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FURTHER IMPROVEMENT OF FINITE-TIME CONSENSUS PROTOCOLS FOR DETAIL-BALANCED NETWORKS

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Abstract This paper presents a new class of protocols to solve finite-time consensus for multi-agent systems. The protocols are induced from the classical finite-time consensus algorithm by using the so-called protocol function. Sufficient conditions are established for networked agents to experience finite-time consensus under time-varying undirected and fixed directed topologies. Numerical simulations show that the proposed protocols can provide more flexibility to improve convergence rate.

Keywords Finite-time consensus, multi-agent systems, convergence rate, graph theory.

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1. Introduction

Recent years have witnessed an increasing number of studies concerned with the consensus problem of multi-agent systems due to its vast potential in applications, including physics [11, 21, 22], biology [1, 20, 25] and management science [3, 5, 16]. In these applications, every system consists of multiple agents whose motions are governed by a first, second or even more higher-order dynamics and aims to reach consensus that states of all agents agree upon a common assessment or certain quantity of interest. To achieve consensus, every individual evolves by comparing its current state with the information coming from its neighbors. Hence, the main challenge in solving the consensus problem lies in how to design the interaction rule, which is called the consensus protocol or algorithm. In the past decades, many systems have been modeled to explore appropriate consensus protocols. In 2004, Olfati-Saber and Murray [19] pioneered a systematic framework of the consensus problem for networked agents, which led to subsequent interesting results. Xiao and Wang [23] studied consensus for discrete-time systems with changing communication topologies and bounded time-varying communication delay. Atay [2] studied consensus problems on networks in the presence of distributed time delays. Yu etc [24] discussed the design of distributed control gains for consensus in multi-agent systems with second-order nonlinear dynamics. Cheng etc [6] studied the mean square consensus of linear multi-agent systems with communication nois-

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es. Zhang etc [26] developed a distributed leader-follower consensus protocol for a class of homogeneous time-varying nonlinear multi-agent systems. Geng etc [10] performed an in-depth study about the consensus problem of heterogeneous multi-agent systems with linear and nonlinear dynamics by the aid of the adaptive method and Lyapunov stability theory, to name just a few.

It should be noted that convergence rate is an important performance indicator in the analysis of consensus. It was shown that the second smallest eigenvalue of the Laplacian matrix of interaction graph, called algebraic connectivity, determines the consensus speed of multi-agent systems. This outcome motivated some researchers to find proper interaction topologies with larger algebraic connectivity. Kim and Mesbahi [15] considered the problem of finding the best vertex positional configuration in the presence of an additional proximity constraint to maximize the algebraic connectivity. Ogiwara etc [18] tackled the problem of finding graphs that maximize or locally maximize the algebraic connectivity in the space of graphs with a fixed number of vertices and edges. Furthermore, based on non-smooth stability analysis, people introduced various protocols to make multi-agent systems reach consensus within finite time. Cortés [8] proposed the normalized and signed gradient dynamical systems associated with a differentiable function to solve finite-time consensus. Zuo and Tie [27] constructed a new class of global continuous time-invariant consensus protocols for each single-integrator agent dynamics with the aid of Lyapunov functions. Hua etc [12] investigated the finite-time consensus of second-order multi-agent systems with unknown velocities and disturbances. All of the above results showed that, compared with asymptotic consensus, finite-time consensus can better meet complex practical cases and has stronger robustness against uncertainties.

This work presents a new class of finite-time consensus protocols for N networked agents written as

$$\dot{x}_i(t) = \sum_{j=1}^{N} a_{ij}(t)\text{sign}(h(x_j(t)) - h(x_i(t)), \alpha), \quad i, j \in \mathcal{I}_N = \{1, 2, ..., N\}, \tag{1.1}$$

where $x_i(t) \in \mathbb{R}$ denotes the state (opinion, voltage, or incremental cost) of agent i at time t, a_{ij} measures the mutual influence of agent j on agent i, $0 < \alpha < 1$, $\text{sign}(r, \alpha) = \text{sign}(r)|r|^\alpha$ for simplicity and the so-called protocol function $h(x) : \mathbb{R} \to \mathbb{R}$ is continuously differentiable as well as strictly increasing. The main contribution of this paper is twofold. First, we investigate finite-time consensus of multiple agents governed by the dynamics (1.1) under detail-balanced networks. Then we find out some concrete protocol functions to improve consensus rate. For this to happen, it is necessary to give the definition of finite-time consensus mathematically at first.

Definition 1.1. Finite-time consensus in (1.1) is said to be reached if for arbitrary initial conditions $\boldsymbol{x}_0 = [x_1(0), x_2(0), ..., x_N(0)]^T$ and all $i, j \in \mathcal{I}_N$, there exists a settling time $T(\boldsymbol{x}_0) \in [0, +\infty)$ such that

$$\begin{cases} \lim\limits_{t \to T(\boldsymbol{x}_0)} |x_i(t) - x_j(t)| = 0, \\ x_i(t) = x_j(t), \quad t \geq T(\boldsymbol{x}_0). \end{cases}$$

The rest of this paper is organized as follows. Section 2 reviews some preliminaries necessary throughout the paper. Section 3 states sufficient conditions for finite-time consensus and technical proofs follow. Numerical simulations are carried out to illustrate consensus performance in Section 4. Conclusions and future research directions end the paper in Section 5.

2. Preliminaries

2.1. Graph theory

A directed graph $G(\boldsymbol{A}) = (V, \varepsilon, \boldsymbol{A})$ consists of a node set $V(G) = \{v_1, v_2, ..., v_N\}$, an edge set $\varepsilon \subseteq V \times V$ and an adjacent matrix $\boldsymbol{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$. For a directed graph, an edge (v_i, v_j) denotes the state of node v_i is available to node v_j, but not necessarily vice versa. In contrast, for an undirected graph, $(v_i, v_j) \in \varepsilon$ implies $(v_j, v_i) \in \varepsilon$. If $(v_i, v_j) \in \varepsilon$, then node v_i is called a neighbor of node v_j or v_i and v_j are adjacent. For an undirected graph, its adjacent matrix $\boldsymbol{A} = [a_{ij}]$ satisfies $a_{ij} = a_{ji} > 0$. Moreover, $a_{ij} > 0$ means $(v_j, v_i) \in \varepsilon$, $a_{ij} = 0$ implies $(v_j, v_i) \notin \varepsilon$, and $a_{ii} \equiv 0$. For a directed graph, properties of the adjacent matrix are similar except for $a_{ij} = a_{ji}$. A path on G from v_i to v_j is a sequence of distinct vertices v_i, ..., v_j if consecutive vertices are adjacent. An undirected graph is called connected if there exists a path for any two distinct nodes. A directed graph is called strongly connected if and only if for any two distinct nodes, there exists a directed path. Furthermore, a directed graph $G(\boldsymbol{A})$ is said to satisfy the detail-balanced condition if there exist some scalars $\omega_i > 0$ such that $\omega_i a_{ij} = \omega_j a_{ji}$ for all $i, j \in \mathcal{I}_N$ [7].

Lemma 2.1 (Remark 4, [19]). *Let $\boldsymbol{L_A} = [l_{ij}] \in \mathbb{R}^{N\times N}$ denote the Laplacian matrix of graph $G(\boldsymbol{A})$ with elements*

$$l_{ij} = \begin{cases} \sum\limits_{k=1,k\neq i}^{N} a_{ik}, & i = j, \\ -a_{ij}, & i \neq j. \end{cases}$$

Then $\boldsymbol{L_A}$ has the following properties:

(i) 0 is an eigenvalue of $\boldsymbol{L_A}$ and $\mathbf{1}_N = [1, 1, ..., 1]^T \in \mathbb{R}^N$ is the associated eigenvector;

(ii) If $G(\boldsymbol{A})$ is undirected, then $\boldsymbol{x}^T \boldsymbol{L_A} \boldsymbol{x} = \frac{1}{2}\sum\limits_{i=1}^{N}\sum\limits_{j=1}^{N} a_{ij}(x_j - x_i)^2$ for any $\boldsymbol{x} = [x_1, x_2, ..., x_N]^T$, and $\boldsymbol{L_A}$ is positive semi-definite, which implies that all eigenvalues of $\boldsymbol{L_A}$ are nonnegative real numbers;

(iii) For an undirected $G(\boldsymbol{A})$, the algebraic connectivity of $G(\boldsymbol{A})$ is given by

$$\lambda_2(\boldsymbol{L_A}) = \min_{\boldsymbol{x}\neq \mathbf{0}, \mathbf{1}_N^T \boldsymbol{x} = 0} \frac{\boldsymbol{x}^T \boldsymbol{L_A} \boldsymbol{x}}{\boldsymbol{x}^T \boldsymbol{x}},$$

where $\lambda_2(\boldsymbol{L_A})$ also equals the second smallest eigenvalue of $\boldsymbol{L_A}$. In addition, $G(\boldsymbol{A})$ is connected if and only if $\lambda_2(\boldsymbol{L_A}) > 0$.

2.2. Mathematical lemmas

Lemma 2.2. *Suppose that* $\boldsymbol{\xi} = [\xi_1, \xi_2, ..., \xi_N]^T \in \mathbb{R}^N$ *and* $\boldsymbol{Q} = [q_{ij}] \in \mathbb{R}^{N\times N}$ *is symmetric. If* $f : \mathbb{R} \to \mathbb{R}$ *is an odd function, then we have*

$$\sum_{i=1}^{N}\sum_{j=1}^{N} q_{ij}\xi_i f(x_j - x_i) = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} q_{ij}(\xi_j - \xi_i) f(x_j - x_i).$$

Proof.

$$\begin{aligned}\sum_{i=1}^{N}\sum_{j=1}^{N} q_{ij}\xi_i f(x_j - x_i) &= \sum_{i=1}^{N}\sum_{j=1}^{N} q_{ji}\xi_j f(x_i - x_j)\\ &= -\sum_{i=1}^{N}\sum_{j=1}^{N} q_{ij}\xi_j f(x_j - x_i),\end{aligned}$$

it follows that

$$\begin{aligned}\sum_{i=1}^{N}\sum_{j=1}^{N} q_{ij}\xi_i f(x_j - x_i) &= \frac{1}{2}\left(\sum_{i=1}^{N}\sum_{j=1}^{N} q_{ij}\xi_i f(x_j - x_i) - \sum_{i=1}^{N}\sum_{j=1}^{N} q_{ij}\xi_j f(x_j - x_i)\right)\\ &= -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} q_{ij}(\xi_j - \xi_i) f(x_j - x_i).\end{aligned}$$

□

Lemma 2.3 (Theorem 19, [13])**.** *If* $\xi_1,\ \xi_2,\ ...,\xi_N \geq 0$ *and* $0 < p \leq 1$*, then*

$$\left(\sum_{i=1}^{N}\xi_i\right)^p \leq \sum_{i=1}^{N}\xi_i^p.$$

Lemma 2.4 (Theorem 4.2, [4])**.** *Assume that a continuous, positive-definite function* $V(t)$ *satisfies*

$$\dot{V}(t) \leq -kV^{\rho}(t),$$

where $k > 0$, $0 < \rho < 1$ *are two constants. Then, for any given* t_0*,* V *will tend to zero within finite time estimated by*

$$t_1 = t_0 + \frac{V^{1-\rho}(t_0)}{k(1-\rho)},$$

and $V(t) \equiv 0$ *for* $t \geq t_1$*.*

Definition 2.1. Assume that $\boldsymbol{A} \in \mathbb{C}^{N\times N}$ is an Hermitian matrix, then for $\boldsymbol{x} \neq 0$,

$$R(\boldsymbol{A};\boldsymbol{x}) = \frac{\boldsymbol{x}^H\boldsymbol{A}\boldsymbol{x}}{\boldsymbol{x}^H\boldsymbol{x}}$$

is called the Rayleigh quotient of $\boldsymbol{A}$.

All eigenvalues of $\boldsymbol{A}$ are real numbers and one can order them as

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N.$$

Then we have

Lemma 2.5 (Theorem 4.2.2, [14])**.**

$$\min_{\boldsymbol{x}\neq\boldsymbol{0}} R(\boldsymbol{A};\boldsymbol{x}) = \lambda_1, \quad \max_{\boldsymbol{x}\neq\boldsymbol{0}} R(\boldsymbol{A};\boldsymbol{x}) = \lambda_N.$$

Definition 2.2. Suppose that $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{C}^{N\times N}$ are Hermitian matrices, then for $\boldsymbol{x} \neq \boldsymbol{0}$,

$$R(\boldsymbol{A}, \boldsymbol{B}; \boldsymbol{x}) = \frac{\boldsymbol{x}^H\boldsymbol{A}\boldsymbol{x}}{\boldsymbol{x}^H\boldsymbol{B}\boldsymbol{x}}$$

is called the generalized Rayleigh quotient of $\boldsymbol{A}$ and $\boldsymbol{B}$.

It is easy to check that all roots of $\det(\mu\boldsymbol{B} - \boldsymbol{A})$ are real numbers and one can order them as

$$\mu_1 \leq \mu_2 \leq \cdots \leq \mu_N.$$

With this, we have

Lemma 2.6. *If the Hermitian matrix $\boldsymbol{B}$ is positive-definite, then*

$$\min_{\boldsymbol{x}\neq\boldsymbol{0}} R(\boldsymbol{A}, \boldsymbol{B};\boldsymbol{x}) = \mu_1, \quad \max_{\boldsymbol{x}\neq\boldsymbol{0}} R(\boldsymbol{A}, \boldsymbol{B};\boldsymbol{x}) = \mu_N.$$

Proof. Denote $\boldsymbol{y} = \boldsymbol{B}^{\frac{1}{2}}\boldsymbol{x}$. Noting that $\boldsymbol{B}^{\frac{1}{2}}(\boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{A}\boldsymbol{B}^{-\frac{1}{2}})\boldsymbol{B}^{-\frac{1}{2}} = \boldsymbol{A}\boldsymbol{B}^{-1}$, according to Lemma 2.5, we have

$$\max_{\boldsymbol{x}\neq\boldsymbol{0}} \frac{\boldsymbol{x}^H\boldsymbol{A}\boldsymbol{x}}{\boldsymbol{x}^H\boldsymbol{B}\boldsymbol{x}} = \max_{\boldsymbol{x}\neq\boldsymbol{0}} \frac{\boldsymbol{y}^H\boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{A}\boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{y}}{\boldsymbol{y}^H\boldsymbol{y}} = \lambda_N(\boldsymbol{A}\boldsymbol{B}^{-1}) = \mu_N.$$

The same reasoning applies to the other case. □

3. Consensus results

This section presents two kinds of networks for a group of N agents to ensure finite-time consensus.

3.1. Network with time-varying undirected topology

In many practical situations, the information exchange may not be available all the time due to special physical devices, limited sensing range or existence of obstacles. Therefore, it is reasonable to assume that the interaction topology is dynamically changing. For this case, we have the following result.

Theorem 3.1. *Suppose that the time-varying network $G(\boldsymbol{A}(t))$ is always undirected and connected. Moreover, we assume that the algebraic connectivity of $G(\boldsymbol{B}(t))$ has a lower bound, that is, there exists $\lambda_2^* > 0$ such that*

$$\inf_{t\geq 0} \lambda_2(\boldsymbol{L}_{\boldsymbol{B}}(t)) \geq \lambda_2^*, \tag{3.1}$$

where we define $\boldsymbol{B}(t) = [b_{ij}(t)] = \left[a_{ij}^{\frac{2}{1+\alpha}}(t)\right] \in \mathbb{R}^{N\times N}$. Then finite-time consensus can be reached in system (1.1).

Proof. Note that $G(\boldsymbol{A}(t))$ is undirected in the sense that $a_{ij}(t) = a_{ji}(t)$, namely, two agents have the same influence on the alignment of each other. The symmetry implies the total momentum in the model

$$\bar{x}(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t)$$

is conserved, that is, $\bar{x}(t) \equiv \bar{x}(0)$. In fact, we can get

$$\dot{\bar{x}}(t) = \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(t)\text{sign}(h(x_j) - h(x_i), \alpha) = 0$$

immediately from Lemma 2.2 with $\boldsymbol{\xi} = \mathbf{1}_N$ in it. Let $\boldsymbol{\delta}(t) = [\delta_1(t), \delta_2(t), ..., \delta_N(t)]^T$ be the group disagreement vector with $\delta_i(t) = x_i(t) - \bar{x}$, then $\dot{\delta}_i(t) = \dot{x}_i(t)$ and $\sum\limits_{i=1}^{N} \delta_i(t) = 0$.

Consider the following Lyapunov function candidate

$$V(\delta) = \frac{1}{2}\sum_{i=1}^{N} \delta_i^2(t).$$

Differentiating V along the protocol versus time yields

$$\begin{aligned}
\dot{V} &= \sum_{i=1}^{N} \delta_i \dot{\delta}_i \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\delta_i \text{sign}(h(x_j) - h(x_i), \alpha) \\
&= -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(\delta_j - \delta_i)\text{sign}(h(x_j) - h(x_i), \alpha) \\
&= -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}|\delta_j - \delta_i||h(x_j) - h(x_i)|^{\alpha},
\end{aligned}$$

where the last equality is derived from $h(x)$ being strictly increasing. From $\dot{V} \leq 0$ and $V \geq 0$ we know that $V(t)$ is bounded for all $t \geq 0$. Then it follows that $x_i(t)$ remains bounded for all $t \geq 0$ and $i \in \mathcal{I}_N$, that is, there exists $M > 0$ such that $|x_i(t)| \leq M$. Since $h(x)$ is continuously differentiable, according to the Lagrange mean value theorem, there exists $\bar{c} = \max\limits_{|x| \leq M} h'(x)$ such that $|\delta_j - \delta_i| = |x_j - x_i| \geq \frac{1}{\bar{c}}|h(x_j) - h(x_i)|$. Invoking Lemma 2.3, we have

$$\begin{aligned}
\dot{V} &\leq -\frac{1}{2\bar{c}}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}|h(x_j) - h(x_i)|^{\alpha+1} \\
&= -\frac{1}{2\bar{c}}\sum_{i=1}^{N}\sum_{j=1}^{N} \left(a_{ij}^{\frac{2}{1+\alpha}}(h(x_j) - h(x_i))^2\right)^{\frac{\alpha+1}{2}}
\end{aligned}$$

$$\leq -\frac{1}{2\bar{c}}\left(\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^{\frac{2}{1+\alpha}}(h(x_j)-h(x_i))^2\right)^{\frac{\alpha+1}{2}}$$

$$= -\frac{1}{2\bar{c}}\left(\frac{\sum\limits_{i=1}^{N}\sum\limits_{j=1}^{N} a_{ij}^{\frac{2}{1+\alpha}}(h(x_j)-h(x_i))^2}{\sum\limits_{i=1}^{N}\sum\limits_{j=1}^{N} a_{ij}^{\frac{2}{1+\alpha}}(\delta_j-\delta_i)^2}\cdot\frac{\sum\limits_{i=1}^{N}\sum\limits_{j=1}^{N} a_{ij}^{\frac{2}{1+\alpha}}(\delta_j-\delta_i)^2}{V}\cdot V\right)^{\frac{\alpha+1}{2}}. \quad (3.2)$$

Let $\underline{c} = \min_{|x|\leq M} h'(x)$, which leads to $|h(x_j)-h(x_i)| \geq \underline{c}|x_j - x_i|$ and

$$\frac{\sum\limits_{i=1}^{N}\sum\limits_{j=1}^{N} a_{ij}^{\frac{2}{1+\alpha}}(h(x_j)-h(x_i))^2}{\sum\limits_{i=1}^{N}\sum\limits_{j=1}^{N} a_{ij}^{\frac{2}{1+\alpha}}(\delta_j-\delta_i)^2} \geq \frac{\sum\limits_{i=1}^{N}\sum\limits_{j=1}^{N} a_{ij}^{\frac{2}{1+\alpha}}\underline{c}^2(\delta_j-\delta_i)^2}{\sum\limits_{i=1}^{N}\sum\limits_{j=1}^{N} a_{ij}^{\frac{2}{1+\alpha}}(\delta_j-\delta_i)^2} = \underline{c}^2. \quad (3.3)$$

Then, from Lemma 2.1 and (3.1), we have

$$\frac{\sum\limits_{i=1}^{N}\sum\limits_{j=1}^{N} a_{ij}^{\frac{2}{1+\alpha}}(t)(\delta_j-\delta_i)^2}{V} = \left.\frac{2\boldsymbol{\delta}^T\boldsymbol{L_B}(t)\boldsymbol{\delta}}{\frac{1}{2}\boldsymbol{\delta}^T\boldsymbol{\delta}}\right|_{\boldsymbol{\delta}\neq\mathbf{0},\mathbf{1}_N^T\boldsymbol{\delta}=0} \geq 4\lambda_2(\boldsymbol{L_B}(t)) \geq 4\lambda_2^*. \quad (3.4)$$

Substituting (3.3) and (3.4) into (3.2), one has

$$\dot{V} \leq -\frac{1}{2\bar{c}}(4\underline{c}^2\lambda_2^*)^{\frac{\alpha+1}{2}}V^{\frac{\alpha+1}{2}}. \quad (3.5)$$

From the above discussion and Lemma 2.4, system (1.1) can achieve consensus at finite time

$$t_1 \leq \frac{4\bar{c}V^{\frac{1-\alpha}{2}}(0)}{(1-\alpha)(4\underline{c}^2\lambda_2^*)^{\frac{\alpha+1}{2}}},$$

where $V(0) = \frac{1}{2}\sum\limits_{i=1}^{N}\delta_i^2(0)$. This completes the proof. □

We have thus obtained a sufficient condition for system (1.1) to experience finite-time consensus when the network topology is time-varying undirected. The following result focuses on the fixed topology case where we prove consensus under more flexible conditions.

3.2. Network with time-invariant directed topology

We are now in a position to discuss the network with time-invariant directed topology. The main result of this part is the following theorem.

Theorem 3.2. *Suppose that the fixed topology $G(\boldsymbol{A})$ is strongly connected, detail-balanced and satisfies $a_{ij} > 0$ for $i \neq j$. Then finite-time consensus can be reached in system* (1.1).

Proof. In this case, there exists a positive vector $\boldsymbol{\omega} = [\omega_1, \omega_2, ..., \omega_N]^T$ satisfying $\omega_i a_{ij} = \omega_j a_{ji}$ for $i, j \in \mathcal{I}_N$. Let $\boldsymbol{D} = \text{diag}(\omega_1, \omega_2, ..., \omega_N)$, then $\boldsymbol{DA}$ is symmetric and $G(\boldsymbol{DA})$ is connected. It is easy to check that the weighted center

$$\bar{x}(t) = \frac{\sum\limits_{i=1}^{N} \omega_i x_i(t)}{\sum\limits_{i=1}^{N} \omega_i}$$

remains time-invariant. In fact, we can get

$$\dot{\bar{x}}(t) = \frac{1}{\sum\limits_{i=1}^{N} \omega_i} \sum_{i=1}^{N} \sum_{j=1}^{N} \omega_i a_{ij} \text{sign}(h(x_j) - h(x_i), \alpha) = 0$$

from Lemma 2.2 with $\boldsymbol{Q} = \boldsymbol{DA}$ and $\boldsymbol{\xi} = \mathbf{1}_N$ in it. Let $\boldsymbol{\delta}(t) = [\delta_1(t), \delta_2(t), ..., \delta_N(t)]^T$ be the group disagreement vector with $\delta_i(t) = x_i(t) - \bar{x}$, then $\dot{\delta}_i(t) = \dot{x}_i(t)$ and $\sum\limits_{i=1}^{N} \omega_i \delta_i = 0$.

Consider the following Lyapunov function candidate

$$V(\delta) = \frac{1}{2} \sum_{i=1}^{N} \omega_i \delta_i^2(t).$$

The time derivative of V is as follows

$$\begin{aligned}
\dot{V} &= \sum_{i=1}^{N} \omega_i \delta_i \dot{\delta}_i \\
&= \sum_{i=1}^{N} \sum_{j=1}^{N} \omega_i a_{ij} \delta_i \text{sign}(h(x_j) - h(x_i), \alpha) \\
&= -\frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \omega_i a_{ij} (\delta_j - \delta_i) \text{sign}(h(x_j) - h(x_i), \alpha) \\
&= -\frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \omega_i a_{ij} |\delta_j - \delta_i| |h(x_j) - h(x_i)|^{\alpha}.
\end{aligned}$$

Choose $\bar{c}$ and $\underline{c}$ as the same meanings in the proof of Theorem 3.1, then

$$\begin{aligned}
\dot{V} &\leq -\frac{1}{2\bar{c}} \sum_{i=1}^{N} \sum_{j=1}^{N} \omega_i a_{ij} |h(x_j) - h(x_i)|^{\alpha+1} \\
&= -\frac{1}{2\bar{c}} \sum_{i=1}^{N} \sum_{j=1}^{N} \left((\omega_i a_{ij})^{\frac{2}{1+\alpha}} (h(x_j) - h(x_i))^2 \right)^{\frac{\alpha+1}{2}} \\
&\leq -\frac{1}{2\bar{c}} \left(\sum_{i=1}^{N} \sum_{j=1}^{N} (\omega_i a_{ij})^{\frac{2}{1+\alpha}} (h(x_j) - h(x_i))^2 \right)^{\frac{\alpha+1}{2}}
\end{aligned}$$

$$= -\frac{1}{2\bar{c}}\left(\frac{\sum_{i=1}^{N}\sum_{j=1}^{N}(\omega_i a_{ij})^{\frac{2}{1+\alpha}}(h(x_j)-h(x_i))^2}{\sum_{i=1}^{N}\sum_{j=1}^{N}(\omega_i a_{ij})^{\frac{2}{1+\alpha}}(\delta_j-\delta_i)^2}\cdot\frac{\sum_{i=1}^{N}\sum_{j=1}^{N}(\omega_i a_{ij})^{\frac{2}{1+\alpha}}(\delta_j-\delta_i)^2}{V}\cdot V\right)^{\frac{\alpha+1}{2}}.$$

Set $\boldsymbol{C} = [c_{ij}] = \left[(\omega_i a_{ij})^{\frac{2}{1+\alpha}}\right] \in \mathbb{R}^{N\times N}$, then

$$\frac{\sum_{i=1}^{N}\sum_{j=1}^{N}(\omega_i a_{ij})^{\frac{2}{1+\alpha}}(h(x_j)-h(x_i))^2}{\sum_{i=1}^{N}\sum_{j=1}^{N}(\omega_i a_{ij})^{\frac{2}{1+\alpha}}(\delta_j-\delta_i)^2} \geq \underline{c}^2$$

and

$$\frac{\sum_{i=1}^{N}\sum_{j=1}^{N}(\omega_i a_{ij})^{\frac{2}{1+\alpha}}(\delta_j-\delta_i)^2}{V} = \frac{2\boldsymbol{\delta}^T\boldsymbol{L_C}\boldsymbol{\delta}}{\frac{1}{2}\boldsymbol{\delta}^T\boldsymbol{D}\boldsymbol{\delta}}.$$

Recalling Lemma 2.6, one has

$$\left.\frac{\boldsymbol{\delta}^T\boldsymbol{L_C}\boldsymbol{\delta}}{\boldsymbol{\delta}^T\boldsymbol{D}\boldsymbol{\delta}}\right|_{\boldsymbol{\delta}\neq\boldsymbol{0},\boldsymbol{\delta}\perp\boldsymbol{\omega}} \geq \min_{\boldsymbol{\delta}\neq\boldsymbol{0}} R(\boldsymbol{L_C},\boldsymbol{D};\boldsymbol{\delta}) = \mu_1^* > 0,$$

where μ_1 is the smallest root of $\det(\mu\boldsymbol{D}-\boldsymbol{L_C}) = 0$. Summarizing what we have obtained leads to

$$\dot{V} \leq -\frac{1}{2\bar{c}}(4\underline{c}^2\mu_1^*)^{\frac{\alpha+1}{2}}V^{\frac{\alpha+1}{2}}.$$

From Lemma 2.4, it follows that system (1.1) can evolve into zero at finite time

$$t_1 \leq \frac{4\bar{c}V^{\frac{1-\alpha}{2}}(0)}{(1-\alpha)(4\underline{c}^2\mu_1^*)^{\frac{\alpha+1}{2}}},$$

where $V(0) = \frac{1}{2}\sum_{i=1}^{N}\omega_i\delta_i^2(0)$. This completes the proof. □

Remark 3.1. In fact, an undirected network itself can be regarded as a special case of a directed network with detail-balanced coefficients $\omega_i = 1$ for all $i \in \mathcal{I}_N$.

4. Numerical simulations

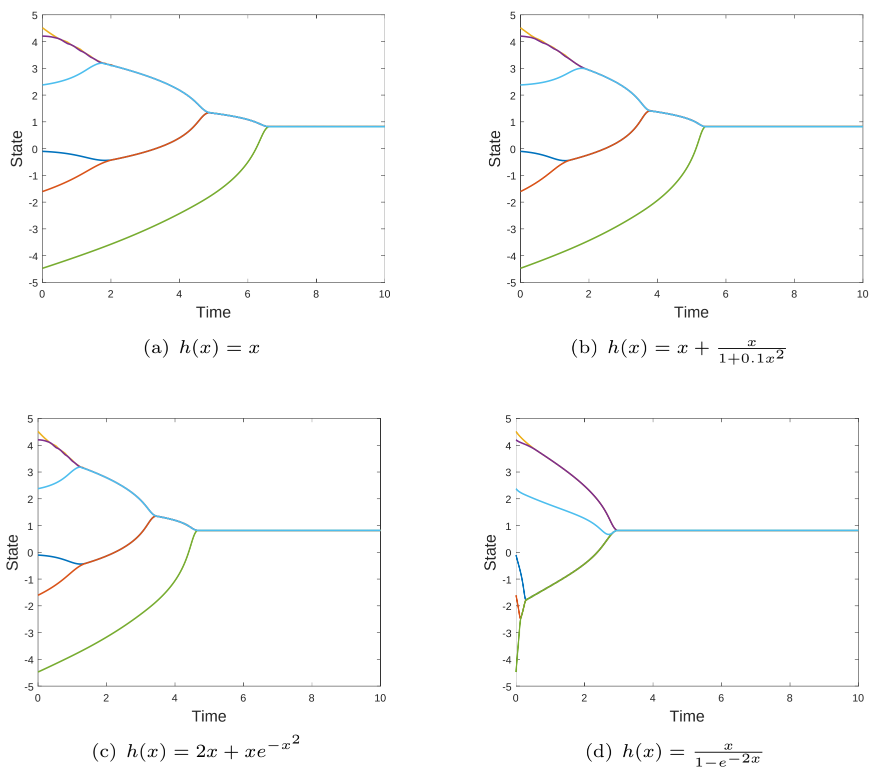
In what follows, we will provide some concrete protocol functions to illustrate theoretical results. For computational convenience and demonstration purpose, we take $N = 6$, $\alpha = 0.5$ and initial data are randomly chosen from $(-5, 5)$. Three different protocol functions $h(x) = x + \frac{x}{1+0.1x^2}$, $h(x) = 2x + xe^{-x^2}$ and $h(x) = \frac{x}{1-e^{-2x}}$ are selected to make comparisons with the existing $h(x) = x$ to demonstrate that the presented protocols can improve convergence rate without increasing control input [17].

To begin with, we take the Cucker-Smale potential

$$a_{ij} = I(|x_j(t) - x_i(t)|) \tag{4.1}$$

with $I(r) = \frac{1}{1+r^2}$ whose symmetry makes the network topology undirected all the time. Consensus behaviors under four protocol functions are shown in Figure 1 respectively. It is clear that all the protocols enable the states of six agents to reach an agreement. Meanwhile, we compute the convergence time out numerically and the results are 6.618 for $h(x) = x$, 5.348 for $h(x) = x + \frac{x}{1+0.1x^2}$, 4.601 for $h(x) = 2x + xe^{-x^2}$ and 2.975 for $h(x) = \frac{x}{1-e^{-2x}}$. Obviously, the three proposed protocols do achieve faster consensus than that with $h(x) = x$.

Remark 4.1. It is reasonable to assume that the mutual influence is a function of the distance between agents. We can refer to [9] where the authors introduced a symmetric pairwise influence function as (4.1), which reflects that the closer two agents are, the more they tend to align with each other, to describe the emergence of flocking behavior.



(a) $h(x) = x$

(b) $h(x) = x + \frac{x}{1+0.1x^2}$

(c) $h(x) = 2x + xe^{-x^2}$

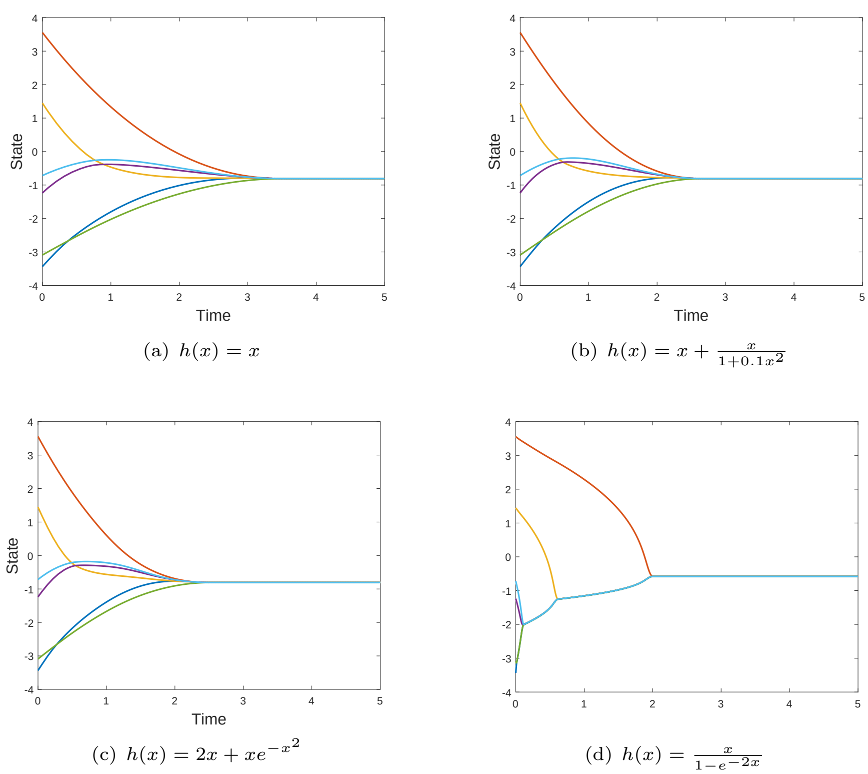
(d) $h(x) = \frac{x}{1-e^{-2x}}$

Figure 1. Consensus of (1.1) with different protocol functions under time-varying undirected topology.

Next we assume that system (1.1) has a fixed directed topology modeled by the detail-balanced adjacent matrix

$$\begin{bmatrix} 0 & 0.3 & 0.2 & 0.4 & 0.5 & 0.1 \\ 0.1 & 0 & 0.1 & 0.2 & 0.5 & 0.25 \\ 0.4 & 0.6 & 0 & 0.5 & 1 & 0.15 \\ 0.8 & 1.2 & 0.5 & 0 & 0.6 & 0.7 \\ 0.1 & 0.3 & 0.1 & 0.06 & 0 & 0.1 \\ 0.2 & 1.5 & 0.15 & 0.7 & 1 & 0 \end{bmatrix}$$

with balanced coefficients $\omega_1 = 1$, $\omega_2 = 3$, $\omega_3 = \omega_4 = \omega_6 = 0.5$ and $\omega_5 = 5$. Initial configurations are also chosen in $(-5, 5)$ but different from those in Figure 1. All the protocols can achieve finite-time consensus, as shown in Figure 2. Under this topology, the three proposed protocols also achieve faster consensus than that with $h(x) = x$. Specifically, the convergence time for $h(x) = x$ is 3.419, for $h(x) = x + \frac{x}{1+0.1x^2}$ is 2.585, for $h(x) = 2x + xe^{-x^2}$ is 2.386 and for $h(x) = \frac{x}{1-e^{-2x}}$ is 1.985.



(a) $h(x) = x$

(b) $h(x) = x + \frac{x}{1+0.1x^2}$

(c) $h(x) = 2x + xe^{-x^2}$

(d) $h(x) = \frac{x}{1-e^{-2x}}$

Figure 2. Consensus of (1.1) with different protocol functions under fixed directed topology.

5. Conclusions

In this paper, a general form of finite-time consensus protocols is proposed to further improve the classical protocol in terms of the convergence rate. We first proved that the presented protocol can admit finite-time consensus for networks with time-varying undirected and time-invariant directed topologies. It is also shown that the convergence time is determined by network parameters and the protocol function $h(x)$. Then by choosing some concrete protocol functions, we numerically illustrate the presented protocol can really provide more flexibility to improve convergence rate. However, much remains to be done and our future work will focus on the case with time-delay and nonlinear dynamics.

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