THE BREATHER WAVE SOLUTIONS, M-LUMP SOLUTIONS AND SEMI-RATIONAL SOLUTIONS TO A (2+1)-DIMENSIONAL GENERALIZED KORTEWEG-DE VRIES EQUATION

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Abstract Under investigation in this work is a (2+1)-dimensional generalized Korteweg-de Vries equation, which can be used to describe many nonlinear phenomena in plasma physics. By using the properties of Bell's polynomial, we obtain the bilinear formalism of this equation. The expression of N-soliton solution is established in terms of the Hirota's bilinear method. Based on the resulting N-soliton solutions, we succinctly show its breather wave solutions. Furthermore, with the aid of the corresponding soliton solutions, the M-lump solutions are well presented by taking a long wave limit. Two types of hybrid solutions are also represented in detail. Finally, some graphic analysis are provided in order to better understand the propagation characteristics of the obtained solutions.

Keywords A (2+1)-dimensional generalized Korteweg-de Vries equation, bilinear form, breather wave solutions, lump solutions, hybrid solutions.

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1. Introduction

It is widely acknowledged that the research of studying integrable properties and constructing exact solutions for the nonlinear evolution equations (NLEEs) is one of the most meaningful and interesting works in the field of mathematical physics. Additionally, finding exact solutions of NLEEs is also a hot topic for research workers all the time. More recently, the breather wave solutions and lump solutions have gradually occupied the eyes of researchers. The breather wave solutions possess periodicity in one direction which could be transformed into rogue wave solutions in view of certain circumstances. The lump solutions, as special localized waves, are a kind of rational solutions in all space directions. It was first discovered in 1977 by Manakov et al. in [21]. In the past few years, it has been reported in many nonlinear fields, such as optic media, the plasma, and shallow

water wave [34, 35, 45, 46]. Meanwhile, various skills, such as taking a long wave limit of the corresponding N-soliton solutions [2, 33], the inverse scattering transformation [1], Darboux transformation method [9, 22], Lie symmetry method [3], Bäcklund transformation [14], and the Hirota's bilinear method [12, 13], etc, have been used to solve exact lump solutions of NLEEs. By utilizing the Hirota's bilinear method and symbolic computation [5, 11, 15, 28, 32, 36–38, 43, 44, 47–49, 51, 54, 56], lots of works about breather wave solutions and lump solutions have been done in the past time [6–8, 10, 16, 18, 23–26, 29, 50, 58–61]. More importantly, taking a long wave limit for the corresponding N-soliton solutions has great significance in the research of lump solutions for NLEEs. Hence, we will mainly focus on this topic.

In this work, we would like to consider a (2+1)-dimensional generalized Korteweg-de Vries (gKdV) equation of the following form

$$\begin{cases} u_t + u_x + \alpha(6uu_x + u_{xxx}) + \beta v_y = 0, \\ u_y = v_x, \end{cases} \tag{1.1}$$

where $u = u(x, y, t), v = v(x, y, t)$, and α, β are both arbitrary constants. Its Bäcklund transformation, infinite conservation laws, soliton solutions and periodic wave solutions have been detailedly reported in [55]. Although some works have been represented for the gKdV equation (1.1), high-order breather solutions, M-lump solutions and relevant hybrid solutions have not been investigated before. Moreover, Eq.(1.1) has widespread application in terms of shallow water waves with weakly nonlinear restoring forces, thus we will focus on its above properties that have not been studied before.

The outline of present paper is given as follows. In section 2, we first obtain N-soliton solutions of Eq.(1.1), then we further derive the nth-order breather wave solutions by taking suitable parameters. Subsequently, we choose the first-order breather wave solution as an example, and carry out the detailed analysis. In section 3, by virtue of the corresponding N-soliton solutions, we systematically construct M-lump solutions of the gKdV equation (1.1) by taking a long wave limit. In section 4, we consider the behavior characteristics of hybrid solutions, which are hybrid of lump solution and soliton solution, and hybrid of lump solution and breather wave solution. Finally, some conclusions and discussions of this work are revealed in the last section.

2. The breather wave solutions

By employing the results provided in [30, 39–42, 52, 53, 57], we first know that Eq.(1.1) can be mapped into

$$(D_x D_t + D_x^2 + \alpha D_x^4 + \beta D_y^2)f \cdot f = 0, \tag{2.1}$$

with a variable transformation

$$u = 2(\ln f)_{xx}, \tag{2.2}$$

where $f = f(x, y, t)$ is a real function, and the derivatives $D_x D_t, D_x^2, D_x^4, D_y^2$ are all the bilinear derivative operators defined by

$$D_x^m D_y^n D_t^l(f \cdot g) =$$

$$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^n \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^l f(x,y,t) \cdot g(x',y',t')|_{x=x',y=y',z=z'}.$$

(2.3)

Then, we further get the N-soliton solution of the gKdV equation (1.1)

$$f = f_N = \sum_{\sigma=0,1} \exp\left(\sum_{1 \le i < j \le N} \sigma_i \sigma_j A_{ij} + \sum_{i=1}^{N} \sigma_i \xi_i\right),$$

(2.4)

with

$$\xi_i = k_i(x + p_i y - (\alpha k_i^2 + \beta p_i^2 + 1)t) + \xi_i^{(0)},$$
$$e^{A_{ij}} = \frac{3\alpha(k_i - k_j)^2 - \beta(p_i - p_j)^2}{3\alpha(k_i + k_j)^2 - \beta(p_i - p_j)^2},$$

(2.5)

where $k_i, p_i, \xi_i^{(0)}$ are all arbitrary parameters, and the notation $\sum_{\sigma=0,1}$ is a summation that takes over all possible combinations $\sigma_i = 0, 1(i = 1, 2, \cdots, N)$.

On the basis of previous works [4,27,31], one can obtain the nth-order breather wave solutions of the (2+1)-dimensional gKdV equation (1.1) by taking those parameters in above expression (2.4)

$$N = 2n, \quad k_j^* = k_{j+1}, \quad p_j^* = p_{j+1}, \quad \xi_j^* = \xi_{j+1}.$$

(2.6)

Without the loss of generality, we focus on the first-order breather wave solution of Eq.(1.1). Therefore, we have

$$N = 2, \quad k_1^* = k_2, \quad p_1^* = p_2, \quad \xi_1^* = \xi_2.$$

(2.7)

If we take parameters

$$k_1 = k_2^* = i, \quad p_1 = p_2^* = 2 + i, \quad \xi_1^{(0)} = \xi_2^{(0)} = 0, \quad \alpha = 1, \quad \beta = -1,$$

(2.8)

it is not difficult to find that the expression f in (2.4) can be rewritten as

$$f = 1 + 2\cos(x + 2y + 3t)\sinh(-y - 4t) + 2\cos(x + 2y + 3t)\cosh(-y - 4t)$$
$$+ 4\sinh(-2y - 8t) + 4\cosh(-2y - 8t).$$

(2.9)

Next, by inserting Eq.(2.9) into Eq.(2.2), the obtained result is called the first-order breather wave solution of Eq.(1.1). In order to understand propagation characteristics of the first-order breather wave solution (2.2) with (2.9) intuitively, we plot the following Figure 1. It is necessary to point out that these three pictures are plotted at $t = 0$. Moreover, it is worth noting that the first-order breather wave solution is periodic in x axis and localized in y axis, as described in Figure 1(a).
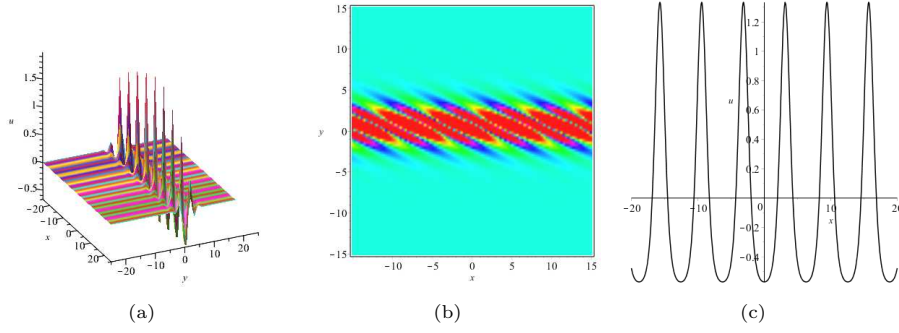
Figure 1.   (Color online) The first-order breather wave solution of Eq.(1.1) with the corresponding parameters selection in (2.8). (a) Three dimensional plot. (b) Density plot. (c) The wave propagation pattern along x axis.

3. M-lump solutions

In this part, our main goal is to construct lump solutions of Eq.(1.1) in detail by taking a long wave limit for the corresponding N-soliton solutions. From the expression (2.4), we easily find that the first three soliton solutions as follows

$$
\begin{aligned}
f_1 &= 1 + \exp \xi_1, \\
f_2 &= 1 + \exp \xi_1 + \exp \xi_2 + \exp(\xi_1 + \xi_2 + A_{12}), \\
f_3 &= 1 + \exp \xi_1 + \exp \xi_2 + \exp \xi_3 + \exp(\xi_1 + \xi_2 + A_{12}) + \exp(\xi_1 + \xi_3 + A_{13}) \\
&\quad + \exp(\xi_2 + \xi_3 + A_{23}) + \exp(\xi_1 + \xi_2 + \xi_3 + A_{12} + A_{13} + A_{23}).
\end{aligned} \tag{3.1}
$$

Subsequently, we want to get M-lump solutions of Eq.(1.1), so we take

$$
\exp(\xi_i^{(0)}) = -1, \quad 1 \le i \le N, \tag{3.2}
$$

and let a limit $k_i \longrightarrow 0$ in (2.4). Then, the following theorem can be presented.

Theorem 3.1. *Eq.(1.1) has the M-lump solutions in the following form*

$$
u = 2(\ln f_N)_{xx}, \tag{3.3}
$$

with

$$
\begin{aligned}
f_N &= \prod_{i=1}^{N} \eta_i + \frac{1}{2} \sum_{i,j}^{N} B_{ij} \prod_{l \ne i,j}^{N} \eta_l + \frac{1}{2!2^2} \sum_{i,j,s,r}^{N} B_{ij} B_{sr} \prod_{l \ne i,j,s,r}^{N} \eta_l + \cdots \\
&\quad + \frac{1}{M!2^M} \sum_{i,j,\cdots,m,n}^{N} \overbrace{B_{ij} B_{vl} \cdots B_{mn}}^{M} \prod_{p \ne i,j,v,l,\cdots,m,n}^{N} \eta_p + \cdots,
\end{aligned} \tag{3.4}
$$

and

$$
\begin{aligned}
\eta_i &= x + p_i y - (\beta p_i^2 + 1)t, \\
B_{ij} &= \frac{12\alpha}{\beta(p_i - p_j)^2},
\end{aligned} \tag{3.5}
$$

where $\sum_{i,j,\cdots,m,n}$ stands for the summation roundly feasible combinations of $i, j,$
$\cdots, m, n$, which are chosen from $1, 2, \cdots, N$ and they are all distinct. A class of
nonsingular lump solutions which were confirmed by Satsuma and Ablowitz (see
Ref. [33]) can be derived, if we choose the parameters $p_{n+i} = p_i^*(i = 1, 2, \cdots, n)$
with $N = 2n$.

3.1. 1-lump solution

In this subsection, 1-lump solution of Eq.(1.1) can be obtained from 2-soliton solution when we take $n = 1, N = 2$. Meanwhile, Eq.(3.4) can be represented as

$$f_2 = \eta_1 \eta_2 + B_{12}, \tag{3.6}$$

with

$$\eta_i = x + p_i y - (\beta p_i^2 + 1)t, \quad i = 1, 2,$$
$$B_{12} = \frac{12\alpha}{\beta(p_1 - p_2)^2}. \tag{3.7}$$

By taking $p_2 = p_1^*$, we have a nonsingular solution

$$f_2 = \eta_1 \eta_1^* + \frac{12\alpha}{\beta(p_1 - p_1^*)^2}. \tag{3.8}$$

Inserting Eq.(3.8) into $u = 2(\ln f_2)_{xx}$ and setting $p_1 = a + bi$, 1-lump solution of Eq.(1.1) is obtained by

$$u = 2\frac{\partial^2}{\partial x^2} \ln[(x' + ay')^2 + (by')^2 - \frac{3\alpha}{\beta b^2}]$$
$$= 4\frac{-(x' + ay')^2 + (by')^2 - \frac{3\alpha}{\beta b^2}}{[(x' + ay')^2 + (by')^2 - \frac{3\alpha}{\beta b^2}]^2}, \tag{3.9}$$

with

$$x' = x + (a^2\beta + b^2\beta - 1)t, \qquad y' = y - 2a\beta t. \tag{3.10}$$

It is necessary to point out that the rational solution (3.9) is a permanent 1-lump solution, this solution decaying as $O(1/x^2, 1/y^2)$ for $|x|, |y| \to \infty$ and moving with the velocity $v_x = 1 - a^2\beta - b^2\beta$ and $v_y = 2a\beta$. As shown in Figure 2, the evolution of the solution (3.9) is plotted with an appropriate choice of the parameters a, b, α and β. Moreover, it is easily find that f_2 is a positive quadratic function, which is consistent with the results in [17, 19, 20].

3.2. Multiple-lump solutions

Here, we will derive multiple-lump solutions of Eq.(1.1). We first take $n = 2, N = 4$, then Eq.(3.4) can be expressed as f_4, given by

$$f_4 = \eta_1\eta_2\eta_3\eta_4 + B_{12}\eta_3\eta_4 + B_{13}\eta_2\eta_4 + B_{14}\eta_2\eta_3 + B_{23}\eta_1\eta_4 + B_{24}\eta_1\eta_3 + B_{34}\eta_1\eta_2$$
$$+ B_{12}B_{34} + B_{13}B_{24} + B_{14}B_{23}, \tag{3.11}$$

with

$$\eta_i = x + p_i y - (\beta p_i^2 + 1)t, \quad i = 1, 2, 3, 4,$$
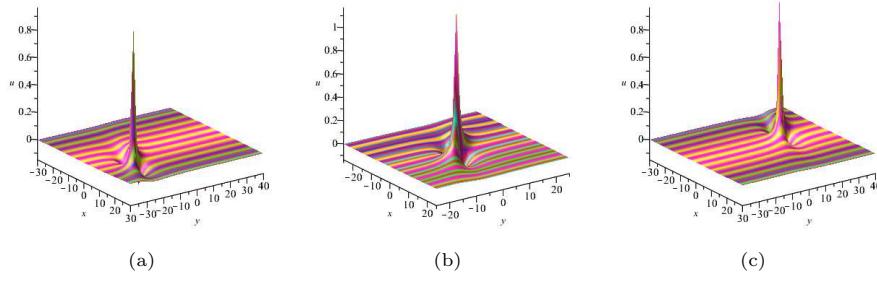
Figure 2. (Color online) Evolution plots of 1-lump solution for Eq.(1.1) by choosing suitable parameters $a = 1, b = 1, \alpha = 1, \beta = 1$. (a) $t = -10$. (b) $t = 0$. (c) $t = 10$.

$$B_{ij} = \frac{12\alpha}{\beta(p_i - p_j)^2}, \quad i < j, j = 2, 3, 4. \tag{3.12}$$

Putting $p_1 = p_R + p_I i, p_2 = \lambda_R + \lambda_I i$, we have $p_3 = p_1^* = p_R - p_I i, p_4 = p_2^* = \lambda_R - \lambda_I i$. It is necessary to illustrate that the parameters $p_R, p_I, \lambda_R, \lambda_I$ are all arbitrary real constants. Noted that the expression f_4 is a positive function consisted of quartic and quadratic perfect square functions. Next, substituting Eq.(3.11) into $u = 2(\ln f_4)_{xx}$, the obtained result is 2-lump solution of Eq.(1.1).

Similarly, Eq.(3.4) can be transformed into f_6 by choosing $n = 3, N = 6$. It is not difficult to find that the expression f_6 has 76 terms according to Eq.(3.4). Substituting f_6 into the transformation $u = 2(\ln f_6)_{xx}$, and putting $p_4 = p_1^* = p_R - p_I i, p_5 = p_2^* = h_R - h_I i, p_6 = p_3^* = q_R - q_I i$, we get a nonsingular rational solution which is called 3-lump solution of Eq.(1.1).

In what follows, the evolution plots of 2-lump solution and 3-lump solution of Eq.(1.1) are drawn by taking suitable parameters in appropriate time, respectively. Figure 3 displays the propagation feature of 2-lump solution at time $t = -10, t = 0, t = 10$. Additionally, we can see the propagation characteristic of 3-lump solution from Figure 4.
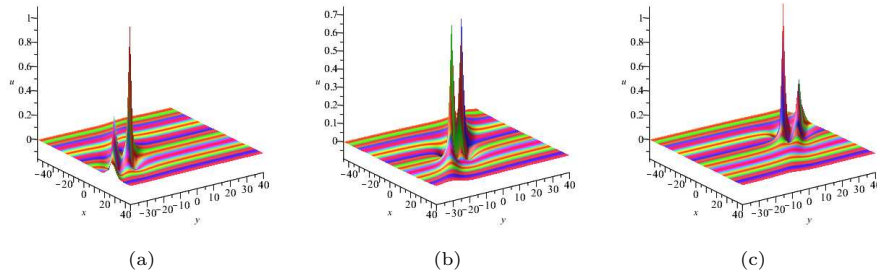

Figure 3. (Color online) Evolution plots of 2-lump solution for Eq.(1.1) by choosing suitable parameters $p_R = 1, p_I = 1, \lambda_R = 1.4, \lambda_I = 0.5, \alpha = -1, \beta = 1$. (a) $t = -10$. (b) $t = 0$. (c) $t = 10$.
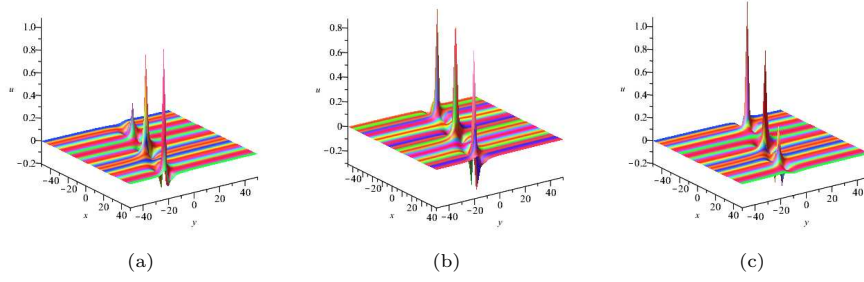
(a)　　　　　　　　　　　　　(b)　　　　　　　　　　　　　(c)

Figure 4. (Color online) Evolution plots of 3-lump solution for Eq.(1.1) by choosing suitable parameters $p_R = 1, p_I = 1, h_R = 1.4, h_I = 1.5, q_R = 1.5, q_I = 1.6, \alpha = -1, \beta = 1$. (a) $t = -2$. (b) $t = 0$. (c) $t = 2$.

4. Semi-rational solutions

Now, we are in a position to investigate the semi-rational solutions of Eq.(1.1). By taking a long wave limit for the partial exponential functions in Eq.(2.4), a combination of polynomial and exponential functions can be derived, which also be called as semi-rational solutions or hybrid solutions. In order to illustrate the solution systematically, we will consider the following two types of hybrid solutions.

4.1. Hybrid of lump solution and soliton solution

In this subsection, we first consider the situation of $N = 3$. Let

$$N = 3, \quad \xi_1^{(0)} = \xi_2^{(0)} = i\pi, \tag{4.1}$$

and take $k_1, k_2 \to 0$ in Eq.(2.4). One can find that

$$f = (\eta_1\eta_2 + B_{12}) + (\eta_1\eta_2 + B_{12} + B_{13}\eta_2 + B_{23}\eta_1 + B_{12}B_{23})e^{\xi_3}, \tag{4.2}$$

with

$$B_{s3} = -\frac{12\alpha k_3}{3\alpha k_3^2 - \beta(p_s - p_3)^2}, \quad s = 1, 2, \tag{4.3}$$

where η_1, η_2, B_{12} are given by Eq.(3.5), and ξ_3 is defined as Eq.(2.5). Then, we let $p_2 = p_1^* = a - bi$, where a, b, k_3, p_3, α and β are all arbitrary real constants. The corresponding hybrid solution u defined by Eq.(2.2) with Eq.(4.2) is derived.

In order to better analysis dynamical behavior of hybrid of 1-lump solution and 1-soliton solution, we give its three dimensional plots in different time. As shown in Figure 5, it is easily to see that the lump moves and passes the soliton and in the interaction domain of the two waveforms the amplitude increases considerably.
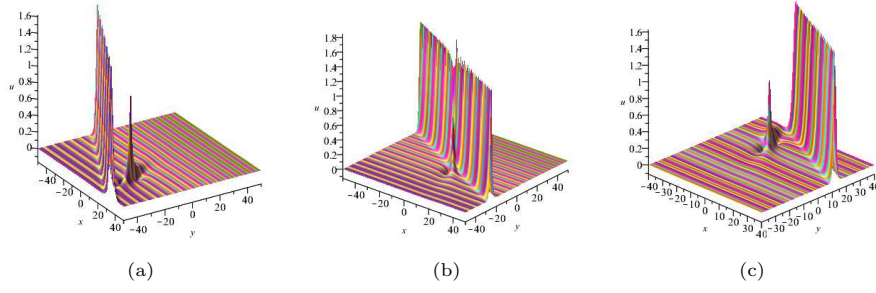
Figure 5. (Color online)Evolution plots of hybrid solution composed of 1-lump and 1-soliton for Eq.(1.1) by choosing suitable parameters $a = 1, b = 1, k_3 = -1, p_3 = 0.5, \xi_3^{(0)} = 0, \alpha = 1, \beta = -1$. (a) $t = -8$. (b) $t = 0$. (c) $t = 8$.

4.2. Hybrid of lump solution and breather wave solution

Higher-order semi-rational solutions consisting of lump solution and breather wave solution can also be generated in a similar way. For instance, we set

$$N = 4, \quad p_2 = p_1^*, \quad p_4 = p_3^*, \quad k_4 = k_3^*, \quad \xi_1^{(0)} = \xi_2^{(0)} = i\pi, \qquad (4.4)$$

and take $k_1, k_2 \to 0$ in Eq.(2.4), one can derive

$$f = e^{A_{34}}(B_{13}B_{23} + B_{13}B_{24} + B_{13}\eta_2 + B_{14}B_{23} + B_{14}B_{24} + B_{14}\eta_2 + B_{23}\eta_1 + B_{24}\eta_1 + \eta_1\eta_2$$
$$+ B_{12})e^{\xi_3+\xi_4} + (B_{13}B_{23} + B_{13}\eta_2 + B_{23}\eta_1 + \eta_1\eta_2 + B_{12})e^{\xi_3} + (B_{14}B_{24} + B_{14}\eta_2 + B_{24}\eta_1$$
$$+ \eta_1\eta_2 + B_{12})e^{\xi_4} + \eta_1\eta_2 + B_{12}, \qquad (4.5)$$

with

$$B_{sj} = -\frac{12\alpha k_j}{3\alpha k_j^2 - \beta(p_s + p_j)^2}, \quad s = 1, 2, j = 3, 4, \qquad (4.6)$$

where η_1, η_2, B_{12} are given by Eq.(3.5), and $\xi_3, \xi_4, e^{A_{34}}$ are defined as Eq.(2.5). By the substitution of Eq.(4.5) into Eq.(2.2), the corresponding hybrid solution u between lump solution and breather wave solution is deduced.

In order to better illustrate the hybrid of lump solution and breather wave solution, we give a special example with the free parameters selection being taken as follows

$$\alpha = -1, \ \beta = 1, \ p_1 = 0.5 + i, \ p_2 = 0.5 - i, \ p_3 = 1 + 0.5i,$$
$$p_4 = 1 - 0.5i, \ k_3 = 1.5i, \ k_4 = -1.5i, \ \xi_3^{(0)} = \xi_4^{(0)} = 4\pi. \qquad (4.7)$$

To observe the dynamic characteristics of the hybrid between lump solution and breather wave solution more intuitively, the corresponding figure is revealed in Figure 6. It is obvious to see that the hybrid solution (2.2) with (4.5) is a mixture of a lump solution and a breather wave solution.
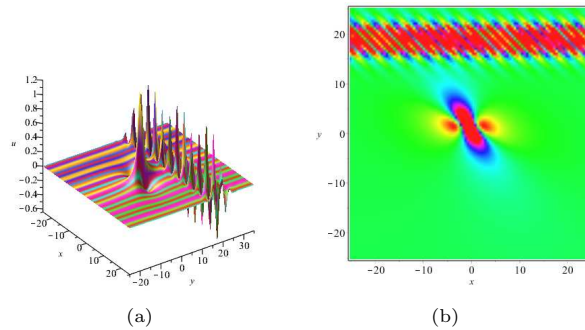
<div align="center">(a)   (b)</div>

Figure 6. (Color online) Hybrid of lump solution and breather wave solution for Eq.(1.1) with the corresponding parameters selection in (4.7). (a) Three dimensional plot ($t = 0$). (b) Density plot ($t = 0$).

5. Conclusions and discussions

In this paper, we have researched a (2+1)-dimensional generalized Korteweg-de Vries equation. Its N-soliton solutions have been constructed by employing the Hirota's bilinear method. We further derived the breather wave solutions of the equation based on the obtained N-soliton solutions. Moreover, M-lump solutions and semi-rational solutions have also been established in detail by taking a long wave limit. Most importantly, the figures of breather wave solution, 1-lump solution, 2-lump solution, 3-lump solution and two types of hybrid solutions have been presented in Figures 1-6 in order to better understand their dynamical behavior characteristics.

The paper shows an effect and powerful method to seek exact solutions of N-LEEs, which is worthy of further exploration to other models in mathematical physics and engineering. Finally, we hope that our results provided in this work are helpful to understand the breather wave solutions, the lump solutions and hybrid solutions for more models.

Acknowledgements

References

[1] M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University Press, Cambridge, 1991.

[2] M. J. Ablowitz and J. Satsuma, *Solitons and rational solutions of nonlinear evolution equations*, J. Math. Phys., 1978, 19(10), 2180–2186.

[3] G. W. Bluman and S. Kumei, *Symmetries and differential equations*, Springer-Verlag, New York, 1989.

[4] Y. L. Cao, J. S. He and D. Mihalache, *Families of exact solutions of a new extended (2+1)-dimensional Boussinesq equation*, Nonlinear Dyn., 2018, 91(4), 2593–2605.

[5] M. J. Dong, S. F. Tian, X. W. Yan and L. Zou, *Nonlocal symmetries, conservation laws and interaction solutions for the classical Boussinesq-Burgers equation*, Nonlinear Dyn., 2018, 95(1), 273–291.

[6] M. J. Dong, S. F. Tian, X. B. Wang and T. T. Zhang, *Lump-type solutions and interaction solutions in the (3+1)-dimensional potential Yu-Toda-Sasa-Fukuyama equation*, Anal. Math. Phys., 2019, 9(3), 1511–1523.

[7] M. J. Dong, S. F. Tian, X. W. Yan and L. Zou, *Solitary waves, homoclinic breather waves and rogue waves of the (3+1)-dimensional Hirota bilinear equation*, Comput. Math. Appl., 2018, 75(3), 957–964.

[8] L. L. Feng, S. F. Tian and T. T. Zhang, *Solitary wave, breather wave and rogue wave solutions of an inhomogeneous fifth-order nonlinear Schrödinger equation from Heisenberg ferromagnetism*, Rocky Mountain J. Math., 2019, 49(1), 29–45.

[9] C. H. Gu, H. S. Hu and Z. X. Zhou, *Darboux Transformation in Soliton Theory and its Geometric Applications*, Shanghai Scientific and Technical Publishers, Shanghai, 1999.

[10] D. Guo, S. F. Tian and T. T. Zhang, *Integrability, soliton solutions and modulation instability analysis of a (2+ 1)-dimensional nonlinear Heisenberg ferromagnetic spin chain equation*, Comput. Math. Appl., 2019, 77(3), 770–778.

[11] D. Guo, S. F. Tian, T. T. Zhang and J. Li, *Modulation instability analysis and soliton solutions of an integrable coupled nonlinear Schrödinger system*, Nonlinear Dyn., 2018, 94(4), 2749–2761.

[12] J. Hietarinta, *Hirota's bilinear method and soliton solutions*, Phys. AUC, 2005, 15(1), 31–37.

[13] R. Hirota, *The Direct Method in Soliton Theory*, Cambridge University Press, 2004.

[14] D. J. Kaup, *The lump solutions and the bäcklund transformation for the three-dimensional three-wave resonant interaction*, J. Math. Phys., 1981, 22(6), 1176–1181.

[15] J. B. Li and Z. J. Qiao, *Explicit soliton solutions of the Kaup-Kupershmidt equation through the dynamical system approach*, J. Appl. Anal. Comput., 2011, 1(2), 243–250.

[16] X. Lü and W. X. Ma, *Study of lump dynamics based on a dimensionally reduced Hirota bilinear equation*, Nonlinear Dyn., 2016, 85(2), 1217–1222.

[17] W. X. Ma, *Lump solutions to the Kadomtsev-Petviashvili equation*, Phys. Lett. A, 2015, 379(36), 1975–1978.

[18] W. X. Ma, Y. Zhou and R. Dougherty, *Lump-type solutions to nonlinear differential equations derived from generalized bilinear equations*, Int. J. Modern Phys. B, 2016, 30(28n29), 1640018.

[19] W. X. Ma, Z. Y. Qin and X. Lü, *Lump solutions to dimensionally reduced p-gKP and p-gBKP equations*, Nonlinear Dyn., 2016, 84(2), 923–931.

[20] W. X. Ma and Y. Zhou, *Lump solutions to nonlinear partial differential equations via Hirota bilinear forms*, J. Differential Equations, 2018, 264(4), 2633–2659.

[21] S. V. Manakov, V. E. Zakharov, L. A. Bordag, A. R. Its and V. B. Matveev, *Two-dimensional solitons of the kadomtsev-petviashvili equation and their interaction*, Phys. Lett. A, 1977, 63(3), 205–206.

[22] V. B. Matveev and M. A. Salle, *Darboux Transformations and Solitons*, Springer, Berlin, 1991.

[23] W. Q. Peng, S. F. Tian, L. Zou and T. T. Zhang, *Characteristics of the solitary waves and lump waves with interaction phenomena in a (2+1)-dimensional generalized Caudrey-Dodd-Gibbon-Kotera-Sawada equation*, Nonlinear Dyn., 2018, 93(4), 1841–1851.

[24] W. Q. Peng, S. F. Tian and T. T. Zhang, *Breather waves and rational solutions in the (3+1)-dimensional Boiti-Leon-Manna-Pempinelli equation*, Comput. Math. Appl., 2019, 77(3), 715–723.

[25] W. Q. Peng, S. F. Tian and T. T. Zhang, *Dynamics of breather waves and higher-order rogue waves in a coupled nonlinear Schrödinger equation*, EPL, 2018, 123(5), 50005.

[26] W. Q. Peng, S. F. Tian and T. T. Zhang, *On the Breather Waves, Rogue Waves and Solitary Waves to a Generalized (2+1)-dimensional Caudrey-Dodd-Gibbon-Kotera-Sawada Equation*, Filomat, 2018, 32(14), 4959–4969.

[27] C. Qian, J. G. Rao, Y. B. Liu and J. S. He, *Rogue Waves in the Three-Dimensional Kadomtsev-Petviashvili Equation*, Chin. Phys. Lett., 2016, 33(11), 110201.

[28] C. Y. Qin, S. F. Tian, L. Zou and T. T. Zhang, *Lie symmetry analysis, conservation laws and exact solutions of the generalized time fractional Burgers equation*, J. Appl. Anal. Comput., 2018, 8(6), 1727–1746.

[29] C. Y. Qin, S. F. Tian, X. B. Wang, T. T. Zhang and J. Li, *Rogue waves, bright-dark solitons and traveling wave solutions of the (3+1)-dimensional generalized Kadomtsev-Petviashvili equation*, Comput. Math. Appl., 2018, 75(12), 4221–4231.

[30] C. Y. Qin, S. F. Tian, L. Zou and W. X. Ma, *Solitary wave and quasi-periodic wave solutions to a (3+1)-dimensional generalized Calogero-Bogoyavlenskii-Schiff equation*, Adv. Appl. Math. Mech., 2018, 10(4), 948–977.

[31] J. G. Rao, Y. Cheng and J. S. He, *Rational and Semirational Solutions of the Nonlocal Davey-Stewartson Equations*, Stud. Appl. Math., 2017, 139(4), 569–598.

[32] S. Sahoo and S. S. Ray, *The new exact solutions of variant types of time fractional coupled schrödinger equations in plasma physics*, J. Appl. Anal. Comput., 2017, 7(3), 824–840.

[33] J. Satsuma and M. J. Ablowitz, *Two-dimensional lumps in nonlinear dispersive systems*, J. Math. Phys., 1979, 20(7), 1496–1503.

[34] D. R. Solli, C. Ropers, P. Koonath and B. Jalali, *Optical rogue waves*, Nature, 2007, 450(7172), 1054–1057.

[35] D. R. Solli, C. Ropers and B. Jalali, *Active Control of Rogue Waves for Stimulated Supercontinuum Generation*, Phys. Rev. Lett., 2008, 101(23), 233902.

[36] S. F. Tian, *Initial-boundary value problems for the coupled modified Korteweg-de Vries equation on the interval*, Commun. Pure Appl. Anal., 2018, 17(3), 923–957.

[37] S. F. Tian, *Asymptotic behavior of a weakly dissipative modified two-component Dullin-Gottwald-Holm system*, Appl. Math. Lett., 2018, 83, 65–72.

[38] S. F. Tian, *Initial-boundary value problems for the general coupled nonlinear Schrödinger equation on the interval via the Fokas method*, J. Differential Equations, 2017, 262(1), 506–558.

[39] S. F. Tian, *Infinite propagation speed of a weakly dissipative modified two-component Dullin-Gottwald-Holm system*, Appl. Math. Lett., 2019, 89, 1-7.

[40] S. F. Tian and H. Q. Zhang, *Riemann theta functions periodic wave solutions and rational characteristics for the nonlinear equations*, J. Math. Anal. Appl., 2010, 371(2), 585–608.

[41] S. F. Tian and H. Q. Zhang, *On the integrability of a generalized variable-coefficient Kadomtsev-Petviashvili equation*, J. Phys. A: Math. Theor., 2012, 45(5), 055203.

[42] S. F. Tian and H. Q. Zhang, *On the Integrability of a Generalized Variable-Coefficient Forced Korteweg-de Vries Equation in Fluids*, Stud. Appl. Math., 2014, 132(3), 212–246.

[43] S. F. Tian and T. T. Zhang, *Long-time asymptotic behavior for the Gerdjikov-Ivanov type of derivative nonlinear Schrödinger equation with time-periodic boundary condition*, Proc. Amer. Math. Soc., 2018, 146(4), 1713–1729.

[44] D. S. Wang, B. L. Guo and X. L. Wang, *Long-time asymptotics of the focusing Kundu-Eckhaus equation with nonzero boundary conditions*, J. Differential Equations, 2019, 266(9), 5209–5253.

[45] D. S. Wang, J. Liu and L. Z. Wang, *Non-autonomous matter-wave solitons in hybrid atomic-molecular Bose-Einstein condensates with tunable interactions and harmonic potential*, Phys. Lett. A, 2018, 382(12), 799–805.

[46] D. S. Wang, Y. R. Shi, W. X. Feng and L. Wen, *Dynamical and energetic instabilities of F=2 spinor Bose-Einstein condensates in an optical lattice*, Physica D, 2017, 351, 30–41.

[47] D. S. Wang and J. Liu, *Integrability aspects of some two-component KdV systems*, Appl. Math. Lett., 2018, 79, 211–219.

[48] D. S. Wang and X. L. Wang, *Long-time asymptotics and the bright N-soliton solutions of the Kundu-Eckhaus equation via the Riemann-Hilbert approach*, Nonlinear Anal. Real World Appl., 2018, 41, 334–361.

[49] D. S. Wang, X. G. Li, C. K. Chan and J. Zhou, *Double Wronskian solution and soliton properties of the nonisospectral BKP equation*, Commun. Theor. Phys., 2016, 65(3), 259–265.

[50] X. B. Wang, S. F. Tian and T. T. Zhang, *Characteristics of the breather and rogue waves in a (2+1)-dimensional nonlinear Schrödinger equation*, Proc. Amer. Math. Soc., 2018, 146(8), 3353–3365.

[51] X. B. Wang and S. F. Tian, *Lie symmetry analysis, conservation laws and analytical solutions of the time-fractional thin-film equation*, Comput. Appl. Math., 2018, 37(5), 6270–6282.

[52] X. B. Wang, S. F. Tian, C. Y. Qin and T. T. Zhang, *Characteristics of the breathers, rogue waves and solitary waves in a generalized (2+1)-dimensional Boussinesq equation*, EPL, 2016, 115(1), 10002.

[53] H. Wang, S. Tian, T. Zhang and Y. Chen, *Lump wave and hybrid solutions of a generalized (3+1)-dimensional nonlinear wave equation in liquid with gas bubbles*, Front. Math. China, 2019, 14(3), 631–643.

[54] A. M. Wazwaz and G. Q. Xu, *Negative-ordermodified KdV equations: multiple soliton and multiple singular soliton solutions*, Math. Methods Appl. Sci., 2016, 39(4), 661–667.

[55] M. J. Xu, S. F. Tian, J. M. Tu, P. L. Ma and T. T. Zhang, *On quasiperiodic wave solutions and integrability to a generalized (2+1)-dimensional Korteweg-de Vries equation*, Nonlinear Dyn., 2015, 82(4), 2031–2049.

[56] X. W. Yan, S. F. Tian, M. J. Dong, X. B. Wang and T. T. Zhang, *Nonlocal Symmetries, Conservation Laws and Interaction Solutions of the Generalised Dispersive Modified Benjamin-Bona-Mahony Equation*, Z. Naturforsch A, 2018, 73(5), 399–405.

[57] X. W. Yan, S. F. Tian, X. B. Wang and T. T. Zhang, *Solitons to rogue waves transition, lump solutions and interaction solutions for the (3+1)-dimensional generalized B-type Kadomtsev-Petviashvili equation in fluid dynamics*, Int. J. Comput. Math., 2019, 96(9), 1839–1848.

[58] J. Y. Yang and W. X. Ma, *Abundant lump-type solutions of the Jimbo-Miwa equation in (3+1)-dimensions*, Comput. Math. Appl., 2017, 73(2), 220–225.

[59] J. Y. Yang, W. X. Ma and Z. Y. Qin, *Lump and lump-soliton solutions to the (2+1)-dimensional Ito equation*, Anal. Math. Phys., 2018, 8(3), 427–436.

[60] J. P. Yu and Y. L. Sun, *Study of lump solutions to dimensionally reduced generalized KP equations*, Nonlinear Dyn., 2017, 87(4), 2755–2763.

[61] Y. Zhang, Y. P. Liu and X. Y. Tang, *M-lump and interactive solutions to a (3+1)-dimensional nonlinear system*, Nonlinear Dyn., 2018, 93(4), 2533–2541.