). $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is called a metric dynamical systems if $\theta : \mathbb{R} \times \Omega \to \Omega$ is $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F})$-measurable, and θ_0 is the identity on Ω, $\theta_{s+t} = \theta_t \circ \theta_s$ for all $t, s \in \mathbb{R}$ and $\theta_t\mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{R}$.

Definition 2.2 ([7, 10, 11, 18, 24–26]). A random dynamical system (RDS)(φ, θ) on X over a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is a mapping

$$\varphi(t, \tau, \omega) : X \to X, \quad (t, \tau, \omega, x) \to \varphi(t, \tau, \omega)x,$$

which represents the dynamics in the state space X and satisfies the properties

(i) $\varphi(\tau, \tau, \omega)$ is the identity on X;

(ii) $\varphi(t, \tau, \omega) = \varphi(t, s, \theta_{s-\tau}\omega)\varphi(s, \tau, \omega)$ for all $\tau \le s \le t$;

(iii) $\omega \to \varphi(t, \tau, \omega)x$ is $\mathcal{F}$-measurable for all $t \ge \tau$ and $x \in X$.

Remark 2.1. A RDS (φ, θ) is called a continuous random dynamical system if $\varphi(t, \tau, \omega) : X \to X$ is continuous for all $t \ge \tau$ and $\omega \in \Omega$. A RDS (φ, θ) is called a norm-to-weak continuous random dynamical system if $x_n \to x, \varphi(t, \tau, \omega)x_n \rightharpoonup \varphi(t, \tau, \omega)x$ for all $t \ge \tau$, and $\omega \in \Omega$.

Obviously, a continuous random dynamical systems is also a norm-to-weak continuous random dynamical system.

Definition 2.3 ([7, 10, 11, 18, 24–26]). A random set D is a multivalued mapping $D : \Omega \to \mathcal{B}(X)$ such that, for every $x \in X$, the mapping $\omega \to d(x, D(\omega))$ is measurable, where $d(x, B)$ is the distance between the element x and the set B. It is said that the random set is bounded(resp., closed or compact) if $D(\omega)$ is bounded(resp., closed or compact) for $\mathbb{P}$- a.s. $\omega \in \Omega$.

In the sequel, we use $\mathcal{D}$ to denote a collection of some families of nonempty subsets of X:

$$D' \in \mathcal{D}, D' = \{D(t, \omega) \in \mathcal{B}(X) : t \in \mathbb{R}, \omega \in \Omega\}.$$

Definition 2.4 ([17, 18, 26]). A set $B' \in \mathcal{D}$ is called a random $\mathcal{D}$-pullback bounded absorbing set for RDS (φ, θ) if for any $t \in \mathbb{R}$ and any $D' \in \mathcal{D}$, there exists $\tau_0(t, D')$ such that $\varphi(t, \tau, \theta_{\tau-t}\omega)D(\tau, \theta_{\tau-t}\omega) \subset B(t, \omega)$ for any $\tau \le \tau_0$.

Definition 2.5 ([17, 18, 26]). A set $\mathcal{A} = \{A(t, \omega) : t \in \mathbb{R}, \omega \in \Omega\}$ is called a random $\mathcal{D}$-pullback attractor for $\{\varphi, \theta\}$ if the following hold:

(i) $A(t, \omega)$ is a random compact set;

(ii) $\mathcal{A}$ is invariant; that is, for $\mathbb{P}$-a.s. $\omega \in \Omega$, and $\tau \le t$, $\varphi(t, \tau, \omega)A(\tau, \omega) = A(t, \theta_{t-\tau}\omega)$;

(iii) $\mathcal{A}$ attracts all set in $\mathcal{D}$; that is, for all $B' \in \mathcal{D}$ and $\mathbb{P}$-a.s. $\omega \in \Omega$,

$$\lim_{\tau\to-\infty} d(\varphi(t, \tau, \theta_{\tau-t}\omega)B(\tau, \theta_{\tau-t}\omega), A(t, \omega)) = 0,$$

where d is the Hausdorff semimetric given by $dist(B, \mathcal{A}) = \sup_{b\in B} \inf_{a\in\mathcal{A}} \| b - a \|_X$.

We will use the following measure of non-compactness.

Definition 2.6 ([9, 12, 20, 22]). Let X be a metric space and B a bounded subset of X. The Kuratowski measure of non-compactness $\alpha(B)$ of B is defined by

$$\alpha(B) = \inf\{\delta > 0 \ : B \text{ admits a finite cover by sets of diameter} \leq \delta\}.$$

The following summarizes some of the basic properties of this measure of non-compactness.

Lemma 2.1 ([7, 9, 12, 20, 22]). *Let X be Banach space, α be the measure of non-compactness. Then*

(1) $\alpha(B) = 0$ if, and only if, $\overline{B}$ is compact;

(2) $\alpha(B_1 + B_2) \leq \alpha(B_1) + \alpha(B_2)$;

(3) $\alpha(B_1) \leq \alpha(B_2)$ for $B_1 \subset B_2$;

(4) $\alpha(B_1 \bigcup B_2) \leq \max\{\alpha(B_1), \alpha(B_2)\}$;

(5) $\alpha(B) = \alpha(\overline{B})$;

(6) If $F_1 \supset F_2 \ldots$ are non-empty closed sets in X such that $\alpha(F_n) \to 0$ as $n \to \infty$, then $F = \bigcap_{n=1}^{\infty} F_n$ is nonempty and compact.

In addition, let X be an infinite dimensional Banach space with a decomposition $X = X_1 \oplus X_2$ and let $P : X \to X_1$, $Q : X \to X_2$ be projectors with $dim X_1 < \infty$. Then

(7) $\alpha(B(\varepsilon))) = 2\varepsilon$, where $B(\varepsilon)$ is a ball of radius ε.

(8) $\alpha(B) < \varepsilon$ for any bounded subset B of X for which the diameter of QB is less than ε.

Definition 2.7 ([7, 9, 12, 17]). A RDS (φ, θ) on a Banach space X is said to be random pullback limit set compact if for any $D' \in \mathcal{D}$, $\varepsilon > 0$ and $\omega \in \Omega$ there exists a $T(D', \varepsilon, \omega) \leq t$ such that

$$\alpha(\bigcup_{\tau \leq T} \varphi(t, \tau, \theta_{\tau - t}\omega) D(\tau, \theta_{\tau - t}\omega)) < \varepsilon,$$

where α is a measure of non-compactness defined on the subsets of X.

Definition 2.8 ([9, 17, 20]). A RDS (φ, θ) on a Banach space X is said to be random pullback asymptotically compact in X if for any $D' \in \mathcal{D}$, each $\omega \in \Omega$ and any sequence $\tau_n \to -\infty$ and $x_n \in D(\tau_n, \theta_{\tau_n - t}\omega), n = 1, 2, \cdots$, the set $\{\varphi(t, \tau_n, \theta_{\tau_n - t}\omega)x_n, k = 1, 2, \cdots.\}$ is precompact in X.

Let $\omega \in \Omega$ be arbitrary but fixed, define $\mathcal{D}$-pullback limit set $A(B', t, \omega)$ of $B' \in \mathcal{D}$ by

$$A(B', t, \omega) = \bigcap_{s \leq t} \overline{\bigcup_{\tau \leq s} \varphi(t, \tau, \theta_{\tau - t}\omega) B(\tau, \theta_{\tau - t}\omega)},$$

which can be characterized, analogously to [5], by

$$y \in A(B', t, \omega) \Leftrightarrow \begin{cases} \text{there exist sequences } \tau_n \leq t, \text{ and } x_n \in B(\tau_n, \theta_{\tau_n - t}\omega) \text{ such} \\ \text{that } \tau_n \to -\infty, \text{ and } \varphi(t, \tau_n, \theta_{\tau_n - t}\omega)x_n \to y \text{ as } n \to \infty. \end{cases} \tag{2.1}$$

3. Existence of random $\mathcal{D}$-pullback attractors

In this section, we study the existence of random $\mathcal{D}$-pullback attractors for norm-to-weak continuous RDS.

Let $(X, \|\cdot\|_X)$ be a separable Banach space with Borel σ-algebra $\mathcal{B}(X)$, $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ be a metric dynamical systems, (φ, θ) be a RDS on X.

Lemma 3.1. *Assume that the RDS (φ, θ) is pullback limit set compact, $D' \in \mathcal{D}, D' = \{D(t,\omega) \in \mathcal{B}(X) : t \in \mathbb{R}, \omega \in \Omega\}$, then for any sequence $\tau_n \leq t$, $\tau_n \to -\infty$ as $n \to \infty$, and any sequence $x_n \in D(\tau_n, \theta_{\tau_n - t}\omega)$, there exists a convergent subsequence of $\{\phi(t, \tau_n, \theta_{\tau_n-t}\omega)x_n\}$ whose limit lies in $A(D', t, \omega)$.*

Proof. For each $\omega \in \Omega$ and $\varepsilon > 0$, there exists $T_\varepsilon = T_\varepsilon(D, \varepsilon, \omega)$ such that

$$\alpha(\bigcup_{\tau \leq T_\varepsilon} \varphi(t, \tau, \theta_{\tau-t}\omega)D(\tau, \theta_{\tau-t}\omega)) < \varepsilon.$$

Now, we choose $\varepsilon = \frac{1}{n}$, $T_n = T_\varepsilon(D, \varepsilon, \omega)$, for $n = 1, 2, \cdots$, with $T_1 > T_2 > \cdots$, we get that

$$\alpha(\bigcup_{\tau_n \leq T_n} \varphi(t, \tau_n, \theta_{\tau_n-t}\omega)D(\tau_n, \theta_{\tau_n-t}\omega)) < \frac{1}{n}.$$

By (3) and (5) of Lemma 2.1, we get

$$\alpha(\bigcup_{\tau_n \leq T_n} \varphi(t, \tau_n, \theta_{\tau_n-t}\omega)x_n) < \frac{1}{n},$$

and

$$\alpha(\overline{\bigcup_{\tau_n \leq T_n} \varphi(t, \tau_n, \theta_{\tau_n-t}\omega)x_n}) < \frac{1}{n}.$$

Property (6) of Lemma 2.1 implies that

$$F(t,\omega) = \bigcap_{n=1}^{\infty} \overline{\bigcup_{\tau_n \leq T_n} \varphi(t, \tau_n, \theta_{\tau_n-t}\omega)x_n}$$

is a nonempty compact set. By (2.1), for any $y \in F(t, \omega)$, there exists $n_k \to \infty$ such that $\varphi(t, \tau_{n_k}, \theta_{\tau_{n_k}-t}\omega)x_{n_k} \to y$. $\square$

Theorem 3.1. *Let (φ, θ) be a RDS on a separable Banach space X and (φ, θ) be pullback limit set compact. Then for any $D' \in \mathcal{D}$ the following conditions hold true:*

(1)$A(D', t, \omega)$ is nonempty and random compact set in X;

(2) $\lim\limits_{\tau \to -\infty} dist(\varphi(t, \tau, \theta_{\tau-t}\omega)D(\tau, \theta_{\tau-t}\omega), A(D', t, \omega))$=0;

(3) if $Y = \{Y(t,\omega) : t \in \mathbb{R}, \omega \in \Omega\}$ is a closed random set attracting D', then $A(D', t, \omega) \subset Y(t, \omega)$.

Proof. (1) By Lemma 3.1, $A(D', t, \omega)$ is nonempty and compact, we only prove that $A(D', t, \omega)$ is a random set in X, that is

$$\omega \to d(x, A(D', t, \omega)$$

is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$ measurable for each $x \in X$.

$A(D', t, \omega)$ is a compact set, hence there exists $y \in A(D', t, \omega)$ such that

$$d(x, A(D', t, \omega)) = ||x - y||_X.$$

By (2.1) there exist $\tau_n \le t$, $\tau_n \to -\infty$ as $n \to \infty$, and $x_n \in D(\tau_n, \theta_{\tau_n - t})$ such that

$$y = \lim_{n\to\infty} \varphi(t, \tau_n, \theta_{\tau_n - t}\omega)x_n.$$

By the Definition 2.1 and 2.2, we know that $\theta_t\omega$ is $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F})$-measurable, $\varphi(t,\tau,\omega)x$ is $\mathcal{F}$-measurable, hence $\varphi(t, \tau_n, \theta_{\tau_n - t}\omega)x_n$ is $\mathcal{F}$-measurable, we get

$$\bigcap_{n=1}^{\infty} \overline{\bigcup_{\tau_m \le \tau_n} \varphi(t, \tau_m, \theta_{\tau_m - t}\omega)x_m}$$

is $\mathcal{F}$-measurable, and by (2.1), we know that

$$y = \bigcap_{n=1}^{\infty} \overline{\bigcup_{\tau_m \le \tau_n} \varphi(t, \tau_m, \theta_{\tau_m - t}\omega)x_m},$$

and

$$\begin{aligned} d(x, A(D', t, \omega)) &= ||x - y||_X \\ &= ||x - \lim_{n\to\infty} \varphi(t, \tau_n, \theta_{\tau_n - t}\omega)x_n||_X \\ &= \lim_{n\to\infty} ||x - \varphi(t, \tau_n, \theta_{\tau_n - t}\omega)x_n||_X \\ &= ||\bigcap_{n=1}^{\infty} \overline{\bigcup_{\tau_m \le \tau_n} (x - \varphi(t, \tau_m, \theta_{\tau_m - t}\omega)x_m)}||_X. \end{aligned}$$

$x - \varphi(t, \tau_m, \theta_{\tau_m - t}\omega)x_m$ is $\mathcal{F}$-measurable, hence $\bigcap_{n=1}^{\infty} \overline{\bigcup_{\tau_m \le \tau_n} (x - \varphi(t, \tau_m, \theta_{\tau_m - t}\omega)x_m)}$ is $\mathcal{F}$-measurable, therefore by the continuous of norm, we get

$$d(x, A(D', t, \omega)) = ||\bigcap_{n=1}^{\infty} \overline{\bigcup_{\tau_m \le \tau_n} (x - \varphi(t, \tau_m, \theta_{\tau_m - t}\omega)x_m)}||_X$$

is $\mathcal{F}$-measurable.

(2) We prove that $A(D', t, \omega)$ pullback attracts D' by contradiction.

Suppose there exists an $\omega \in \Omega$, $\varepsilon_0 > 0$, sequence $\tau_n \to -\infty$ and $x_n \in D(t, \tau_n, \theta_{\tau_n - t}\omega)$ such that

$$d(\varphi(t, \tau_n, \theta_{\tau_n - t}\omega)x_n, A(D', t, \omega)) \ge \varepsilon_0, \quad \forall N \in \mathbb{N}. \tag{3.1}$$

Then, by the pullback limit set compact of (φ, θ) and Lemma 3.1, there exists a convergent subsequence of $\{\varphi(t, \tau_n, \theta_{\tau_n - t}\omega)x_n\}$ such that $\varphi(t, \tau_{n_k}, \theta_{\tau_{n_k} - t}\omega)x_{n_k} \to y$. By Lemma 3.1, we have $y \in A(D', t, \omega)$, contradicting (3.1). Hence, $A(D', t, \omega)$ pullback attracts D'.

(3) For any $x \in A(D', t, \omega)$, there exist τ_n, $\tau_n \to -\infty$ as $n \to \infty$, and $x_n \in D(\tau_n, \theta_{\tau_n - t}\omega)$ such that

$$x = \lim_{n\to\infty} \varphi(t, \tau_n, \theta_{\tau_n - t}\omega)x_n.$$

Since $Y(t, \omega)$ is a closed random set attracting D', we get

$$\lim_{n\to\infty} d(\varphi(t, \tau_n, \theta_{\tau_n - t}\omega)x_n, Y(t, \omega)) = 0,$$

i.e., $d(x, Y(t, \omega)) = 0$, thus we have $x \in Y(t, \omega)$. Therefore, $A(D', t, \omega) \subset Y(t, \omega)$. □

Theorem 3.2. *Suppose (φ,θ) is a norm-to-weak continuous RDS on a separable Banach space X. If (φ,θ) is pullback limit set compact and have a random $\mathcal{D}-$pullback absorbing sets $B'=\{B(t,\omega):t\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$, i.e., for any $D'\in\mathcal{D}$, there exists a $T(t,D')\le t$ such that $\varphi(t,\tau,\theta_{\tau-t}\omega)D(\tau,\theta_{\tau-t}\omega)\subset B(t,\omega)$ for any $\tau\le T$, then there exists a random $\mathcal{D}$-pullback attractor $\mathcal{A}=\{A(t,\omega):t\in\mathbb{R},\omega\in\Omega\}$ and*

$$A(t,\omega)=\bigcap_{s\le t}\overline{\bigcup_{\tau\le s}\varphi(t,\tau,\theta_{\tau-t}\omega)B(\tau,\theta_{\tau-t}\omega)}. \tag{3.2}$$

By Theorem 3.1, we know that $A(t,\omega)$ is a nonempty and random compact set in X. Next, we only prove that $A(t,\omega)$ is invariant and pullback atrracts any set in $\mathcal{D}$.

Proof. First we prove that $A(t,\omega)$ is invariant, i.e., $\varphi(t,\tau,\omega)A(\tau,\omega)=A(t,\theta_{t-\tau}\omega)$, $\forall t\in\mathbb{R},\omega\in\Omega$.

For any $y\in\varphi(t,\tau,\omega)A(\tau,\omega)$, then $y=\varphi(t,\tau,\omega)x$ for some $x\in A(\tau,\omega)$, hence there exist two sequences $\tau_n\le t$ and $x_n\in B(\tau_n,\theta_{\tau_n-\tau}\omega)$ such that $\varphi(\tau,\tau_n,\theta_{\tau_n-\tau}\omega)x_n\to x$. By the norm-to-weak continuous of φ, we get

$$\begin{aligned}&\varphi(t,\tau_n,\theta_{\tau_n-\tau}\omega)x_n=\varphi(t,\tau_n,\theta_{\tau_n-t}\theta_{t-\tau}\omega)x_n=\varphi(t,\tau,\omega)\varphi(\tau,\tau_n,\theta_{\tau_n-\tau}\omega)x_n\\&\rightharpoonup\varphi(t,\tau,\omega)x=y.\end{aligned}$$

Since φ is $\mathcal{D}$-pullback limit set compact, by Lemma 3.1, $\varphi(t,\tau_n,\theta_{\tau_n-\tau}\omega)x_n$ has a convergent subsequence $\varphi(t,\tau_{n_j},\theta_{\tau_{n_j}-\tau}\omega)x_{n_j}$. Let $\varphi(t,\tau_{n_j},\theta_{\tau_{n_j}-\tau}\omega)x_{n_j}\to\phi$, by (2.1), we know that $\phi\in A(t,\theta_{t-\tau}\omega)$. Obviously $y=\phi$, which implies

$$\varphi(t,\tau,\omega)A(\tau,\omega)\subset A(t,\theta_{t-\tau}\omega). \tag{3.3}$$

Conversely, if $\phi\in A(t,\theta_{t-\tau}\omega)$, by (2.1) there exist $\tau_n\le t$ and $x_n\in B(\tau_n,\theta_{\tau_n-\tau}\omega)=B(\tau_n,\theta_{\tau_n-t}\theta_{t-\tau}\omega)$ such that

$$\varphi(t,\tau_n,\theta_{\tau_n-t}\theta_{t-\tau}\omega)x_n=\varphi(t,\tau_n,\theta_{\tau_n-\tau}\omega)x_n\to\phi,$$

and

$$\varphi(t,\tau_n,\theta_{\tau_n-\tau}\omega)x_n=\varphi(t,\tau,\omega)\varphi(\tau,\tau_n,\theta_{\tau_n-\tau}\omega)x_n.$$

φ is $\mathcal{D}$-pullback limit set compact, by Lemma 3.1, $\varphi(\tau,\tau_n,\theta_{\tau_n-\tau}\omega)x_n$ has a convergent subsequence, we denote by $\varphi(\tau,\tau_{n_j},\theta_{\tau_{n_j}-\tau}\omega)x_{n_j}$ and $\varphi(\tau,\tau_{n_j},\theta_{\tau_{n_j}-\tau}\omega)x_{n_j}\to\psi$, we know $\psi\in A(\tau,\omega)$, and

$$\varphi(t,\tau_{n_j},\theta_{\tau_{n_j}-\tau}\omega)x_{n_j}=\varphi(t,\tau,\omega)\varphi(\tau,\tau_{n_j},\theta_{\tau_{n_j}-\tau}\omega)x_{n_j}\rightharpoonup\varphi(t,\tau,\omega)\psi,$$

hence, $\varphi(t,\tau,\omega)\psi=\phi$, which shows that

$$\varphi(t,\tau,\omega)A(\tau,\omega)\supset A(t,\theta_{t-\tau}\omega). \tag{3.4}$$

Combinbing with (3.3), we have

$$\varphi(t,\tau,\omega)A(\tau,\omega)=A(t,\theta_{t-\tau}\omega).$$

Now we prove that

$$d(\varphi(t,\tau,\theta_{\tau-t}\omega)D(\tau,\theta_{\tau-t}\omega),A(t,\omega))=0,\ \forall D'\in\mathcal{D}.$$

Since $B' = \{B(t,\omega) : t \in \mathbb{R}, \omega \in \Omega\}$ is a $\mathcal{D}$-pullback absorbing set of φ, for any $D' \in \mathcal{D}$, there exists $T \le t$ such that $\varphi(t,\tau,\theta_{\tau-t}\omega)D(\tau,\theta_{\tau-t}\omega) \subset B(t,\omega)$ for any $\tau \le T$. For any $s \ge t$

$$\begin{aligned}&d(\varphi(s,\tau,\theta_{\tau-s}\omega)D(\tau,\theta_{\tau-s}\omega), A(s,\omega))\\ =\,&d(\varphi(s,t,\theta_{t-s}\omega)\varphi(t,\tau,\theta_{\tau-s}\omega)D(\tau,\theta_{\tau-s}\omega), A(s,\omega))\\ \le\,&d(\varphi(s,t,\theta_{t-s}\omega)B(t,\theta_{t-s}\omega), A(s,\omega)).\end{aligned}$$

By (2) of Theorem 3.1, we know that $d(\varphi(s,t,\theta_{t-s}\omega)B(t,\theta_{t-s}\omega), A(s,\omega)) \to 0$ as $t \to -\infty$. Hence

$$\lim_{\tau\to-\infty} d(\varphi(s,\tau,\theta_{\tau-s}\omega)D(\tau,\theta_{\tau-s}\omega), A(s,\omega)) = 0,$$

which shows that $A(s,\omega)$ attracts D'. □

Remark 3.1. By (3) of Theorem 3.1, we see that the random $\mathcal{D}$-pullback attractors $\mathcal{A} = \{A(t,\omega) : t \in \mathbb{R}, \omega \in \Omega\}$ given in (3.2) is unique and minimal.

Now we present a method to verify the pullback limit set compact of the RDS (φ,θ) in X.

Definition 3.1. A RDS (φ,θ) on a Banach space X is said to be pullback flattening if for every random bounded set $B' = \{B(t,\omega) : t \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, for any $\varepsilon > 0$ and $\omega \in \Omega$ there exists a $T(B',\varepsilon,\omega) < t$ and a finite dimensional subspace X_ϵ such that

(i) $P(\bigcup_{\tau\le T_\varepsilon} \varphi(t,\tau,\theta_{\tau-t}\omega)B(\tau,\theta_{\tau-t}\omega))$ is bounded, and

(ii)$||(I-P)(\bigcup_{\tau\le T_\varepsilon} \varphi(t,\tau,\theta_{\tau-t}\omega)B(\tau,\theta_{\tau-t}\omega))||_X < \varepsilon$,

where $P : X \to X_\varepsilon$ is a bounded projector.

Theorem 3.3. *A RDS (φ,θ) on a Banach space X satisfying pullback flattening is pullback limit set compact. Moreover, if X is a uniform convex Banach space then the converse is true.*

Proof. From (2) of Lemma 2.1, for any $t \in \mathbb{R}$, $\omega \in \Omega$, $B' \in \mathcal{D}$, $\varepsilon > 0$, there exists $T(B',\varepsilon,\omega) < t$ such that

$$\begin{aligned}\alpha(\textstyle\bigcup_{\tau\le T} \varphi(t,\tau,\theta_{\tau-t}\omega)B(\tau,\theta_{\tau-t}\omega)) &\le \alpha(P\textstyle\bigcup_{\tau\le T} \varphi(t,\tau,\theta_{\tau-t}\omega)B(\tau,\theta_{\tau-t}\omega))\\ &\quad + \alpha((I-P)\textstyle\bigcup_{\tau\le T} \varphi(t,\tau,\theta_{\tau-t}\omega)B(\tau,\theta_{\tau-t}\omega))\\ &\le 0 + \alpha(B_X(0,\varepsilon)) = 2\varepsilon,\end{aligned}$$

where $B_X(0,\varepsilon)$ is the open ball in X with centre 0 and radius ε, $P : X \to X_\varepsilon$ and the dimension of X_ε is finite. This means that the RDS (φ,θ) is pullback limit-set compact.

On the other hand, if (φ,θ) is pullback limit set compact, for any $\varepsilon > 0$, there exists $T(B',\varepsilon,\omega) < t$ such that $\alpha(\bigcup_{\tau\le T}\varphi(t,\tau,\theta_{\tau-t}\omega)B(\tau,\theta_{\tau-t}\omega)) < \varepsilon$. By Definition 2.6, there exist finite subsets of $A_1, A_2, \cdots, A_n$ with diameters less than ε. Let $x_i \in A_i$ and $X_\varepsilon = span\{x_1, x_2, \cdots, x_n\}$. Since X is a uniformly convex Banach space, there exists a projector $P : X \to X_\varepsilon$ such that for any $x \in X$,

$||x-Px|| = d(x, X_\varepsilon)$. So $||(I-P)x|| \le dist(x, \{x_1, x_2, \cdots, x_n\}) \le \varepsilon$ for any $x \in \bigcup_{\tau \le T} \varphi(t, \tau, \theta_{\tau-t}\omega)B(\tau, \theta_{\tau-t}\omega)$. Hence (φ, θ) is pullback flattening.

In fact, if X is a uniformly convex Banach space, the following three properties of a random dynamical system on X are equivalent:

(1) pullback limit set compact;

(2) pullback flattening;

(3) pullback asymptotically compact.

The detailed proof can be found in [7]. □

By Theorem 3.2 and Theorem 3.3, we get the following result:

Theorem 3.4. *Suppose that (φ, θ) is a norm-to-weak continuous RDS on a uniformly convex Banach space X. If (φ, θ) possesses a random $\mathcal{D}-$pullback bounded absorbing sets $B' = \{B(t, \omega) : t \in \mathbb{R}, \omega \in \Omega\}$ and (φ, θ) is pullback flattening, i.e., for any $\varepsilon > 0$ and $\omega \in \Omega$ there exists a $T(B', \varepsilon, \omega) < t$ and a finite dimensional subspace X_ϵ such that*

(i) $P(\bigcup_{\tau \le T_\varepsilon} \varphi(t, \tau, \theta_{\tau-t}\omega)B(\tau, \theta_{\tau-t}\omega))$ is bounded,

(ii)$||(I-P)(\bigcup_{\tau \le T_\varepsilon} \varphi(t, \tau, \theta_{\tau-t}\omega)B(\tau, \theta_{\tau-t}\omega))||_X < \varepsilon$,

where $P : X \to X_\varepsilon$ is a bounded projector, then there exists a random $\mathcal{D}$-pullback attractor $\mathcal{A} = \{A(t, \omega) : t \in \mathbb{R}, \omega \in \Omega\}$ and

$$A(t, \omega) = \bigcap_{s \le t} \overline{\bigcup_{\tau \le s} \varphi(t, \tau, \theta_{\tau-t}\omega)B(\tau, \theta_{\tau-t}\omega)}.$$

4. Random $\mathcal{D}$-pullback attractors for reaction diffusion equations

In this section, we will use our abstract theory developed in Section 3 to obtain the $\mathcal{D}$-pullback attractors for non-autonomous stochastic reaction-diffusion equation (SRDE) with multiplicative noise on the bounded domian $D \subset \mathbb{R}^n$, i.e., we consider the following equation

$$du - (\Delta u - f(u) + g(t))dt = bu(t) \circ dW(t), \quad x \in D, t \in \mathbb{R}, \tag{4.1}$$

with the initial boundary value conditions

$$\begin{cases} u(x, \tau) = u_\tau(x), \quad x \in D, \tau \le t, \\ u(x, t)|_{\partial D} = 0, \end{cases} \tag{4.2}$$

where $g(\cdot) \in L^2_{loc}(\mathbb{R}, L^2(D))$ and there exist $p \ge 2, l > 0, \alpha_i, k_i > 0 (i = 1, 2, 3)$ such that

$$\alpha_1|u|^p - k_1 \le f(u)u \le \alpha_2|u|^p + k_2, \tag{4.3}$$

$$|f(u)| \le \alpha_3|u|^{p-1} + k_3, \tag{4.4}$$

$$f_u(u) \ge -l, \tag{4.5}$$

for all $u \in \mathbb{R}$, $W(t)$ is a two-sided real-valued Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where

$$\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\},$$

$\mathcal{F}$ is the Borel algebra induced by the compact open topology of Ω, and $\mathbb{P}$ is the corresponding Wiener measure on $\{\Omega, \mathcal{F}\}$. We identify $\omega(t)$ with $W(t)$, i.e.,

$$W(t) = W(t, \omega) = \omega(t), t \in \mathbb{R}.$$

Define the Wiener time shift by

$$\theta_t \omega(s) = \omega(s+t) - \omega(t), \omega \in \Omega, t, s \in \mathbb{R}.$$

Then $(\Omega, \mathcal{F}, \mathbb{P}, \theta_t)$ is an ergodic metric dynamical system. We introduce an Ornstein-Uhlenbeck process,

$$z(\theta_t(\omega)) := -\int_{-\infty}^{0} e^{\tau}(\theta_t \omega)(\tau) d\tau, t \in \mathbb{R}.$$

We known from [6], it is the solution of Langevin equation

$$dz + z dt = dW(t).$$

From [2, 10, 11, 18–21, 24–26], it is known that the random variable $z(\omega)$ is tempered and there exists a θ_t-invariant set of full measure $\widetilde{\Omega} \subset \Omega$ such that for all $\omega \in \widetilde{\Omega}$:

$$\lim_{t \to \pm\infty} \frac{z(\theta_t \omega)}{t} = 0, \quad \lim_{t \to \pm\infty} \frac{1}{t} \int_0^t z(\theta_s \omega) ds = 0. \tag{4.6}$$

We set $A := -\Delta$, λ denotes the first eigenvalue of A, and $H = L^2(D)$ with a scalar product $(\cdot, \cdot)$ and a norm $|\cdot|$. Let $\|\cdot\|$ the norm of $H_0^1(D)$, $|\cdot|_p$ the norm of $L^p(D)$. Moreover, we suppose for any $t \in \mathbb{R}$, there exists $0 < \varepsilon_0 \leq \frac{\lambda}{4}$ such that

$$\int_{-\infty}^{t} e^{(\lambda - \varepsilon_0)s} |g(s)|^2 ds < \infty, \text{ for all } t \in \mathbb{R}, \tag{4.7}$$

which implies that

$$\int_{-\infty}^{t} e^{\alpha s} |g(s)|^2 ds < \infty, \text{ for all } t \in \mathbb{R}, \quad \alpha \geq \lambda - \varepsilon_0.$$

Let $v(s) = e^{-bz(\theta_{s-t}\omega)} u(s)$ for $s \leq t$, then $dv = -be^{-bz(\theta_{s-t}\omega)} u(s) dz + e^{-bz(\theta_{s-t}\omega)} du$, by Langevin equation and (4.1), we get the following evolution equation with random parameter but without white noise:

$$\frac{dv}{ds} - \Delta v - bz(\theta_{s-t}\omega) v(s) + e^{-bz(\theta_{s-t}\omega)} f(u) = e^{-bz(\theta_{s-t}\omega)} g(s), \ x \in D, \ s \leq t, \tag{4.8}$$

and with the initial boundary value conditions

$$\begin{cases} v(x, \tau) = e^{-bz(\theta_{\tau-t}\omega)} u(\tau), & x \in D, \\ v(x, s)|_{\partial D} = 0, \end{cases} \tag{4.9}$$

where $v(t) = e^{-bz(\omega)} u(t)$. By using Proposition 4.3.3 in [1], there exists random variable $r(\omega) > 0$ such that

$$\frac{1}{r(\omega)} \leq e^{-bz(\omega)} \leq r(\omega).$$

By a standard Galerkin approximation method, we can show that for all $v_\tau \in L^2(D)$ and $\omega \in \widetilde{\Omega}$, the equations (4.8)-(4.9) admit a unique weak solution (see [13, 14, 16, 22])

$$v(t,\tau,\omega)v_\tau \in C([\tau,T];L^2(D)) \cap L^2(\tau,T;H_0^1(D)) \cap L^p(\tau,T;L^p(D)),$$

and $v(t,\tau,\omega)v_\tau$ is norm-to-weak continuous with respect to v_τ in $L^2(D)$ and $\mathcal{F}$-measurable for all $t \geq \tau$. The proof is the same as in the autonomous case and can be found in [10, 13, 16, 22, 25, 26], the first reference where this has been done in the autonomous case suffices.

For convenience, let $\varphi(t,\tau,\omega)u_\tau = u(t,\tau,\omega)$ for $u_\tau \in L^2(D)$, $t \geq \tau$ and for all $\omega \in \tilde{\Omega}$. Obviously $\varphi(t,\tau,\theta_{\tau-t}\omega)u_\tau = u(t,\tau,\theta_{\tau-t}\omega) = v(t,\tau,\theta_{\tau-t}\omega)e^{bz(\omega)}$ is a solution of (4.1)-(4.2) and hence $\{\varphi(t,\tau,\omega)\}$ is a norm-to-weak continuous RDS on H_0^1 and $u(t,\tau,\theta_{\tau-t}\omega)u_\tau$ is $\mathcal{F}$-measurable for all $t \geq \tau$ and $u_\tau \in L^2(D)$.

Let $\mathcal{D}$ denote the collection of families of nonemptey subsets of H, $D \in \mathcal{D}$, $D = \{D(t,\omega) : t \in \mathbb{R}, \omega \in \tilde{\Omega}, D(t,\omega) \subset H\}$ satisfies

$$\lim_{t\to-\infty} e^{\frac{\lambda}{2}t} r(t,\omega) = 0, \quad r(t,\omega) = \sup\{|\varphi(x)|^2 : \varphi(x) \in D(t,\omega)\}.$$

$B(r_0(t,\omega))$ denote the closed ball in H_0^1 with radius

$$r_0(t,\omega) = (\rho(\omega)(1 + \int_{-\infty}^t e^{\frac{\lambda}{4}(s-t)}|g(s)|^2 ds))^{\frac{1}{2}}.$$

Hereafter, c or $c(\omega)$ be an arbitrary positive constant, which depends on ω and may be different from line to line and even in the same line.

Theorem 4.1. *Assume that (4.3)-(4.7) hold, $u_\tau \in H$. Then for $\omega \in \widetilde{\Omega}$, there exists $T > 0$, for all $t - \tau \geq T$, the weak solution of (4.1)-(4.2) satisfies*

$$|v(t)|^2 \leq \rho(\omega)(1 + e^{-\frac{3}{4}\lambda(t-\tau)}|u(\tau)|^2 + \int_{-\infty}^t e^{\frac{3}{4}\lambda(s-t)}|g(s)|^2 ds), \tag{4.10}$$

and

$$\int_\tau^t e^{\frac{\lambda}{2}(s-t)+2b\int_s^t z(\theta_{r-t}\omega)dr}\|v\|^2 ds \leq \rho(\omega)(1 + e^{-\frac{\lambda}{4}(t-\tau)}|u_\tau|^2 + c\int_\tau^t e^{\frac{\lambda}{4}(s-t)}|g(s)|^2 ds). \tag{4.11}$$

Proof. Taking the inner product of Eq.(4.8) with $v(s)$, we obtain

$$\frac{1}{2}\frac{d}{ds}|v|^2 + \|v\|^2 - bz(\theta_{s-t}\omega)|v|^2 + e^{-2bz(\theta_{s-t}\omega)}(f(u),u) = e^{-bz(\theta_{s-t}\omega)}(g(s),v),$$

thanks to Young's inequality and Poincaré inequality, we get

$$|e^{-bz(\theta_{s-t}\omega)}(g(s),v)| \leq \frac{\lambda}{4}|v|^2 + \frac{1}{\lambda}e^{-2bz(\theta_{s-t}\omega)}|g(s)|^2, \quad \lambda|v|^2 \leq \|v\|^2.$$

It follows that

$$\frac{d}{ds}|v|^2 + \frac{3}{2}\|v\|^2 - 2bz(\theta_{s-t}\omega)|v|^2 + 2e^{-2bz(\theta_{s-t}\omega)}(f(u,u) \leq \frac{2}{\lambda}e^{-2bz(\theta_{s-t}\omega)}|g(s)|^2.$$

By (4.3), we get

$$\begin{aligned}&\frac{d}{ds}|v|^2+\frac{3}{2}\|v\|^2-2bz(\theta_{s-t}\omega)|v|^2+2\alpha_1 e^{-2bz(\theta_{s-t}\omega)}|u|_p^p\\ &\leq 2k_1 m(D)e^{-2bz(\theta_{s-t}\omega)}+\frac{2}{\lambda}e^{-2bz(\theta_{s-t}\omega)}|g(s)|^2,\end{aligned}$$

here $m(D)$ denotes the measure of D, which implies that

$$\frac{d}{ds}|v|^2+(\lambda-2bz(\theta_{s-t}\omega))|v|^2+2\alpha_1 e^{-2bz(\theta_{s-t}\omega)}|u|_p^p\leq ce^{-2bz(\theta_{s-t}\omega)}(1+|g(s)|^2). \tag{4.12}$$

$$\frac{d}{ds}|v|^2+(\frac{\lambda}{2}-2bz(\theta_{s-t}\omega))|v|^2+\|v\|^2\leq ce^{-2bz(\theta_{s-t}\omega)}(1+|g(s)|^2). \tag{4.13}$$

By (4.12), we obtain

$$\begin{aligned}&\frac{d}{ds}(e^{\lambda s-2b\int_\tau^s z(\theta_{r-t}\omega)dr}|v|^2)+2\alpha_1 e^{\lambda s-2b\int_\tau^s z(\theta_{r-t}\omega)dr-2bz(\theta_{s-t}\omega)}|u|_p^p\\ &\leq ce^{\lambda s-2b\int_\tau^s z(\theta_{r-t}\omega)dr-2bz(\theta_{s-t}\omega)}(1+|g(s)|^2).\end{aligned} \tag{4.14}$$

Integrating (4.14) with respect to s over (τ,t), we get

$$\begin{aligned}&|v(t)|^2+2\alpha_1\int_\tau^t e^{\lambda(s-t)+2b\int_s^t z(\theta_{r-t}\omega)dr-2bz(\theta_{s-t}\omega)}|u|_p^p ds\\ &\leq e^{-\lambda(t-\tau)+2b\int_\tau^t z(\theta_{r-t}\omega)dr}|v(\tau)|^2\\ &\quad+c\int_\tau^t e^{\lambda(s-t)+2b\int_s^t z(\theta_{r-t}\omega)dr-2bz(\theta_{s-t}\omega)}(1+|g(s)|^2)ds\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}|v(t)|^2\leq & e^{-\lambda(t-\tau)+2b\int_\tau^t z(\theta_{r-t}\omega)dr-2bz(\theta_{\tau-t}\omega)}|u(\tau)|^2\\ &+c\int_\tau^t e^{\lambda(s-t)+2b\int_s^t z(\theta_{r-t}\omega)dr-2bz(\theta_{s-t}\omega)}(1+|g(s)|^2)ds.\end{aligned} \tag{4.16}$$

By (4.6), for $\omega\in\widetilde{\Omega}$ and $\varepsilon>0$, there exists $B(\varepsilon)>0$, such that for any $|t|\geq B$, we have

$$\frac{2b}{|t|}(|\int_0^t z(\theta_s\omega)ds|+|z(\theta_t\omega)|)<\varepsilon. \tag{4.17}$$

Taking $\varepsilon=\frac{\lambda}{4}$, for any $t\in\mathbb{R}$, we find that

$$|2b|(|\int_0^t z(\theta_s\omega)ds|+|z(\theta_t\omega)|)<c(\omega)+\frac{\lambda}{4}|t|, \tag{4.18}$$

and

$$\begin{aligned}&\int_\tau^t e^{\lambda(s-t)+2b\int_s^t z(\theta_{r-t}\omega)dr-2bz(\theta_{s-t}\omega)}(1+|g(s)|^2)ds\\ &=\int_{\tau-t}^0 e^{\lambda s+2b\int_s^0 z(\theta_r\omega)dr-2bz(\theta_s\omega)}(1+|g(s+t)|^2)ds\end{aligned}$$

$$\begin{aligned}
&\le e^{c(\omega)}\int_{\tau-t}^{0} e^{3\lambda s/4}(1+|g(s+t)|^2)ds\\
&= e^{c(\omega)}\int_{\tau}^{t} e^{3\lambda(s-t)/4}(1+|g(s)|^2)ds\\
&\le \rho(\omega)(1+\int_{-\infty}^{t} e^{3\lambda(s-t)/4}|g(s)|^2ds),
\end{aligned}$$

where $\rho(\omega)=\max\{\frac{1}{\lambda},\frac{c}{\lambda},1,c\}e^{c(\omega)}$. By (4.15), we get

$$|v(t)|^2 \le \rho(\omega)(1+e^{-3\lambda(t-\tau)/4}|u(\tau)|^2+\int_{-\infty}^{t} e^{3\lambda(s-t)/4}|g(s)|^2ds), \tag{4.19}$$

and

$$\begin{aligned}
&\int_{\tau}^{t} e^{\lambda(s-t)+2b\int_s^t z(\theta_{r-t}\omega)dr-2bz(\theta_{s-t}\omega)}|u|_p^p ds\\
&\le \rho(\omega)(1+e^{-3\lambda(t-\tau)/4}|u(\tau)|^2+\int_{-\infty}^{t} e^{3\lambda(s-t)/4}|g(s)|^2ds).
\end{aligned} \tag{4.20}$$

By (4.13), we obtain

$$\begin{aligned}
&\frac{d}{ds}e^{\frac{\lambda}{2}s-2b\int_\tau^s z(\theta_{r-t}\omega)dr}|v|^2\\
&\le -e^{\frac{\lambda}{2}s-2b\int_\tau^s z(\theta_{r-t}\omega)dr}\|v\|^2+ce^{\frac{\lambda}{2}s-2b\int_\tau^s z(\theta_{r-t}\omega)dr-2bz(\theta_{s-t}\omega)}(1+|g(s)|^2).
\end{aligned}$$

Integrating with respect to s over (τ,t), we get

$$\begin{aligned}
&\int_{\tau}^{t} e^{\frac{\lambda}{2}(s-t)+2b\int_s^t z(\theta_{r-t}\omega)dr}\|v\|^2ds\\
\le& e^{-\frac{\lambda}{2}(t-\tau)+2b\int_\tau^t z(\theta_{r-t}\omega)dr-2bz(\theta_{\tau-t}\omega)}|u_\tau|^2\\
&+c\int_{\tau}^{t} e^{\frac{\lambda}{2}(s-t)+2b\int_s^t z(\theta_{r-t}\omega)dr-2bz(\theta_{s-t}\omega)}(1+|g(s)|^2)\\
\le&\rho(\omega)(1+e^{-\frac{\lambda}{4}(t-\tau)}|u_\tau|^2+\int_{\tau}^{t} e^{\frac{\lambda}{4}(s-t)}|g(s)|^2ds).
\end{aligned}$$

The proof of Theorem 4.1 has been completed. □

Theorem 4.2. *Assume that (4.3)-(4.7) hold, $u_\tau\in H$. Then for $\omega\in\widetilde{\Omega}$, there exists $T>0$, for all $t-\tau\ge T$, the weak solution of (4.1)-(4.2) satisfies*

$$\|v(t)\|^2 \le \rho(\omega)(e^{-\frac{\lambda}{4}(t-\tau)}|u_\tau|^2+\int_{-\infty}^{t} e^{\frac{\lambda}{4}(s-t)}|g(s)|^2ds). \tag{4.21}$$

Proof. Taking the inner product of Eq.(4.8) with $-\Delta v(s)$, we have

$$\frac{1}{2}\frac{d}{ds}\|v\|^2+|\Delta v|^2-bz(\theta_{s-t}\omega)\|v\|^2+e^{-bz(\theta_{s-t}\omega)}(f(u),-\Delta v)=e^{-bz(\theta_{s-t}\omega)}(g(s),-\Delta v)).$$

By (4.5), we find that

$$e^{-bz(\theta_{s-t}\omega)}(f(u),-\Delta v)=e^{-2bz(\theta_{s-t}\omega)}(f(u),-\Delta u)\ge le^{-2bz(\theta_{s-t}\omega)}|\nabla u|^2=l\|v\|^2.$$

Since

$$|e^{-bz(\theta_{s-t}\omega)}(g(s),-\Delta v))| \leq \frac{1}{2}|\Delta v|^2 + \frac{1}{2}e^{-2bz(\theta_{s-t}\omega)}|g(s)|^2,$$

it follows that

$$\frac{d}{ds}\|v\|^2 + |\Delta v|^2 - 2bz(\theta_{s-t}\omega)\|v\|^2 \leq 2l\|v\|^2 + e^{-2bz(\theta_{s-t}\omega)}|g(s)|^2$$

and

$$\frac{d}{ds}\|v\|^2 + (\lambda - 2bz(\theta_{s-t}\omega))\|v\|^2 \leq 2l\|v\|^2 + e^{-2bz(\theta_{s-t}\omega)}|g(s)|^2, \tag{4.22}$$

which implies that

$$\begin{aligned}&\frac{d}{ds}((s-\tau)e^{\lambda s-2b\int_\tau^s z(\theta_{r-t}\omega)dr}\|v\|^2)\\ =& e^{\lambda s-2b\int_\tau^s z(\theta_{r-t})dr}((1+2l(s-\tau))\|v\|^2 + (s-\tau)e^{-2bz(\theta_{s-t}\omega)}|g(s)|^2).\end{aligned} \tag{4.23}$$

Integrating from τ to t and using (4.11), we find that

$$\begin{aligned}\|v(t)\|^2 \leq& \frac{1}{t-\tau}\int_\tau^t e^{\lambda(s-t)+2b\int_s^t z(\theta_{r-t}\omega)dr}((1+2l(s-\tau))\|v\|^2\\ &+(s-\tau)e^{-2bz(\theta_{s-\tau}\omega)}|g(s)|^2)ds\\ \leq&(2l+\frac{1}{t-\tau})\int_\tau^t e^{\lambda(s-t)+2b\int_s^t z(\theta_{r-t}\omega)dr}\|v\|^2 ds\\ &+\int_\tau^t e^{\lambda(s-t)+2b\int_s^t z(\theta_{r-t}\omega)dr-2bz(\theta_{s-t}\omega)}|g(s)|^2 ds\\ \leq&\rho(\omega)(2l+\frac{1}{t-\tau})(1+e^{-\frac{\lambda}{4}(t-\tau)}|u_\tau|^2+\int_\tau^t e^{\frac{\lambda}{4}(s-t)}|g(s)|^2 ds)\\ &+\rho(\omega)\int_\tau^t e^{\frac{\lambda}{2}(s-t)}|g(s)|^2 ds\\ \leq&\rho(\omega)(1+e^{-\frac{\lambda}{4}(t-\tau)}|u_\tau|^2+\int_{-\infty}^t e^{\frac{\lambda}{4}(s-t)}|g(s)|^2 ds).\end{aligned}$$

By Theorem 4.2, we know that for the RDS $u(t,\tau,\omega)$ exists a family of random $\mathcal{D}$-pullback bounded absorbing sets $\{B(r_0(t,\omega):t\in\mathbb{R},\omega\in\tilde{\Omega}\}\in\mathcal{D}$ in H_0^1. Using Theorem 3.3, we come to the following result.

Theorem 4.3. *Assume that (4.3)-(4.7) hold, then the RDS $u(t,\tau,\omega)$ generated by Eq. (4.1)-(4.2) possesses a random $\mathcal{D}$-pullback attractor in H.*

Next, we will prove that the RDS $u(t,\tau,\omega)$ generated by Eq. (4.1)-(4.2) possesses a random $\mathcal{D}$-pullback attractor in H_0^1, i.e., we will get the following Theorem.

Theorem 4.4. *Assume that (4.3)-(4.7) hold, then the RDS $u(t,\tau,\omega)$ generated by Eq.(4.1)-(4.2) possesses a random $\mathcal{D}$-pullback attractor in H_0^1.*

By Theorem 4.2, we know that the RDS $u(t,\tau,\omega)$ exists a family of random bounded absorbing sets in H_0^1, we only proof that the RDS $u(t,\tau,\omega)$ satisfies Theorem 3.6.

Proof. Taking the inner product of Eq.(4.8) with $|v|^{p-2}v$, we get

$$\frac{1}{p}\frac{d}{ds}|v|_p^p + (p-1)\int_D |v|^{p-2}|\nabla v|^2 dx - bz(\theta_{s-t}\omega)|v|_p^p$$
$$+e^{-pbz(\theta_{s-t}\omega)}(f(u), |u|^{p-2}u) = e^{-bz(\theta_{s-t}\omega)}(g(s), |v|^{p-2}v).$$

By (4.3) and Young's inequality, we obtain

$$\frac{d}{ds}|v|_p^p - pbz(\theta_{s-t}\omega)|v|_p^p + \frac{\alpha_1}{2}pe^{-pbz(\theta_{s-t}\omega)}|u|_{2p-2}^{2p-2}$$
$$\leq ce^{-pbz(\theta_{s-t}\omega)}(1+|g(s)|^2),$$

it follows that

$$\frac{d}{ds}((s-\tau)e^{\lambda s - pb\int_\tau^s z(\theta_{r-t}\omega)dr}|v|_p^p) + \frac{\alpha_1}{2}p(s-\tau)e^{\lambda s - pb\int_\tau^s z(\theta_{r-t}\omega)dr - pbz(\theta_{s-t}\omega)}|u|_{2p-2}^{2p-2}$$
$$\leq e^{\lambda s - pb\int_\tau^s z(\theta_{r-t}\omega)dr}(1+\lambda(s-\tau)|v|_p^p + c(s-\tau)(e^{-pbz(\theta_{s-t}\omega)}(1+|g(s)|^2))).$$

Using (4.16),(4.17) and (4.20), we get

$$\begin{aligned}&\frac{1}{t-\tau}\int_\tau^t (s-\tau)e^{\lambda(s-t)+pb\int_s^t z(\theta_{r-t}\omega)dr - pbz(\theta_{s-t}\omega)}|u|_{2p-2}^{2p-2}\\ &\leq \rho(\omega)(1+\frac{1}{t-\tau} + e^{-\frac{\lambda}{4}(t-\tau)}|u(\tau)|^2 + \int_{-\infty}^t e^{\frac{\lambda}{4}(s-t)}|g(s)|^2 ds.\end{aligned} \tag{4.24}$$

Since A^{-1} is a continuous compact operator in H, by the classical spectral theorem, there exists a sequence $\{\lambda_j\}_{j=1}^\infty$ satisfing

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \lambda_j \leq \cdots, \lambda_j \to +\infty, \text{as } j \to +\infty,$$

and a family of elements $\{e_j\}_{j=1}^\infty$ of H_0^1 which are orthonormal in H such that

$$Ae_j = \lambda_j e_j,\ \forall j \in \mathbb{N}^+.$$

Let $H_m = span\{e_1, e_2, \cdots, e_m\}$ in H and $P_m : H \to H_m$ be an orthogonal projector. For any $v \in H$ we write

$$v = P_m v + (I - P_m)v = v_1 + v_2.$$

Taking the inner product of (4.8) with $-\Delta v_2$, we have

$$\frac{1}{2}\frac{d}{ds}\|v_2\|^2 + |\Delta v_2|^2 - bz(\theta_{s-t}\omega)\|v_2\|^2 + e^{-bz(\theta_{s-t}\omega)}(f(u), -\Delta v_2)$$
$$= e^{-bz(\theta_{s-t}\omega)}(g(s), -\Delta v_2)).$$

Thanks to Poincaré inequality and Young's inequality we get

$$\lambda_m\|v_2\|^2 \leq |\Delta v_2|^2,\ |e^{-bz(\theta_{s-t}\omega)}(f(u), -\Delta v_2)| \leq \frac{1}{4}|\Delta v_2|^2 + e^{-2bz(\theta_{s-t}\omega)}\int_D |f(u)|^2 dx,$$

and

$$|e^{-bz(\theta_{s-t}\omega)}(g(s),-\Delta v_2))| \leq \frac{1}{4}|\Delta v_2|^2 + e^{-2bz(\theta_{s-t}\omega)}|g(s)|^2.$$

By (4.4), we obtain

$$\int_D |f(u)|^2 dx \leq c(1+|u|_{2p-2}^{2p-2}).$$

Hence

$$\frac{d}{ds}\|v_2\|^2 + (\lambda_m - 2bz(\theta_{s-t}\omega))\|v_2\|^2 \leq ce^{-2bz(\theta_{s-t}\omega)}(1+|u|_{2p-2}^{2p-2}+|g(s)|^2).$$

Thus we have

$$\begin{aligned}&\frac{d}{ds}((s-\tau)e^{\lambda_m s-2b\int_\tau^s z(\theta_{r-t}\omega)dr}\|v_2\|^2)\\ &\leq ce^{\lambda_m s-2b\int_\tau^s z(\theta_{r-t}\omega)dr}(\|v_2\|^2+(s-\tau)e^{-2bz(\theta_{s-t}\omega)}(1+|u|_{2p-2}^{2p-2}+|g(s)|^2).\end{aligned} \tag{4.25}$$

Integrating (4.26) with respect to s from τ to t, we have

$$\begin{aligned}\|v_2(t)\|^2 \leq & c(\frac{1}{t-\tau}\int_\tau^t e^{\lambda_m(s-t)+2b\int_s^t z(\theta_{r-t}\omega)dr}\|v(s)\|^2)\\ &+\int_\tau^t e^{\lambda_m(s-t)+2b\int_s^t z(\theta_{r-t}\omega)dr-2bz(\theta_{s-t}\omega)}(1+|u|_{2p-2}^{2p-2}+|g(s)|^2).\end{aligned} \tag{4.26}$$

For the right-hand side of (4.26), using (4.7),(4.11),(4.24) and Lebesgue's dominated convergence theorem, we obtain

$$\begin{aligned}&\frac{1}{t-\tau}\int_\tau^t e^{\lambda_m(s-t)+2b\int_s^t z(\theta_{r-t}\omega)dr}\|v(s)\|^2 ds\\ &\leq \frac{\rho(\omega)}{t-\tau}\int_\tau^t e^{(\lambda_m-\frac{\lambda}{2})(s-t)}e^{\frac{\lambda}{2}(s-t)+2b\int_s^t z(\theta_{r-t}\omega)}\|v\|^2 ds \to 0,\ as\ m\to+\infty,\end{aligned} \tag{4.27}$$

$$\begin{aligned}&\int_\tau^t e^{\lambda_m(s-t)+2b\int_s^t z(\theta_{r-t}\omega)dr-2bz(\theta_{s-t}\omega)}ds\\ &\leq \rho(\omega)\int_\tau^t e^{(\lambda_m-\frac{\lambda}{4})(s-t)}ds\\ &\leq \frac{4\rho(\omega)}{4\lambda_m-\lambda}\to 0 \quad \text{as} \quad m\to+\infty,\end{aligned} \tag{4.28}$$

$$\begin{aligned}&\int_\tau^t e^{\lambda_m(s-t)+2b\int_s^t z(\theta_{r-t}\omega)dr-2bz(\theta_{s-t}\omega)}|u|_{2p-2}^{2p-2}ds\\ &=\int_\tau^t e^{(\lambda_m-\lambda)(s-t)-(p-2)b\int_s^t z(\theta_{r-t}\omega)dr+(p-2)bz(\theta_{s-t}\omega)},\end{aligned} \tag{4.29}$$

$$\begin{aligned}&e^{\lambda(s-t)+pb\int_s^t z(\theta_{r-t}\omega)dr-pbz(\theta_{s-t}\omega)}|u|_{2p-2}^{2p-2}ds\\ &\leq \rho(\omega)\int_\tau^t e^{(\lambda_m-\frac{3\lambda}{2})(s-t)}e^{\lambda(s-t)+pb\int_s^t z(\theta_{r-t}\omega)dr-pbz(\theta_{s-t}\omega)}|u|_{2p-2}^{2p-2}ds\\ &\to 0 \text{ as } m\to+\infty,\end{aligned} \tag{4.30}$$

and

$$\begin{aligned}&\int_{\tau}^{t} e^{\lambda_m(s-t)+2b\int_s^t z(\theta_{r-t}\omega)dr-2bz(\theta_{s-t}\omega)}|g(s)|^2 ds\\ \le\ &\rho(\omega)\int_{-\infty}^{t} e^{(\lambda_m-\frac{\lambda}{4})(s-t)}|g(s)|^2 ds\\ \to\ &0 \text{ as } m\to+\infty.\end{aligned}\tag{4.31}$$

Then by (4.26)-(4.30), for any $\varepsilon > 0$, there exists $B \in \mathbb{N}^+$, for any $m \ge B$, we have

$$\|v_2(t,\tau,\omega)\|^2 < \varepsilon,$$

which means that the RDS $u(t,\tau,\omega)$ satisfies pullback flattening. □

Acknowledgements. The authors would like to express their sincere thanks to the referee for his/her valuable comments and suggestions.

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