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EXISTENCE OF NONOSCILLATORY SOLUTIONS FOR SYSTEM OF FRACTIONAL DIFFERENTIAL EQUATIONS WITH POSITIVE AND NEGATIVE COEFFICIENTS*

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Abstract In this paper we consider the system of fractional differential equations with positive and negative coefficients. We use the *Banach* contraction principle to obtain new sufficient conditions for the existence of nonoscillatory solutions.

Keywords System, fractional differential equation, Liouville derivative, positive and negative coefficients, nonoscillatory solutions.

MSC(2010) 34A08, 34K11, 35K99.

1. Introduction

In this paper, we consider the system of fractional differential equations with positive and negative coefficients

$$D_t^\alpha \left[r(t)\mathbf{x}(t) + P(t)\mathbf{x}(t-\theta)\right]' - Q_1(t)\mathbf{x}(t-\tau) + Q_2(t)\mathbf{x}(t-\sigma) = \mathbf{h}(t), \tag{1.1}$$

where D_t^α is Liouville fractional derivatives of order $\alpha \geq 0$ on the half-axis, $\theta, \tau, \sigma > 0$, $r \in C\big([t_0,\infty), R^+\big), P \in C\big([t_0,\infty)\times[a,b], R\big), \mathbf{h} \in C\big([t_0,\infty), \mathbf{R}^n\big)$, $\mathbf{x} \in \mathbf{R}^n, Q_i$ is continuous $n \times n$ matrix on $[t_0,\infty), i = 1, 2$.

Fractional differential equations have attracted extensive attention because of their wide application covering multiple fields of chemical physics, control theory of dynamical systems, rheology, fluid flows,electrical networks and economics. As lately reported, various achievements on the partial differential equations as well as fractional-order ordinary have been attained [3, 8, 9, 12–14].

As the significance of oscillation theory in achieving favorable information on the qualitative properties of solutions of differential equations, during the past decades, oscillation theory has been widely investigated for classical functional differential equations [1, 4–7].

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In 2013, Candan [2] studied the existence of nonoscillatory solutions for system of higher order nonliear neutral differential equations

$$[\mathbf{x}(t)+P(t)\mathbf{x}(t-\theta)]^{(n)}+(-1)^{n+1}[Q_1(t)\mathbf{x}(t-\sigma_1)-Q_2(t)\mathbf{x}(t-\sigma_2)]=\mathbf{0}, \quad (1.2)$$

However, the discussed condition for coefficient $P(t)$ was $(-\infty,-2)$, $(-\frac{1}{2},0)$, $[0,\frac{1}{2})$, $(2,+\infty)$. Recently, We noticed that the nonoscillatory theory for fractional differential equations [10, 11]. Nevertheless, as far as we are acquainted, the nonoscillatory theory for system of fractional differential equations with positive and negative coefficients has not been reported yet.

Hence, in this paper, we considered the system of fractional differential equations, skillfully introduced coefficient $r(t)$ and constructed the new operator, where the scope of the coefficient $P(t)$ of neutral section in literature was expanded to $(-\infty,-1),(-1,0],[0,1),(1,+\infty)$, and the sufficient condition for the existence of nonoscillatory solutions of fractional differential equation was obtained.Thus, this paper may present its theoretical value as well as practical application value.

2. Preliminaries

In this section,we will introduce the preliminary details which are used throughout this paper.

Definition 2.1. As usual, a continuous function $x(t)$ defined on $[t_0,\infty)$ is said to be oscillatory if it has arbitrarily large zeros. Otherwise the solution is said to be nonoscillatory.

Definition 2.2. The vector solution $\mathbf{x}(t)=\{x_1(t),x_2(t),\cdots,x_n(t)\}^\top$ of equation (1.1) is said to be oscillatory in $[t_0,\infty)$ if at least one of its nontrivial components is oscillatory based on Definition 1. Otherwise, the vector solution $\mathbf{x}(t)$ is said to be nonoscillatory.

Definition 2.3. A solution of system of equation (1.1) is a continuous vector function $\mathbf{x}(t)$ defined on $([t_1-\mu,\infty),\mathbf{R}^n)$, for some $t_1>t_0$, such that $D_t^\alpha[r(t)\mathbf{x}(t)+P(t)\mathbf{x}(t-\theta)]'$ exist on $[t_0,\infty)$ and system of equation (1.1) holds for all $t_1>t_0$. Here, $\mu=\max\{\theta,\tau,\sigma\}$.

Definition 2.4 ([8])**.** The Liouville fractional derivative on the half-axis is defined by

$$D_t^{-\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_t^\infty(s-t)^{\alpha-1}f(s)ds,$$

where $t\in R$ and $\alpha\in[0,\infty)$.

Definition 2.5 ([8])**.** The Liouville fractional derivative on the half-axis is defined by

$$D_t^\alpha f(t)=\frac{d^n}{dt^n}(D_t^{-(n-\alpha)}f(t))=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_t^\infty(s-t)^{n-\alpha-1}f(s)ds,$$

where $n=[\alpha]+1,\alpha\in[0,\infty),[\alpha]$ denotes the integer part of α and $t\in R$. In particular, if $\alpha=n\in N$, then$D_t^n f(t)=f^{(n)}(t)$, where $f^{(n)}(t)$ is the usual derivative of $f(t)$ of order n.

Property 2.1. ([8]) *For* $\alpha > 0$,

$$D_t^{\alpha}(D_t^{-\alpha} f)(t) = f(t).$$

3. Main results

Theorem 3.1. *Assume that* $0 \le P(t) \le p_1 < 1$ *and*

$$\int_{t_0}^{\infty} s^{\alpha}\|Q_i(s)\|ds < \infty, i = 1,2, \qquad \int_{t_0}^{\infty} s^{\alpha}\|\boldsymbol{h}(s)\|ds < \infty. \tag{3.1}$$

Then equation (1.1) has a bounded nonoscillatory solution.

Proof. Let Λ be the set of all continuous and bounded vector functions on $[t_0,\infty)$ with the sup norm. Let $\mathbf{x}(t) = \{x_1(t), x_2(t), \cdots, x_n(t)\}^{\top}$. Set$A = \{\mathbf{x} \in \Lambda, x_i(t) > 0$ or $x_i(t) < 0, M_1 \le \|\mathbf{x}(t)\| \le M_2, t \ge t_0, i = 1,2,\cdots,n\}$, where M_1, M_2 are two positive constants and $\mathbf{c}$ is a constant vector, such that $p_1M_2 + \dfrac{M_1}{p_1} < \|\mathbf{c}\| < M_2, 1 \le r(t) \le \dfrac{1}{p_1}$. From (3.1), one can choose a $t_1 \ge t_0 + \mu$, sufficiently large $t \ge t_1$ such that

$$\int_t^{\infty} \frac{(s-t)^{\alpha}}{\Gamma(\alpha+1)}[M_2\|Q_1(s)\| + \|\mathbf{h}(s)\|]ds \le M_2 - \|\mathbf{c}\|, \tag{3.2}$$

$$\int_t^{\infty} \frac{(s-t)^{\alpha}}{\Gamma(\alpha+1)}[M_2\|Q_2(s)\| + \|\mathbf{h}(s)\|]ds \le \|\mathbf{c}\| - p_1M_2 + \frac{M_1}{p_1}, \tag{3.3}$$

$$\int_t^{\infty} \frac{(s-t)^{\alpha}}{\Gamma(\alpha+1)}[\|Q_1(s)\| + \|Q_2(s)\|]ds < 1 - p_1, \tag{3.4}$$

and define an operator T on A as follows

$$(T\mathbf{x})(t) = \begin{cases} \dfrac{1}{r(t)}\{\mathbf{c} - P(t)\mathbf{x}(t-\theta) + \displaystyle\int_t^{\infty} \frac{(s-t)^{\alpha}}{\Gamma(\alpha+1)}[Q_1(s)\mathbf{x}(s-\tau) \\ \qquad\qquad -Q_2(s)\mathbf{x}(s-\sigma) + \mathbf{h}(s)]ds\}, & t \ge t_1, \\ (T\mathbf{x})(t_1), & t_0 \le t \le t_1. \end{cases}$$

It is easy to see that $T\mathbf{x}$ is continuous, for $t \ge t_1, \mathbf{x} \in A$, by using (3.2), we have

$$\begin{aligned} \|(T\mathbf{x})(t)\| &\le \frac{1}{r(t)}\left\{\|\mathbf{c}\| + \left\|\int_t^{\infty} \frac{(s-t)^{\alpha}}{\Gamma(\alpha+1)}[Q_1(s)\mathbf{x}(s-\tau) + \mathbf{h}(s)]ds\right\|\right\} \\ &\le \|\mathbf{c}\| + \int_t^{\infty} \frac{(s-t)^{\alpha}}{\Gamma(\alpha+1)}[M_2\|Q_1(s)\| + \|\mathbf{h}(s)\|]ds \\ &\le M_2, \end{aligned}$$

and taking (3.3) into account, we have

$$\|(T\mathbf{x})(t)\| \ge \frac{1}{r(t)}\left\{\|\mathbf{c}\| - P(t)\|\mathbf{x}(t-\theta)\| - \left\|\int_t^{\infty} \frac{(s-t)^{\alpha}}{\Gamma(\alpha+1)}[Q_2(s)\mathbf{x}(s-\sigma) + \mathbf{h}(s)]ds\right\|\right\}$$

$$\geq p_1 \left\{ \|\mathbf{c}\| - p_1 M_2 - \int_t^\infty \frac{(s-t)^\alpha}{\Gamma(\alpha+1)} (M_2\|Q_2(s)\| + \|\mathbf{h}(s)\|) ds \right\}$$
$$\geq M_1,$$

these show that $TA \subset A$. Since A is bounded, close, convex subset of Λ, in order to apply the contraction principle we have to show that T is a contraction mapping on A. For $\forall \mathbf{x}_1, \mathbf{x}_2 \in A$, and $t \geq t_1$,

$$\begin{aligned}
&\|(T\mathbf{x}_1)(t) - (T\mathbf{x}_2)(t)\| \\
\leq & \frac{1}{r(t)} \{ P(t)\|\mathbf{x}_1(t-\theta) - \mathbf{x}_2(t-\theta)\| \\
&+ \left\| \int_t^\infty \frac{(s-t)^\alpha}{\Gamma(\alpha+1)} [Q_1(s)\mathbf{x}_1(s-\tau) - Q_2(s)\mathbf{x}_1(s-\sigma) + \mathbf{h}(s)] ds \right\| \\
&- \left\| \int_t^\infty \frac{(s-t)^\alpha}{\Gamma(\alpha+1)} [Q_1(s)\mathbf{x}_2(s-\tau) - Q_2(s)\mathbf{x}_2(s-\sigma) + \mathbf{h}(s)] ds \right\| \} \\
\leq & \frac{1}{r(t)} \{ p_1\|\mathbf{x}_1 - \mathbf{x}_2\| + \int_t^\infty \frac{(s-t)^\alpha}{\Gamma(\alpha+1)} [\|Q_1(s)\| \|\mathbf{x}_1(s-\tau) - \mathbf{x}_2(s-\tau)\| \\
&+ \|Q_2(s)\| \|\mathbf{x}_1(s-\sigma) - \mathbf{x}_2(s-\sigma)\|] ds \},
\end{aligned}$$

using (3.4),

$$\begin{aligned}
\|(T\mathbf{x}_1)(t) - (T\mathbf{x}_2)(t)\| &\leq \|\mathbf{x}_1 - \mathbf{x}_2\| (p_1 + \int_t^\infty \frac{(s-t)^\alpha}{\Gamma(\alpha+1)} [\|Q_1(s)\| + \|Q_2(s)\|] ds) \\
&< \|\mathbf{x}_1 - \mathbf{x}_2\|,
\end{aligned}$$

which shows that T is a contraction mapping on A and therefore there exists a unique solution, obviously a bounded positive solution of (1.1) $\tilde{\mathbf{x}} \in A$, such that $T\tilde{\mathbf{x}} = \tilde{\mathbf{x}}$, that is

$$\tilde{\mathbf{x}}(t) = \frac{1}{r(t)} \left\{ \mathbf{c} - P(t)\tilde{\mathbf{x}}(t-\theta) + \int_t^\infty \frac{(s-t)^\alpha}{\Gamma(\alpha+1)} [Q_1(s)\tilde{\mathbf{x}}(s-\tau) - Q_2(s)\tilde{\mathbf{x}}(s-\sigma) + \mathbf{h}(s)] ds \right\},$$

which implies that

$$r(t)\tilde{\mathbf{x}}(t) - \mathbf{c} + P(t)\tilde{\mathbf{x}}(t-\theta) = \frac{1}{\Gamma(\alpha)} \int_t^\infty ds \int_t^s (s-u)^{\alpha-1} [Q_1(s)\tilde{\mathbf{x}}(s-\tau) - Q_2(s)\tilde{\mathbf{x}}(s-\sigma) + \mathbf{h}(s)] du,$$

hence

$$[r(t)\tilde{\mathbf{x}}(t) + P(t)\tilde{\mathbf{x}}(t-\theta)]' = \frac{1}{\Gamma(\alpha)} \int_t^\infty (s-t)^{\alpha-1} [Q_1(s)\tilde{\mathbf{x}}(s-\tau) - Q_2(s)\tilde{\mathbf{x}}(s-\sigma) + \mathbf{h}(s)] ds.$$

By Property 1, it is easy to see that $\tilde{x}(t)$ is a nonoscillatory solution of the equation (1.1). The proof is complete. □

Theorem 3.2. *Assume that* $1 < p_3 \leq P(t) \leq p_2 < +\infty$*, and that (3.1) holds.Then equation (1.1) has a bounded nonoscillatory solution.*

Proof. Let Λ be the set of all continuous and bounded vector functions on $[t_0, \infty)$ with the sup norm. Let $\mathbf{x}(t) = \{x_1(t), x_2(t), \cdots, x_n(t)\}^\top$. Set $A = \{x \in \Lambda, x_i(t) > 0$ or $x_i(t) < 0, M_3 \leq x(t) \leq M_4, t \geq t_0, i = 1, 2, \cdots, n\}$, where M_3, M_4 is a positive constants such that $p_2M_3 + M_4 < \|\mathbf{c}\| < p_3M_4, r(t) \le 1$. From (3.1), one can choose a $t_1 \ge t_0 + \mu$, sufficiently large $t \ge t_1$ such that

$$\int_t^\infty \frac{(s-t)^\alpha}{\Gamma(\alpha+1)}[M_4\|Q_1(s)\| + \|\mathbf{h}(s)\|]ds \le p_3M_4 - \|\mathbf{c}\|, \tag{3.5}$$

$$\int_t^\infty \frac{(s-t)^\alpha}{\Gamma(\alpha+1)}[M_4\|Q_2(s)\| + \|\mathbf{h}(s)\|]ds \le \|\mathbf{c}\| - M_4 - p_2M_3, \tag{3.6}$$

$$\int_t^\infty \frac{(s-t)^\alpha}{\Gamma(\alpha+1)}[\|Q_1(s)\| + \|Q_2(s)\|]ds < p_3 - 1, \tag{3.7}$$

and define an operator T on A as follows:

$$(T\mathbf{x})(t) = \begin{cases} \dfrac{1}{P(t+\theta)}\{\mathbf{c} - r(t+\theta)\mathbf{x}(t+\theta) + \displaystyle\int_{t+\theta}^\infty \frac{(s-t-\theta)^\alpha}{\Gamma(\alpha+1)}[Q_1(s)\mathbf{x}(s-\tau) \\ \qquad -Q_2(s)\mathbf{x}(s-\sigma) + \mathbf{h}(s)]ds\}, \quad t \ge t_1, \\ (T\mathbf{x})(t_1), \qquad t_0 \le t \le t_1. \end{cases}$$

It is easy to see that T is continuous, for $t \ge t_1, \mathbf{x} \in A$, By using (3.5), we have

$$\begin{aligned} \|(T\mathbf{x})(t)\| &\le \frac{1}{p_3}\left\{\|\mathbf{c}\| + \left\|\int_{t+\theta}^\infty \frac{(s-t-\theta)^\alpha}{\Gamma(\alpha+1)}[Q_1(s)\mathbf{x}(s-\tau) + \mathbf{h}(s)]ds]\right\|\right\} \\ &\le \frac{1}{p_3}\left\{\|\mathbf{c}\| + \int_t^\infty \frac{(s-t)^\alpha}{\Gamma(\alpha+1)}[M_4\|Q_1(s)\| + \|\mathbf{h}(s)\|]ds\right\} \\ &\le M_4, \end{aligned}$$

and taking (3.6) into account, we have

$$\begin{aligned} \|(T\mathbf{x})(t)\| &\ge \frac{1}{p_2}\left\{\|\mathbf{c}\| - r(t+\theta)\|\mathbf{x}(t+\theta)\| + \left\|\int_{t+\theta}^\infty \frac{(s-t-\theta)^\alpha}{\Gamma(\alpha+1)}[Q_2(s)\mathbf{x}(s-\sigma) + \mathbf{h}(s)]ds\right\|\right\} \\ &\ge \frac{1}{p_2}\left\{\|\mathbf{c}\| - M_4 - \int_t^\infty \frac{(s-t)^\alpha}{\Gamma(\alpha+1)}(M_4\|Q_2(s)\| + \|\mathbf{h}(s)\|)ds\right\} \\ &\ge M_3. \end{aligned}$$

These show that $TA \subset A$. Since A is bounded, close, convex subset of Λ, in order to apply the contraction principle, we have to show that T is a contraction mapping on A. For $\forall \mathbf{x}_1, \mathbf{x}_2 \in A$, and $t \ge t_1$,

$$\begin{aligned} &\|(T\mathbf{x}_1)(t) - (T\mathbf{x}_2)(t)\| \\ \le &\frac{1}{P(t+\theta)}\{r(t+\theta)\|\mathbf{x}_1(t+\theta) - \mathbf{x}_2(t+\theta)\| \\ &+ \left\|\int_{t+\theta}^\infty \frac{(s-t-\theta)^\alpha}{\Gamma(\alpha+1)}[Q_1(s)\mathbf{x}_1(s-\tau) - Q_2(s)\mathbf{x}_1(s-\sigma) + \mathbf{h}(s)]ds\right\| \\ &- \left\|\int_{t+\theta}^\infty \frac{(s-t-\theta)^\alpha}{\Gamma(\alpha+1)}[Q_1(s)\mathbf{x}_2(s-\tau) - Q_2(s)\mathbf{x}_2(s-\sigma) + \mathbf{h}(s)]ds\right\|\} \\ \le &\frac{1}{p_3}\{\|\mathbf{x}_1 - \mathbf{x}_2\| + \int_{t+\theta}^\infty \frac{(s-t-\theta)^\alpha}{\Gamma(\alpha+1)}[\|Q_1(s)\|\|\mathbf{x}_1(s-\tau) - \mathbf{x}_2(s-\tau)\| \\ &+ \|Q_2(s)\|\|\mathbf{x}_1(s-\sigma) - \mathbf{x}_2(s-\sigma)\|]ds\}, \end{aligned}$$

using (3.7),

$$\|(T\mathbf{x}_1)(t)-(T\mathbf{x}_2)(t)\| \le \frac{1}{p_3}\|\mathbf{x}_1-\mathbf{x}_2\|\left\{1+\int_{t+\theta}^{\infty}\frac{(s-t-\theta)^{\alpha}}{\Gamma(\alpha+1)}[\|Q_1(s)\|+\|Q_2(s)\|]ds\right\}$$
$$< \|\mathbf{x}_1-\mathbf{x}_2\|,$$

which shows that T is a contraction mapping on A and therefore there exists a unique solution, obviously a bounded positive solution of (1.1) $\mathbf{x} \in A$, such that $T\mathbf{x} = \mathbf{x}$. The proof is complete. □

Theorem 3.3. *Assume that* $-1 < p_4 \le P(t) \le 0$ *and that (3.1) holds. Then equation (1.1) has a bounded nonoscillatory solution.*

Proof. Let Λ be the set of all continuous and bounded vector functions on $[t_0,\infty)$ with the sup norm. Let $\mathbf{x}(t) = \{x_1(t), x_2(t), \cdots, x_n(t)\}^{\top}$. Set $A = \{x \in \Lambda, x_i(t) > 0$ or $x_i(t) < 0, M_5 \le \|\mathbf{x}(t)\| \le M_6, t \ge t_0, i = 1,2,\cdots,n\}$, where M_5, M_6 is two positive constants such that $\frac{M_5}{-p_4} < \|\mathbf{c}\| < (1+p_4)M_6, 1 \le r(t) \le \frac{1}{-p_4}$. From (3.1), one can choose a $t_1 \ge t_0 + \mu$, sufficiently large $t \ge t_1$ such that

$$\int_t^{\infty}\frac{(s-t)^{\alpha}}{\Gamma(\alpha+1)}[M_6\|Q_1(s)\|+\|\mathbf{h}(s)\|]ds \le (1+p_4)M_6 - \|\mathbf{c}\|, \tag{3.8}$$

$$\int_t^{\infty}\frac{(s-t)^{\alpha}}{\Gamma(\alpha+1)}[M_6\|Q_2(s)\|+\|\mathbf{h}(s)\|]ds \le \|\mathbf{c}\| + \frac{M_5}{p_4}, \tag{3.9}$$

$$\int_t^{\infty}\frac{(s-t)^{\alpha}}{\Gamma(\alpha+1)}[\|Q_1(s)\|+\|Q_2(s)\|]ds < 1+p_4,$$

and define an operator T on A as follows

$$(T\mathbf{x})(t) = \begin{cases} \frac{1}{r(t)}\{\mathbf{c} - P(t)\mathbf{x}(t-\theta) + \int_t^{\infty}\frac{(s-t)^{\alpha}}{\Gamma(\alpha+1)}[Q_1(s)\mathbf{x}(s-\tau) \\ \qquad -Q_2(s)\mathbf{x}(s-\sigma)+\mathbf{h}(s)]ds\}, & t \ge t_1, \\ (T\mathbf{x})(t_1), & t_0 \le t \le t_1. \end{cases}$$

It is easy to see that T is continuous, for $t \ge t_1, \mathbf{x} \in A$, by using (3.8), we have

$$\|(T\mathbf{x})(t)\| \le \frac{1}{r(t)}\left\{\|\mathbf{c}\|-P(t)\|\mathbf{x}(t-\theta)\|+\left\|\int_t^{\infty}\frac{(s-t)^{\alpha}}{\Gamma(\alpha+1)}[Q_1(s)\mathbf{x}(s-\tau)+\mathbf{h}(s)]ds\right\|\right\}$$
$$\le \|\mathbf{c}\| - p_4M_6 + \int_t^{\infty}\frac{(s-t)^{\alpha}}{\Gamma(\alpha+1)}[M_6\|Q_1(s)\|+\|\mathbf{h}(s)\|]ds$$
$$\le M_6,$$

and taking (3.9) into account, we have

$$\|(T\mathbf{x})(t)\| \ge \frac{1}{r(t)}\left\{\|\mathbf{c}\| - \left\|\int_t^{\infty}\frac{(s-t)^{\alpha}}{\Gamma(\alpha+1)}[Q_2(s)\mathbf{x}(s-\sigma)+\mathbf{h}(s)]ds\right\|\right\}$$
$$\ge -p_4\left\{\|\mathbf{c}\| - \int_t^{\infty}\frac{(s-t)^{\alpha}}{\Gamma(\alpha+1)}[M_6\|Q_2(s)\|+\|\mathbf{h}(s)\|]ds\right\}$$
$$\ge M_5.$$

The remaining part of the proof is similar to that of Theorem 3.1; therefore it is omitted. The proof is complete. □

Theorem 3.4. *Assume that* $-\infty < p_6 \le P(t) \le p_5 < -1$ *and that (3.1) holds. Then equation (1.1) has a bounded nonoscillatory solution.*

Proof. Let Λ be the set of all continuous and bounded vector functions on $[t_0, \infty)$ with the sup norm. Let $\mathbf{x}(t) = \{x_1(t), x_2(t), \cdots, x_n(t)\}^\top$. Set $A = \{x \in \Lambda, x_i(t) > 0$ or $x_i(t) < 0, M_7 \le x(t) \le M_8, t \ge t_0, i = 1, 2, \cdots, n\}$, where M_7, M_8 is a positive constants such that $-p_6 M_7 < \|\mathbf{c}\| < (-p_5 - 1)M_8, r(t) \le 1$, From (3.1), one can choose a $t_1 \ge t_0 + \mu$, sufficiently large $t \ge t_1$ such that

$$\int_t^\infty \frac{(s-t)^\alpha}{\Gamma(\alpha+1)}[M_8\|Q_1(s)\| + \|\mathbf{h}(s)\|]ds \le \|\mathbf{c}\| + p_6 M_7, \tag{3.10}$$

$$\int_t^\infty \frac{(s-t)^\alpha}{\Gamma(\alpha+1)}[M_8\|Q_2(s)\| + \|\mathbf{h}(s)\|]ds \le (-p_5 - 1)M_8 - \|\mathbf{c}\|, \tag{3.11}$$

$$\int_t^\infty \frac{(s-t)^\alpha}{\Gamma(\alpha+1)}[\|Q_1(s)\| + \|Q_2(s)\|]ds < -p_5 - 1,$$

and define an operator T on A as follows

$$(T\mathbf{x})(t) = \begin{cases} \dfrac{1}{P(t+\theta)}\{-\mathbf{c} - r(t+\theta)\mathbf{x}(t+\theta) + \displaystyle\int_{t+\theta}^\infty \frac{(s-t-\theta)^\alpha}{\Gamma(\alpha+1)}[Q_1(s)\mathbf{x}(s-\tau) \\ \qquad -Q_2(s)\mathbf{x}(s-\sigma) + \mathbf{h}(s)]ds\}, \quad t \ge t_1, \\ (T\mathbf{x})(t_1), \quad t_0 \le t \le t_1. \end{cases}$$

It is easy to see that T is continuous, for $t \ge t_1, \mathbf{x} \in A$, by using (3.10), we have

$$\begin{aligned} \|(T\mathbf{x})(t)\| &\ge \frac{1}{p_6}\left\{-\|\mathbf{c}\| - r(t+\theta)\mathbf{x}(t+\theta) + \left\|\int_{t+\theta}^\infty \frac{(s-t-\theta)^\alpha}{\Gamma(\alpha+1)}[Q_1(s)\mathbf{x}(s-\tau) + \mathbf{h}(s)]ds\right\|\right\} \\ &\ge \frac{1}{p_3}\left\{-\|\mathbf{c}\| + \int_{t+\theta}^\infty \frac{(s-t-\theta)^\alpha}{\Gamma(\alpha+1)}[M_8\|Q_1(s)\| + \|\mathbf{h}(s)\|]ds\right\} \\ &\ge M_7, \end{aligned}$$

and taking (3.11) into account, we have

$$\begin{aligned} \|(T\mathbf{x})(t)\| &\le \frac{1}{p_5}\left\{-\|\mathbf{c}\| - r(t+\theta)\mathbf{x}(t+\theta) - \left\|\int_{t+\theta}^\infty \frac{(s-t-\theta)^\alpha}{\Gamma(\alpha+1)}[Q_2(s)\mathbf{x}(s-\sigma) + \mathbf{h}(s)]ds\right\|\right\} \\ &\le \frac{1}{p_5}\left\{-\|\mathbf{c}\| - M_8 - \int_t^\infty \frac{(s-t)^\alpha}{\Gamma(\alpha+1)}[M_8\|Q_2(s)\| + \|\mathbf{h}(s)\|]ds\right\} \\ &\le M_8. \end{aligned}$$

The remaining part of the proof is similar to that of Theorem 3.2; therefore it is omitted. The proof is complete. □

4. Remark

When $\alpha = n \in N, r(t) \equiv 1, \mathbf{h}(t) \equiv \mathbf{0}$, equation (1.1) become equation (1.2), thus this paper improve results of Candan[2].

5. Competing Interests

The authors declare that they have no competing interests.

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References

[1] R. P. Agarwal, M. Bohner, W. T. Li, *Nonoscillation and Oscillation: Theory for Functional Differential Equations*, Marcel Dekker Inc, New York, 2004.

[2] T. Candan, *Existence of nonoscillatory solutions for system of higher order nonliear neutral differential equations*, Mathematical and Computer Modelling, 2013, 57, 375–381.

[3] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer, Berlin, 2010.

[4] L. H. Erbe, Q. K. Kong, B. G. Zhang, *Oscillation Theory for Functional Differential Equations*, Marcel Dekker Inc, New York, 1995.

[5] I. Györi, G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, Clarendon Presss, Oxford, 1991.

[6] K. Gopalsamy, *Stability and Oscillation in Delay Differential Equations of Population Dynamics*, Kluwer Academic, Boston, 1992.

[7] G. S. Ladde, V. Lakshmikantham, B. G. Zhang, *Oscillation Theory of Differential Equations with Deviation Arguments*, Dekker, New York, 1989.

[8] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, In: North-Holland Mathematics Studies, vol. 204. Elsevier Science B.V., Amsterdam, 2006.

[9] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[10] Y. Zhou, B. Ahmad, A. Alsaedi, *Existence of nonoscillatory solutions for fractional functional differential equations*, Bulletin of the Malaysian Mathematical Sciences Society, 2017, 2017, 1–16.

[11] Y. Zhou, B. Ahmad, A. Alsaedi, *Existence of nonoscillatory solutions for fractional neutral differential equations*, Appl. Math. Lett., 2017, 72, 70–74.

[12] Y. Zhou, L. Peng, *On the time-fractional Navier-Stokes equations*, Comput. Math. Appl. 2017, 73(6), 874–891.

[13] Y. Zhou, L. Peng, *Weak solutions of the time-fractional Navier-Stokes equations and optimal control*, Comput. Math. Appl., 2017, 73(6), 1016–1027.

[14] Y. Zhou, L. Zhang, *Existence and multiplicity results of homoclinic solutions for fractional Hamiltonian systems*, Comput. Math. Appl., 2017, 73(6), 1325–1345.