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DYNAMIC BEHAVIOR OF A DELAY CHOLERA MODEL WITH CONSTANT INFECTIOUS PERIOD*

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Abstract In this paper, a delay cholera model with constant infectious period is investigated. By analyzing the characteristic equations, the local stability of a disease-free equilibrium and an endemic equilibrium of the model is established. It is proved that if the basic reproductive number $\mathcal{R}_0 > 1$, the system is permanent. If $\mathcal{R}_0 < 1$, by means of an iteration technique, sufficient conditions are obtained for the global asymptotic stability of the disease-free equilibrium. If $\mathcal{R}_0 > 1$, also by means of an iteration technique, sufficient conditions are obtained for the global asymptotic stability of the endemic equilibrium. Numerical simulations are carried out to illustrate the main theoretical results.

Keywords Cholera model, permanence, stability, delay.

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1. Introduction

Cholera is an acute bacterial illness caused by infection of the intestinal tract with the bacterium Vibrio cholerae. Cholera may produce severe gastrointestinal symptoms, including profuse, watery diarrhea, as well as vomiting and dehydration [13, 17, 22, 29]. It has long been, and continues to be, a world health issue. Cholera usually occurs in areas where there's poor sanitation, over-crowding, war or famine [30].

Mathematical models can describe the dynamic character of infectious diseases to show the likely outcome of an epidemic. And they have played an important role in the disease control in epidemiological aspect and help inform public health interventions. Waterborne diseases such as cholera, diarrheal disease, dysentery, giardia, are caused by pathogenic microorganisms that most commonly are transmitted in contaminated fresh water. Few researchers have contributed towards the mathematical study of the eradication of waterborne diseases, for example, distributed delay model [27], spatially explicit model [9, 10], time-varying model [2, 25], case

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studies model [24]. Recently, there have been several efforts in the mathematical modelling of cholera dynamics. The first mathematical model of cholera was developed by Capasso and Paveri-Fontana [5]. In [5], they proposed a mathematical model to describe the 1973 cholera epidemic in Bari (a city in Italy). In their version, two equations describe the dynamics of infected people in the community and the dynamics of the aquatic population of pathogenic bacteria. In 2001, Codeço [6] extended the model in reference [5]. She added an equation for the dynamics of the susceptible population. She studied the role of the aquatic reservoir in the endemic-epidemic dynamics of cholera. In [23], Pascual et al. generalized Codeço's model by including a fourth equation for the volume of water in which the formative live following Codeço's [6]. In [32], Zhou X.Y. et al. considered a cholera model with vaccination on the base of the model of Codeço [6]. They added an equation for the dynamics of the vaccinated populations. They analyzed the locally and globally asymptotical stability of the disease-free and endemic equilibria of their system. In [28], Jianjun Paul Tian et al. presented several nonlinear ordinary differential systems of cholera, which incorporated both human population and pathogen Vibrio cholerae concentration. They employed three different techniques, including the monotone dynamical systems, the geometric approach, and Lyapunov functions, to investigate the endemic global stability for several biologically important cases. We may find other mathematical studies on modeling cholera dynamics in references [18–20, 33]. To the best of our knowledge, these studies do not explicitly confider a delay cholera model with constant infectious period.

In the natural world, there are many diseases which the infected population recover and become susceptible or removed population by itself after they are infected by some certain time. The phenomenon was studied by Hethcote et. al. [14]. For cholera, the incubation period ranges from a few hours to 5 days, usually 2-3 days [15]. Hence, in this paper, we will present a delay cholera model with constant infectious period. We consider the total human population sizes denoted by $N(t)$, which including susceptible individuals $S(t)$, infected individuals $I(t)$ and recovered individuals $R(t)$. The pathogen population at time t, is given by $B(t)$. The susceptible human population is increased by births and/or immigration at a constant rate A (> 0). Natural death occurs in the human classes at a rate μ_1 (> 0). Infected individuals may die due to cholera at a rate δ (> 0). Infected people contribute to the concentration of vibrios at a rate η (> 0). The pathogen population is generated at a rate $\hat{\mu}$ (> 0) and the cholera pathogen has a natural death rate $\check{\mu}$ (> 0) in the aquatic environment, which in this case, is the set of untreated water consumed by the population. According to Islam [16], we know that *Vibrio cholerae* population decay does not necessarily imply death but also the transition towards a non-culturable state. Hence, we assume $\check{\mu} > \hat{\mu}$, and vibrios have a net death rate $\mu_2 = \check{\mu} - \hat{\mu}$.

We assume that susceptible people becomes infected at a rate $\beta\lambda(B)$, where β is the rate of contact with untreated water and $\lambda(B)$ is the probability of such person to catch cholera. And $\lambda(B)$ depends on the concentration of Vibrio cholerae, B, which is given by the dose-response function $\frac{B}{K+B}$, where K is the concentration of V. cholera in water that yields 50% chance of catching cholera [6]. We also assume that when a susceptible individual is infected, there is a time τ (> 0) during which the infectious individual develops, and only after that time the infected individual becomes the removed one [8,31]. The time τ is called infection time. The probability that an individual remains in the infectious period at least t time units before developing Cholera is given by a step function with value 1 for $0 \le t \le \tau$ and value zero for $t > \tau$. The probability that an individual in the infectious period time t units has survived to develop cholera is $e^{-(\mu_1+\delta)\tau}$.

The model is given in the following:

$$\begin{cases} \frac{dS(t)}{dt} = A - \frac{\beta S(t)B(t)}{K+B(t)} - \mu_1 S(t), \\ \frac{dI(t)}{dt} = \frac{\beta S(t)B(t)}{K+B(t)} - \frac{\beta e^{-(\mu_1+\delta)\tau} S(t-\tau)B(t-\tau)}{K+B(t-\tau)} - (\mu_1+\delta)I, \\ \frac{dR(t)}{dt} = \frac{\beta e^{-(\mu_1+\delta)\tau} S(t-\tau)B(t-\tau)}{K+B(t-\tau)} - \mu_1 R(t), \\ \frac{dB(t)}{dt} = \eta I(t) - \mu_2 B(t). \end{cases} \tag{1.1}$$

The second equation of system (1.1) can be rewritten by formally integrating the delay differential equations for $I(t)$ as follows:

$$I(t) = \int_0^\tau \frac{\beta S(t-\theta)B(t-\theta)}{K+B(t-\theta)} e^{-(\mu_1+\delta)\theta} d\theta. \tag{1.2}$$

Furthermore, from the last two equations of system (1.1), we can obtain

$$R(t) = R(0)e^{-(\mu_1+\delta)t} + e^{-(\mu_1+\delta)\tau} \int_0^t \frac{\beta S(t-\theta)B(t-\theta)}{K+B(t-\theta)} e^{-(\mu_1+\delta)(t-\theta)} d\theta \tag{1.3}$$

and

$$B(t) = B(0)e^{-\mu_2 t} + \eta e^{-\mu_2 t} \int_0^t I(\varsigma) e^{-\mu_2 \varsigma} d\varsigma. \tag{1.4}$$

The initial conditions for system (1.1) take the form

$$\begin{aligned} &S(\theta) = \varphi_1(\theta), I(\theta) = \varphi_2(\theta), R(\theta) = \varphi_3(\theta), B(\theta) = \varphi_4(\theta), \\ &\varphi_i(\theta) \ge 0, \theta \in [-\tau, 0], \varphi_i(0) > 0, i = 1,2,3,4, \end{aligned} \tag{1.5}$$

where $(\varphi_1(\theta), \varphi_2(\theta), \varphi_3\theta), \varphi_4(\theta)) \in C([-\tau, 0], \mathbb{R}^4_{+0})$, the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^4_{+0}$, where $\mathbb{R}^4_{+0} = \{(x_1, x_2, x_3, x_4) : x_i \ge 0, i = 1,2,3,4\}$.

In this paper, we will discuss the dynamical behavior of system (1.1). The remainder of this paper is originated as follows. In the next section, we present some basic results, for example, the positive invariance of system (1.1), the existence of equilibria, the boundedness of solutions. In Section 3, we derive the local and global stability of the disease-free equilibrium. A set of conditions which assure the permanence of the system (1.1) are obtained in Section 4. In Section 5, we derive the local and global stability of the endemic equilibrium. Numerical simulations are carried out to illustrate the main theoretical results in Section 6. A brief discussion is given in Section 7 to conclude this work.

2. Some basic results

In this section, we present some basic results, such as the positive invariance of system (1.1), the existence of equilibria, the boundedness of solutions. It is important to show positivity and boundedness for the system (1.1) as they represent populations. Positivity implies that populations survives and boundedness may be interpreted as a natural restriction to growth as a consequence of limited resources.

2.1. Positivity

Since all the state variables (i.e. S, I, R, B) in system (1.1); susceptibles, infectious, recovered and the v. cholerae population are number densities, then they are required to be non-negative. Hence, we need to show the positivity of solutions of the system (1.1).

Theorem 2.1. *The solutions $S(t)$, $I(t)$, $R(t)$, $B(t)$ of system (1.1) are positive for all $t \geq 0$ with initial conditions (1.5).*

Proof. Let $T = \sup\{t \geq 0 : S > 0,\ I > 0,\ R > 0,\ B > 0 \text{ in } [0,t]\}$. Clearly, $T > 0$, and if $T < \infty$ then one of $S(t)$, $I(t)$, $R(t)$, $B(t)$ must be zero. We have from system (1.1) that

$$\frac{d}{dt}[S(t)\exp\{\int_0^t \frac{\beta B(u)}{K+B(u)}du + \mu_1 t\}] = A\exp\{\int_0^t \frac{\beta B(u)}{K+B(u)}du + \mu_1 t\}.$$

Thus,

$$S(T)\exp\{\int_0^t \frac{\beta B(u)}{K+B(u)}du + \mu_1 T\} - S(0) = \int_0^T A\exp\int_0^x \{\frac{\beta B(v)}{K+B(v)}dv + \mu_1 x\}dx,$$

so that

$$\begin{aligned} S(T) = S(0)\exp\{\int_0^t -\frac{\beta B(u)}{K+B(u)}du - \mu_1 T\} + \exp\{\int_0^t -\frac{\beta B(u)}{K+B(u)}du - \mu_1 T\} \\ \times \int_0^T A\exp\int_0^x \{\frac{\beta B(v)}{K+B(v)}dv + \mu_1 x\}dx > 0. \end{aligned}$$

From the second equation of system (1.1), we have

$$I(t) = \int_{t-\tau}^t \frac{\beta S(u)B(u)}{K+B(u)}e^{-(\mu_1+\delta)(t-u)}du > 0,$$

which is strictly positive in $[0, \epsilon]$ for small $\epsilon > 0$.

From (1.3) and (1.4), we can conclude that $R(t) > 0$ and $B(t) > 0$. Thus we can conclude that solutions of system (1.1) remain positive for all $t > 0$.

2.2. Boundedness

Theorem 2.2. *All solutions of system (1.1) satisfying conditions (1.5) are bounded for all $t \geq 0$ at which the solution exists.*

Proof. From the first equations of system (1.1), we can obtain

$$0 < \frac{d(S(t)+I(t)+R(t))}{dt} \leq A - \mu_1(S(t)+I(t)+R(t))$$

for $t \geq 0$ with initial condition $S(0)+I(0)+R(0) > 0$. Thus, we can get

$$0 < S(t)+I(t)+R(t) \leq (S(0)+I(0)+R(0) - \frac{A}{\mu_1})e^{-(\mu_1+\delta)\tau} + \frac{A}{\mu_1}$$

for $t \geq 0$.

From the last equation of system (1.1), we can get

$$\frac{dB(t)}{dt} \leq \eta \frac{A}{\mu_1} - \mu_2 B(t)$$

for $t \geq 0$.

Hence,

$$0 < B(t) \leq (B(0) - \frac{\eta A}{\mu_1 \mu_2}) e^{-\mu_2 t} + \frac{\eta A}{\mu_1 \mu_2}$$

for $t \geq 0$.

We can obtain that

$$\Omega = \{(S, I, R, B) \in \mathbb{R}_+^4 \,|\, 0 < S + I + R \leq \frac{A}{\mu_1}, 0 < B \leq \frac{\eta A}{\mu_1 \mu_2}\}.$$

We complete the proof of the theorem.

2.3. Equilibria

It is easy to see that the system (1.1) always exists a disease-free equilibrium $E_0(S_0, 0, 0, 0)$ (where $S_0 = \frac{A}{\mu_1}$), which exists for all values of the parameters.

In order to consider the existence of the endemic equilibrium E^*, we need define the basic reproduction number $\mathcal{R}_0$ according to the definition in [7], extended to a delay epidemic model. Let $\mathcal{R}_0 = \frac{A\beta\eta(1-e^{-(\mu_1+\delta)\tau})}{K\mu_1\mu_2(\mu_1+\delta)}$. $\mathcal{R}_0$ is called the basic reproduction number.

System (1.1) has an endemic equilibrium $E^*(S^*, I^*, R^*, B^*)$ when $\mathcal{R}_0 > 1$, where $S^* = \frac{K(\mu_1+\delta)\mu_2 + A\eta(1-e^{-(\mu_1+\delta)\tau})}{\eta(\beta+\mu_1)(1-e^{-(\mu_1+\delta)\tau})}$, $I^* = \frac{A\beta\eta(1-e^{-(\mu_1+\delta)\tau}) - K\mu_1\mu_2(\mu_1+\delta)}{\eta(\beta\delta+\mu_1^2+\mu_1\delta+\mu_1\beta)}$, $B^* = \frac{\eta}{\mu_2} I^*$, $R^* = \frac{1}{\mu_1} \frac{\beta e^{-(\mu_1+\delta)\tau} S^* B^*}{K+B^*}$.

As $\mathcal{R}_0 = 1$, E^* becomes coincident with E_0. Hence, E^* exists iff $\mathcal{R}_0 > 1$. In this case, the infectious period τ must be larger than the threshold $\tau^* = \frac{1}{\mu_1+\delta} \ln(\frac{A\beta\eta}{A\beta\eta - K\mu_1\mu_2(\mu_1+\delta)})$.

2.4. Characteristic equation

Let $\bar{E}(\bar{S}, \bar{I}, \bar{R}, \bar{B})$ be arbitrarily equilibrium of system (1.1). To study the locally asymptotic stability of the steady states $\bar{E}$, let us define $x_1(t) = S(t) - \bar{S}$, $x_2(t) = I(t) - \bar{I}$, $x_3(t) = R(t) - \bar{R}$ and $x_4(t) = B(t) - \bar{B}$. Then the linearized system of (1.1) at $\bar{E}$ is given by

$$\begin{cases} \frac{dx_1(t)}{dt} = -(\mu_1 + \frac{\beta\bar{B}}{K+\bar{B}})x_1(t) - \frac{\beta K\bar{S}}{(K+\bar{B})^2}x_4(t), \\ \frac{dx_2(t)}{dt} = \frac{\beta\bar{B}}{K+\bar{B}}x_1(t) - (\mu_1+\delta)x_2(t) + \frac{\beta K\bar{S}}{(K+\bar{B})^2}x_4(t) - \frac{\beta\bar{B}e^{-(\mu_1+\delta)\tau}}{K+\bar{B}}x_1(t-\tau) \\ \qquad -\frac{\beta K\bar{S}e^{-(\mu_1+\delta)\tau}}{(K+\bar{B})^2}x_4(t-\tau), \\ \frac{dx_3(t)}{dt} = \frac{\beta\bar{B}e^{-(\mu_1+\delta)\tau}}{K+\bar{B}}x_1(t-\tau) - \mu_1 x_3(t) + \frac{\beta K\bar{S}e^{-(\mu_1+\delta)\tau}}{(K+\bar{B})^2}x_4(t-\tau), \\ \frac{dx_4(t)}{dt} = \eta x_2(t) - \mu_2 x_4(t). \end{cases} \tag{2.1}$$

We then express system (2.1) in matrix form as follows:

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} = \mathcal{A}_1 \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} + \mathcal{A}_2 \begin{pmatrix} x_1(t-\tau) \\ x_2(t-\tau) \\ x_3(t-\tau) \\ x_4(t-\tau) \end{pmatrix},$$

where $\mathcal{A}_1$ and $\mathcal{A}_2$ are 4×4 matrices given by

$$\mathcal{A}_1 = \begin{pmatrix} -(\mu_1 + \frac{\beta\bar{B}}{K+\bar{B}}) & 0 & 0 & -\frac{\beta K\bar{S}}{(K+\bar{B})^2} \\ \frac{\beta\bar{B}}{K+\bar{B}} & -(\mu_1+\delta) & 0 & \frac{\beta K\bar{S}}{(K+\bar{B})^2} \\ 0 & 0 & -\mu_1 & 0 \\ 0 & \eta & 0 & -\mu_2 \end{pmatrix},$$

$$\mathcal{A}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{\beta\bar{B}e^{-(\mu_1+\delta)\tau}}{K+\bar{B}} & 0 & 0 & -\frac{\beta K\bar{S}e^{-(\mu_1+\delta)\tau}}{(K+\bar{B})^2} \\ \frac{\beta\bar{B}e^{-(\mu_1+\delta)\tau}}{K+\bar{B}} & 0 & 0 & \frac{\beta K\bar{S}e^{-(\mu_1+\delta)\tau}}{(K+\bar{B})^2} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The characteristic equation of system (1.1) is given by

$$\Delta(\lambda) = |\lambda\mathcal{I} - \mathcal{A}_1 - e^{-\lambda\tau}\mathcal{A}_2| = 0, \tag{2.2}$$

where $\mathcal{I}$ is the 4×4 identity matrix.

3. Stability of disease-free equilibrium E_0

In this section, we will discuss the local and global stability of the disease-free equilibrium E_0 of system (1.1), respectively.

For the disease-free equilibrium E_0, (2.2) becomes

$$\begin{vmatrix} -\mu_1-\lambda & 0 & 0 & -\frac{\beta S_0}{K} \\ 0 & -(\mu_1+\delta)-\lambda & 0 & \frac{\beta S_0}{K} - \frac{\beta S_0}{K}e^{-(\mu_1+\delta)\tau}e^{-\lambda\tau} \\ 0 & 0 & -\mu_1-\lambda & \frac{\beta S_0}{K}e^{-(\mu_1+\delta)\tau}e^{-\lambda\tau} \\ 0 & \eta & 0 & -\mu_2-\lambda \end{vmatrix} = 0,$$

i.e.,

$$(\lambda+\mu_1)^2 f_1(\lambda,\tau) = 0, \tag{3.1}$$

where

$$f_1(\lambda,\tau) = \lambda^2 + (\mu_1+\delta+\mu_2)\lambda + (\mu_1+\delta)\mu_2 - \frac{\eta\beta S_0}{K} + \frac{\eta\beta S_0}{K}e^{-(\mu_1+\delta)\tau}e^{-\lambda\tau}. \tag{3.2}$$

Clearly, (3.1) always has two negative roots $\lambda_1 = \lambda_2 = -\mu_1$. Other roots of (3.1) are determined by the equation $f_1(\lambda, \tau) = 0$.

From

$$f_1(\lambda, 0) = \lambda^2 + (\mu_1 + \delta + \mu_2)\lambda + (\mu_1 + \delta)\mu_2 = 0, \tag{3.3}$$

it is apparent that (3.3) has two negative roots $-\mu_2$ and $-\mu_1 - \delta$ if $\mathcal{R}_0 < 1$.

Now, when $\mathcal{R}_0 < 1$, we need to show that all the eigenvalues in $f_1(\lambda, 0) = 0$ have negative real parts. First, note that any eigenvalue in $f_1(\lambda, 0) = 0$ satisfies

$$(\lambda + \mu_1 + \delta)(\lambda + \mu_2) = \frac{\eta\beta S_0}{K} - \frac{\eta\beta S_0}{K} e^{-(\mu_1+\delta)\tau} e^{-\lambda\tau},$$

which is equivalent to

$$(\frac{\lambda}{\mu_1 + \delta} + 1)(\frac{\lambda}{\mu_2} + 1) = \mathcal{R}_0 \frac{1 - e^{-(\mu_1+\delta)\tau} e^{-\lambda\tau}}{1 - e^{-(\mu_1+\delta)\tau}}.$$

Assume that there exists a zero in $f_1(\lambda, \tau) = 0$ with $Re(\lambda) \geq 0$, then $|\frac{\lambda}{\mu+\delta} + 1| \geq 1$, $|\frac{\lambda}{\mu_2} + 1| \geq 1$. We can get, $|\frac{1-e^{-(\mu_1+\delta)\tau} e^{-\lambda\tau}}{1-e^{-(\mu_1+\delta)\tau}}| \geq 1$, which leads to a contradiction. Hence, all the eigenvalues in (3.1) have negative real parts, implying E_0 is locally asymptotically stable.

Then we have the following theorem.

Theorem 3.1. *If $\mathcal{R}_0 < 1$, the disease-free equilibrium E_0 is locally asymptotically stable for all $\tau \geq 0$.*

In the following, we will consider the attraction of the disease-free equilibrium for system (1.1). In order to consider the attraction of the equilibria and the permanence of the solutions of system (1.1), we need the following important lemmas.

Lemma 3.1. (Fatou Lemma) [11] *Let $\{f_n\}_{n\in\mathbb{N}_0}$ be a measurable sequence of non-negative function defied on a measurable set Ω. Then*

$$\int_\Omega \liminf_{n\to+\infty} f_n dx \leq \liminf_{n\to+\infty} \int_\Omega f_n dx.$$

Lemma 3.2. (Inverse Fatou Lemma) [11] *Let $\{f_n\}_{n\in\mathbb{N}_0}$ be a measurable sequence of non-negative function defied on a measurable set Ω. If there exists a non-negative integrable function g defined on Ω and such that $f_n \leq g$ on Ω for all n, then*

$$\int_\Omega \limsup_{n\to+\infty} f_n dx \geq \limsup_{n\to+\infty} \int_\Omega f_n dx.$$

Lemma 3.3. *(1)* $\limsup_{t\to+\infty} B(t) \leq \frac{\eta}{\mu_2} \limsup_{t\to+\infty} I(t)$;

(2) $\liminf_{t\to+\infty} B(t) \geq \frac{\eta}{\mu_2} \liminf_{t\to+\infty} I(t)$.

Proof. From the last equation of system (1.1), we can obtain

$$B(t) = e^{-\mu_2 t}(B(0) + \int_0^t \eta I(s) e^{\mu_2 s} ds).$$

From Lemma 3.1, we can get

$$\begin{aligned}\limsup_{t\to+\infty} B(t) &= \limsup_{t\to+\infty} e^{-\mu_2 t}B(0) + \limsup_{t\to+\infty} e^{-\mu_2 t}\int_0^t \eta I(s)e^{\mu_2 s}ds.\\ &\le \eta \limsup_{t\to+\infty} e^{-\mu_2 t}\int_0^t \limsup_{s\to+\infty} I(s)e^{\mu_2 s}ds.\end{aligned}$$

Hence, $\limsup_{t\to+\infty} B(t) \le \frac{\eta}{\mu_2}\limsup_{t\to+\infty} I(t)$.

Similarly, we can get $\liminf_{t\to+\infty} B(t) \ge \frac{\eta}{\mu_2}\liminf_{t\to+\infty} I(t)$.

Theorem 3.2. *If $\mathcal{R}_0 < 1$, the disease-free equilibrium E_0 is globally attractive.*

Proof. Let $(S(t), I(t), R(t), B(t))$ be any positive solution of system (1.1) with initial conditions (1.5). It follows from the first equations of system (1.1) that

$$\frac{dS}{dt} \le A - \mu_1 S.$$

A standard comparison argument shows that

$$\limsup_{t\to+\infty} S(t) \le \frac{A}{\mu_1}.$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_1 > 0$ such that if $t > T_1$, $S(t) \le \frac{A}{\mu_1}+\varepsilon$. It follows from the second equation of system (1.1) that

$$I(t) = \int_{t-\tau}^t \frac{\beta S(u)B(u)}{K+B(u)}e^{-(\mu_1+\delta)(t-u)}du = \int_0^\tau \frac{\beta S(t-\varsigma)B(t-\varsigma)}{K+B(t-\varsigma)}e^{-(\mu_1+\delta)\varsigma}d\varsigma. \tag{3.4}$$

From (3.4), we can obtain

$$\limsup_{t\to+\infty} I(t) = \limsup_{t\to+\infty}\int_0^\tau \frac{\beta S(t-\varsigma)B(t-\varsigma)}{K+B(t-\varsigma)}e^{-(\mu_1+\delta)\varsigma}d\varsigma.$$

By Lemmas 3.2 and 3.3, we can obtain

$$\begin{aligned}\frac{\mu_2}{\eta}\limsup_{t\to+\infty} B(t) &\le \limsup_{t\to+\infty} I(t)\\ &\le \int_0^\tau \frac{\beta \limsup_{t\to+\infty} B(t)}{K+\limsup_{t\to+\infty} B(t)}\limsup_{t\to+\infty} S(t)e^{-(\mu_1+\delta)\varsigma}d\varsigma\\ &\le \frac{A}{\mu_1}\frac{\beta \limsup_{t\to+\infty} B(t)}{K+\limsup_{t\to+\infty} B(t)}\int_0^\tau e^{-(\mu_1+\delta)\varsigma}d\varsigma\\ &= \tfrac{\beta A}{\mu_1(\mu_1+\delta)}(1-e^{-(\mu_1+\delta)\tau})\frac{\beta \limsup_{t\to+\infty} B(t)}{K+\limsup_{t\to+\infty} B(t)}.\end{aligned} \tag{3.5}$$

From (3.5), we can get

$$\frac{\beta \limsup_{t\to+\infty} B(t)}{K+\limsup_{t\to+\infty} B(t)}(\mathcal{R}_0 - 1 - \limsup_{t\to+\infty} B(t)) \ge 0. \tag{3.6}$$

Noting that $\mathcal{R}_0 < 1$, it follows from (3.6) that

$$\limsup_{t\to+\infty} B(t) = 0.$$

Therefore, for $\varepsilon > 0$ sufficiently small, there is a $T_2 > T_1$ such that if $t > T_2$, $B(t) \le \varepsilon$.

For $\varepsilon > 0$ sufficiently small, we obtain from the third equation of system (1.1) that, for $t > T_2 + \tau$,

$$\frac{dR(t)}{dt} \le \frac{\beta e^{-(\mu_1+\delta)\tau}(\frac{A}{\mu_1} + \varepsilon)\varepsilon}{K + \varepsilon} - \mu_1 R(t).$$

Hence,

$$\limsup_{t\to+\infty} R(t) \le \frac{\beta e^{-(\mu_1+\delta)\tau}(\frac{A}{\mu_1} + \varepsilon)\varepsilon}{\mu_1(K + \varepsilon)}.$$

Letting $\varepsilon \to 0$, it follows that $\lim\limits_{t\to+\infty} R(t) = 0$.

From the first equation of system (1.1), for $t > T_2$, we can obtain

$$\frac{dS(t)}{dt} \ge A - \mu_1 S(t) - \frac{\beta\varepsilon}{K+\varepsilon} S(t).$$

By comparison it follows that

$$\liminf_{t\to+\infty} S(t) \ge \frac{A}{\mu_1 + \frac{\beta\varepsilon}{K+\varepsilon}}.$$

Letting $\varepsilon \to 0$, we get

$$\liminf_{t\to+\infty} S(t) \ge \frac{A}{\mu_1}.$$

Therefore,

$$\lim_{t\to+\infty} S(t) = \frac{A}{\mu_1}.$$

From (3.5) and $\limsup\limits_{t\to+\infty} S(t) \le \dfrac{A}{\mu_1}$, for $t \ge T_2 + \tau$, we can obtain

$$\begin{aligned}\lim_{t\to+\infty} I(t) &\le \frac{\beta A}{\mu_1(\mu_1+\delta)}(1 - e^{-(\mu_1+\delta)\tau})\frac{\beta \limsup\limits_{t\to+\infty} B(t)}{K + \limsup\limits_{t\to+\infty} B(t)} \\ &\le \tfrac{\beta A}{\mu_1(\mu_1+\delta)}(1 - e^{-(\mu_1+\delta)\tau})\frac{\beta\varepsilon}{K+\varepsilon}.\end{aligned}$$

Letting $\varepsilon \to 0$, we get

$$\limsup_{t\to+\infty} I(t) = 0.$$

This completes the proof.

From Theorems 3.1 and 3.2, we can obtain the following result.

Theorem 3.3. *If $\mathcal{R}_0 < 1$, the disease-free equilibrium E_0 is globally asymptotically stable for all $\tau \ge 0$.*

Theorem 3.4. *If $\mathcal{R}_0 > 1$, the disease-free equilibrium E_0 is unstable.*

Proof. Let

$$f_1(\lambda) = \lambda^2 + (\mu_1 + \delta + \mu_2)\lambda + (\mu_1 + \delta)\mu_2 - \frac{\eta\beta S_0}{K} + \frac{\eta\beta S_0}{K} e^{-(\mu_1+\delta)\tau} e^{-\lambda\tau}.$$

If $\mathcal{R}_0 > 1$, then it is easy to show that, for λ real,

$$\begin{aligned} f_1(0) &= (\mu_1 + \delta)\mu_2 - \tfrac{\eta\beta S_0}{K} + \tfrac{\eta\beta S_0}{K} e^{-(\mu_1+\delta)\tau} \\ &= (\mu_1 + \delta)\mu_2(1 - \mathcal{R}_0) \\ &< 0 \end{aligned}$$

and

$$\lim_{\lambda\to+\infty} f_1(\lambda) = +\infty.$$

Hence, (3.2) has a positive root at least. Accordingly, the disease-free equilibrium E_0 is unstable if $\mathcal{R}_0 > 1$. □

4. Permanence

In this section, we will investigate the permanence of system (1.1).

Definition 4.1. System (1.1) is said to be permanence if there exists a compact region $\Omega_0 \subset \operatorname{int} \mathbb{R}^4_+$ such that every solution $(S(t), I(t), R(t), B(t))$ of system (1.1) with initial conditions (1.5) will eventually enter and remain in the region Ω_0.

Theorem 4.1. *If $\mathcal{R}_0 > 1$, system (1.1) is permanent.*

Proof. Let $(S(t), I(t), R(t), B(t))$ be any positive solution of system (1.1) with initial conditions (1.5). Recalling $N(t) = S(t) + I(t) + R(t)$, it follows from the first three equations of system (1.1) that

$$\frac{dN(t)}{dt} = A - \mu_1 N(t) - \delta I(t) \leq A - \mu_1 N(t).$$

A standard comparison argument shows that

$$\limsup_{t\to+\infty} N(t) \leq \frac{A}{\mu_1}, \tag{4.1}$$

which yields

$$\limsup_{t\to+\infty} I(t) \leq \frac{A}{\mu_1}.$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_1 > 0$ such that if $t > T_1$, $I(t) \leq \frac{A}{\mu_1} + \varepsilon$. It follows from the last equation of system (1.1) that, for $t > T_1$,

$$\frac{dB(t)}{dt} \leq \eta(\frac{A}{\mu_1} + \varepsilon) - \mu_2 B(t),$$

which yields

$$\limsup_{t\to+\infty} B(t) \leq \frac{\eta}{\mu_2}(\frac{A}{\mu_1} + \varepsilon).$$

Since the above inequality holds for arbitrary $\varepsilon > 0$ sufficiently small, it follows that

$$\limsup_{t\to+\infty} B(t) \leq \frac{\eta}{\mu_1}\frac{A}{\mu_2}.$$

Hence, for $\varepsilon > 0$ sufficiently small there is a $T_2 > 0$ such that if $t > T_2$, $B(t) \leq \frac{\eta}{\mu_1}\frac{A}{\mu_2} + \varepsilon$.

It follows from the first equation of system (1.1) that, for $t > T_2$,

$$\frac{dS(t)}{dt} \geq A - (\mu_1 + \frac{\beta(\frac{\eta}{\mu_1}\frac{A}{\mu_2} + \varepsilon)}{K + \frac{\eta}{\mu_1}\frac{A}{\mu_2} + \varepsilon})S(t),$$

which yields

$$\liminf_{t\to+\infty} S(t) \geq A/(\mu_1 + \frac{\beta(\frac{\eta}{\mu_1}\frac{A}{\mu_2} + \varepsilon)}{K + \frac{\eta}{\mu_1}\frac{A}{\mu_2} + \varepsilon}).$$

Since this inequality holds for arbitrary $\varepsilon > 0$ sufficiently small, it follows that

$$\liminf_{t\to+\infty} S(t) \geq A/(\mu_1 + \frac{\beta\frac{\eta}{\mu_1}\frac{A}{\mu_2}}{K + \frac{\eta}{\mu_1}\frac{A}{\mu_2}}) \triangleq v_1. \tag{4.2}$$

Hence, for $\varepsilon>0$ sufficiently small there is a $T_3>T_2$ such that if $t>T_3$, $S(t)\geq v_1-\varepsilon$.

Choose positive constants B_0 large enough and d small enough such that

$$1 < \frac{A\beta\eta(1 - e^{-(\mu_1+\delta)\tau})[1 - e^{-(\mu_1+\frac{\beta B^0}{K+B^0})d}]}{(\mu_1 + \delta)\mu_2[K\mu_1 + B^0(\mu_1 + \beta)]} \triangleq q. \tag{4.3}$$

We now claim that if $\mathcal{R}_0 > 1$, there does not exist any $\bar{t}_1 > 0$ such that $B(t) \leq B^0$ for all $t > \bar{t}_1$. Otherwise, there exists a t_0 such that

$$B(t) \leq B^0, t \geq t_0. \tag{4.4}$$

It follows from the first equation of system (1.1) and (4.4) that, for $t > t_0$,

$$\frac{dS(t)}{dt} \geq A - (\mu_1 + \frac{\beta B^0}{K + B^0})S(t).$$

Thus, for $t \geq t_0 + d$, we have

$$\begin{aligned} S(t) &\geq S(t_0)e^{-(\mu_1+\frac{\beta B^0}{K+B^0})(t-t_0)} + A\int_{t_0}^{t} e^{-(\mu_1+\frac{\beta B^0}{K+B^0})(t-\varsigma)}d\varsigma \\ &\geq \frac{A(K + B^0)}{\beta B^0 + \mu_1(K + B^0)}[1 - e^{-(\mu_1+\frac{\beta B^0}{K+B^0})d}] \\ &\triangleq S^{\Delta}. \end{aligned} \tag{4.5}$$

For $t > 0$, define a differentiable function

$$V(t) = \frac{\mu_1 + \delta}{\eta}B(t) + I(t) - \beta e^{-(\mu_1+\delta)\tau}\int_{t-\tau}^{t} \frac{S(\varsigma)B(\varsigma)}{K + B(\varsigma)}d\varsigma. \tag{4.6}$$

Calculating the derivative of $V(t)$ along solutions of system (1.1) we derive that

$$\frac{dV(t)}{dt} = \frac{\mu_2(\mu_1+\delta)}{\eta}[\frac{\eta\beta(1-e^{-(\mu_1+\delta)\tau})}{(\mu_1+\delta)\mu_2}\frac{S(t)}{K+B(t)} - 1]B(t). \tag{4.7}$$

It follows from (4.4), (4.5) and (4.7) that

$$\begin{aligned}\frac{dV(t)}{dt} &\geq \frac{\mu_2(\mu_1+\delta)}{\eta}[\frac{\eta\beta(1-e^{-(\mu_1+\delta)\tau})}{(\mu_1+\delta)\mu_2}\frac{S^\Delta}{K+B^0} - 1]B(t) \\ &= \tfrac{\mu_2(\mu_1+\delta)}{\eta}(q-1)B(t), t \geq t_0+d.\end{aligned} \tag{4.8}$$

Set

$$B_l^0 = \min_{\varsigma\in[-\tau,0]} I(t_0+d+\tau+\varsigma) > 0.$$

The second equation of (1.1) can be rewritten as

$$I(t) = \beta\int_{t-\tau}^{t}\frac{S(u)B(u)}{K+B(u)}e^{-(\mu_1+\delta)(t-u)}du, t \geq \tau. \tag{4.9}$$

It follows from (4.9) that

$$I(t) \geq \beta\int_{t^*-\tau}^{t^*}\frac{S^\Delta B_l^0}{K+B_l^0}e^{-(\mu_1+\delta)(t-u)}du = \frac{\beta S^\Delta B_l^0}{K+B_l^0}\frac{1}{\mu_1+\delta}(1-e^{-(\mu_1+\delta)}). \tag{4.10}$$

From the last equation of system (1.1) we can get

$$\frac{dB(t)}{dt} \geq \eta\frac{\beta S^\Delta B_l^0}{K+B_l^0}\frac{1}{\mu_1+\delta}(1-e^{-(\mu_1+\delta)}) - \mu_2 B(t).$$

Hence,

$$B(t) \geq \frac{\eta}{\mu_2}\frac{\beta S^\Delta B_l^0}{K+B_l^0}\frac{1}{\mu_1+\delta}(1-e^{-(\mu_1+\delta)}).$$

Since $B_l^0 \leq B^0$, we derive from (4.2) that

$$\frac{\eta\beta(1-e^{-(\mu_1+\delta)\tau})}{(\mu_1+\delta)\mu_2}\frac{S^\Delta}{K+B_l^0} \geq \frac{\eta\beta(1-e^{-(\mu_1+\delta)\tau})}{(\mu_1+\delta)\mu_2}\frac{S^\Delta}{K+B^0} = q > 1. \tag{4.11}$$

We deduce from (4.10) and (4.11) that $B(t^*) > B_l^0$, which is a contradiction. This proves the claim.

Therefore, we obtain from (4.3) and (4.8) that

$$\frac{dV(t)}{dt} > \frac{\mu_2(\mu_1+\delta)}{\eta}(q-1)B_l^0 > 0,$$

which implies that $V(t)\to\infty$ as $t\to\infty$. On the other hand, it follow from (4.1) and (4.6) that

$$\limsup_{t\to+\infty} V(t) \leq \frac{A(\mu_1+\delta)}{\mu_1\mu_2} + \frac{A}{\mu_1} + \frac{\eta A^2 e^{-(\mu_1+\delta)\tau}\tau}{\mu_1(K\mu_1\mu_2+\eta A)}.$$

A contradiction occurs. Hence, the claim is proved.

By the claim, we are left to consider two possibilities. First, $B(t) \geq B^0$ for all t sufficiently large. Second, $B(t)$ oscillates about B^0 for all t sufficiently large.

For the second case, we assume that

$$B(t_1) = B(t_1 + \gamma) = B^0 \text{ and } B(t) < B^0 \text{ for } t_1 < t < t_1 + \gamma,$$

where t_1 sufficiently large such that $S(t) \geq v_1 - \varepsilon$ for $\varepsilon > 0$ being sufficiently small. Since $B(t)$ is uniformly continuous, there is a $0 < T < \tau$ (independent of the choice of t_1) such that $B(t) > \frac{B^0}{2}$ for $t_1 < t <_1 +T$. If $\gamma \leq T$, there is nothing to prove. Let us consider the case that $T < \gamma <\leq \tau$. For $t_1 + T < t \leq t_1 + \gamma$, we have

$$\begin{aligned} I(t) &\geq \beta(v_1 - \varepsilon) \textstyle\int_{t-\tau}^{t} \frac{B(u)}{K+B(u)} e^{-(\mu_1+\delta)(t-u)} du \\ &\geq \beta(v_1 - \varepsilon) \textstyle\int_{t_1}^{t_1+T} \frac{B(u)}{K+B(u)} e^{-(\mu_1+\delta)(t-u)} du \\ &\geq \beta(v_1 - \varepsilon) \frac{B^0 T}{2K+B^0} e^{-(\mu_1+\delta)\tau}. \end{aligned} \tag{4.12}$$

From the last equation of system (1.1), we have

$$\frac{dB(t)}{dt} \geq \eta\beta(v_1 - \varepsilon) \frac{B^0 T}{2K + B^0} e^{-(\mu_1+\delta)\tau} - \mu_2 B(t).$$

Hence,

$$B(t) \geq \frac{1}{\mu_2} \eta\beta(v_1 - \varepsilon) \frac{B^0 T}{2K + B^0} e^{-(\mu_1+\delta)\tau} := B_0.$$

Define

$$B_1 = \min\{\frac{B^0}{2}, B_0\}. \tag{4.13}$$

We get $I(t) \geq I_1$ for $t \in [t_1, t_1 + \gamma]$. For $t \in (t_1 + \tau, t_1 + \frac{3\tau}{2}]$, from (4.12), we have

$$\begin{aligned} I(t) &\geq \beta(v_1 - \varepsilon) \textstyle\int_{t_1+\frac{\tau}{2}}^{t_1+\tau} \frac{B_1}{K+B_1} e^{-(\mu_1+\delta)(t-u)} du \\ &\geq \beta(v_1 - \varepsilon) \frac{B_1 \tau}{2(K+B_1)} e^{-(\mu_1+\delta)\tau}. \end{aligned} \tag{4.14}$$

From the last equation of system (1.1) and (4.14), we have

$$\frac{dB(t)}{dt} \geq \eta\beta(v_1 - \varepsilon) \frac{B_1 \tau}{2(K + B_1)} e^{-(\mu_1+\delta)\tau} - \mu_2 B(t).$$

Hence,

$$B(t) \geq \frac{1}{\mu_2} \eta\beta(v_1 - \varepsilon) \frac{B_1 \tau}{2(K + B_1)} e^{-(\mu_1+\delta)\tau} \triangleq B_2. \tag{4.15}$$

For $t \in (t_1 + \tau, t_1 + \frac{3\tau}{2}]$, from (4.15), we have

$$\begin{aligned} I(t) &\geq \beta(v_1 - \varepsilon) \textstyle\int_{t_1+\frac{\tau}{2}}^{t_1+\tau} \frac{B_2}{K+B_2} e^{-(\mu_1+\delta)(t-u)} du \\ &\geq \beta(v_1 - \varepsilon) \frac{B_2 \tau}{2(K+B_2)} e^{-(\mu_1+\delta)\tau}. \end{aligned} \tag{4.16}$$

From the last equation of system (1.1) and (4.16), we have

$$\frac{dB(t)}{dt} \geq \eta\beta(v_1 - \varepsilon) \frac{B_2 \tau}{2(K + B_2)} e^{-(\mu_1+\delta)\tau} - \mu_2 B(t).$$

Hence,

$$B(t) \geq \frac{1}{\mu_2}\eta\beta(v_1-\varepsilon)\frac{B_2\tau}{2(K+B_2)}e^{-(\mu_1+\delta)\tau} \triangleq B_3. \tag{4.17}$$

Denote

$$B_j = \frac{1}{\mu_2}\eta\beta(v_1-\varepsilon)\frac{B_{j-1}\tau}{2(K+B_{j-1})}e^{-(\mu_1+\delta)\tau}, j=2,3,\cdots,2k-1. \tag{4.18}$$

Continuing the process above, we derive that

$$B(t) \geq B_{2m-2}, t \in (t_1+(m-1)\tau, t_1+(m-\frac{1}{2})\tau];$$

$$B(t) \geq B_{2m-1}, t \in (t_1+(m-\frac{1}{2})\tau, t_1+m\tau], m=2,3,\cdots,k.$$

Denote

$$v_2 = \min_{1\leq i\leq 2k-1} B_i,$$

where $k=[\frac{d}{\tau}]$ ($[x]$ is the minimum integer being greater than or equal to x), and B_i ($1 \leq i \leq 2k-1$) are defined in (4.13), (4.15), (4.17) and (4.18).

Obviously, if $\gamma \leq d$, $B(t) \geq v_2$ for $t \in [t_1, t_1+\gamma]$. If $\gamma > d$, then $B(t) \geq v_2$ for $t \in [t_1, t_1+d]$. Furthermore, we can show that $B(t) \geq v_2$ for $t \in (t_1+d, t_1+\gamma]$. In fact, if not, there exists a T^* such that $B(t) \geq v_2$ for $t_1 \leq t \leq t_1+d+T^* \leq t_1+\gamma$ and $B(t_1+d+T^*) = v_2$. On the other hand, $B(t) \leq B^0$ for $t_1 \leq t \leq t_1+\gamma$. Then (4.5) holds true, i.e., $S(t) > S^\Delta$ for $t \geq t_1+d$. It follows that, for $t = t_1+d+T^*$,

$$\begin{aligned} I(t) &\geq \tfrac{\beta S^\Delta v_2}{K+v_2}\int_{t-\tau}^{t} e^{-(\mu_1+\delta)(t-u)}du \\ &= \tfrac{\beta S^\Delta v_2}{(K+v_2)(\mu_1+\delta)}(1-e^{-(\mu_1+\delta)\tau}). \end{aligned} \tag{4.19}$$

From the last equation of system (1.1) and (4.19), we have

$$\frac{dB(t)}{dt} \geq \frac{\eta\beta S^\Delta v_2}{(K+v_2)(\mu_1+\delta)}(1-e^{-(\mu_1+\delta)\tau}) - \mu_2 B(t). \tag{4.20}$$

Hence,

$$B(t) \geq \frac{1}{\mu_2}\frac{\eta\beta S^\Delta v_2}{(K+v_2)(\mu_1+\delta)}(1-e^{-(\mu_1+\delta)\tau}).$$

Noting $v_2 \leq B^0$, we derive from (4.3) that

$$\frac{\eta\beta(1-e^{-(\mu_1+\delta)\tau})}{(\mu_1+\delta)\mu_2}\frac{S^\Delta}{K+v_2} \geq \frac{\eta\beta(1-e^{-(\mu_1+\delta)\tau})}{(\mu_1+\delta)\mu_2}\frac{S^\Delta}{K+B^0} = q > 1. \tag{4.21}$$

Hence, we can deduce from (4.20) and (4.21) that $B(t_1+d+T^*) > v_2$, which is a contradiction. Therefore, we have that $B(t) \geq v_2$ for $t \in [t_1+d, t_1+\gamma]$. Since this kind of interval $[t_1+d, t_1+\gamma]$ is chosen in an arbitrary way (we only need t_1 to be large), we conclude that $B(t) \geq v_2$ for all t sufficiently large in the second case. Accordingly, $\liminf_{t\to+\infty} B(t) \geq v_2$.

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_4 > T_3$ such that if $t > T_4$, $B(t) \geq v_2 - \varepsilon$. For $\varepsilon > 0$ sufficiently small, it follows from the third equation of system (1.1) that, for $t > T_4$,

$$\frac{dR(t)}{dt} \geq \frac{\beta e^{-(\mu_1+\delta)\tau}(v_1-\varepsilon)(v_2-\varepsilon)}{K+v_2-\varepsilon} - \mu_1 R(t),$$

which yields

$$\liminf_{t\to+\infty} R(t) \geq \frac{\beta e^{-(\mu_1+\delta)\tau}(v_1-\varepsilon)(v_2-\varepsilon)}{\mu_2(K+v_2-\varepsilon)}.$$

Since the inequality holds true for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that

$$\liminf_{t\to+\infty} R(t) \geq \frac{\beta e^{-(\mu_1+\delta)\tau}v_1v_2}{\mu_2(K+v_2)} = v_3.$$

From the proof we can see that $\liminf_{t\to+\infty} I(t) \geq v_4$, where $v_4 = \frac{\mu_2}{\eta}v_2$. This completes the proof.

5. Stability of endemic equilibrium E^*

In this section, we will discuss the locally and globally asymptotical stability of the endemic equilibrium E^* of system (1.1), respectively.

For the endemic equilibrium E^*, (2.2) becomes

$$\begin{vmatrix} -\mu_1-\frac{\beta B^*}{K+B^*}-\lambda & 0 & 0 & -\frac{\beta KS^*}{(K+B^*)^2} \\ \frac{\beta B^*}{K+B^*}(1-e^{-(\mu_1+\delta)\tau}) & -(\mu_1+\delta)-\lambda & 0 & \frac{\beta KS^*}{(K+B^*)^2}(1-e^{-(\mu_1+\delta)\tau}) \\ \frac{\beta B^*}{K+B^*}e^{-(\mu_1+\delta)\tau} & 0 & -\mu_1-\lambda & \frac{\beta KS^*}{(K+B^*)^2}e^{-(\mu_1+\delta)\tau} \\ 0 & \eta & 0 & -\mu_2-\lambda \end{vmatrix} = 0,$$

i.e.,

$$(\lambda+\mu_1)[\lambda^3+\kappa_1(\tau)\lambda^2+\kappa_2(\tau)\lambda+\kappa_3(\tau)+e^{-\lambda\tau}(\kappa_4(\tau)\lambda+\kappa_5(\tau))] = 0, \tag{5.1}$$

where

$$\kappa_1(\tau) = 2\mu_1+\mu_2+\delta+\frac{\beta B^*}{K+B^*},$$

$$\kappa_2(\tau) = 2\mu_1\mu_2+\delta\mu_1+\mu_1^2+\frac{\beta B^*}{K+B^*}(\mu_1+A\mu_2+\delta)+\delta\mu_2-\eta\frac{\beta KS^*}{(K+B^*)^2},$$

$$\kappa_3(\tau) = \delta\mu_2\mu_1+\mu_1^2\mu_2-\eta\mu_1\frac{\beta KS^*}{(K+B^*)^2}+\frac{\beta B^*}{K+B^*}(\mu_1\mu_2+\delta\mu_2),$$

$$\kappa_4(\tau) = e^{-(\mu_1+\delta)\tau}\eta\frac{\beta KS^*}{(K+B^*)^2},$$

$$\kappa_5(\tau) = e^{-(\mu_1+\delta)\tau}\eta\mu_1\frac{\beta KS^*}{(K+B^*)^2}.$$

Clearly, (5.1) always has a negative roots $\lambda_1 = -\mu_1$. Other roots of (5.1) are determined by the following equation

$$\lambda^3+\kappa_1(\tau)\lambda^2+\kappa_2(\tau)\lambda+\kappa_3(\tau)+e^{-\lambda\tau}(\kappa_4(\tau)\lambda+\kappa_5(\tau)) = 0. \tag{5.2}$$

When $\tau = 0$, (5.2) becomes

$$\lambda^3+\kappa_1(0)\lambda^2+(\kappa_2(0)+\kappa_4(0))\lambda+\kappa_3(0)+\kappa_5(0) = 0. \tag{5.3}$$

It is easy to see that $\kappa_1(0) > 0$, $\kappa_2(0)+\kappa_4(0) > 0$, $\kappa_3(0)+\kappa_5(0) > 0$ and $\kappa_1(0)(\kappa_2(0)+\kappa_4(0)) - (\kappa_3(0) + \kappa_5(0)) > 0$ when $\tau = 0$. Hence, the endemic equilibrium E^* is locally asymptotically stable when $\mathcal{R}_0 > 1$ and $\tau = 0$. In the following, we will consider the local stability of E^* when $\mathcal{R}_0 > 1$ and $\tau > 0$.

If $i\omega$ $(\omega > 0)$ is a solution of (5.2) if and only if

$$-i\omega^3 - \kappa_1\omega^2 + i\kappa_2\omega + \kappa_3 + (\cos\omega\tau - \sin\omega\tau)(i\kappa_4\omega + \kappa_5) = 0.$$

Separating real and imaginary parts, we have

$$\begin{cases} -\omega^3 + \kappa_2\omega + \kappa_4\omega\cos\omega\tau - \kappa_5\sin\omega\tau = 0, \\ -\kappa_1\omega^2 + \kappa_3 + \kappa_5\omega\cos\omega\tau + \kappa_4\sin\omega\tau = 0. \end{cases} \tag{5.4}$$

It follows from (5.4) that

$$\omega^6 + \vartheta_1\omega^4 + \vartheta_2\omega^2 + \vartheta_3 = 0, \tag{5.5}$$

where

$$\begin{aligned} \vartheta_1 &= \kappa_1^2 - 2\kappa_2, \\ \vartheta_2 &= \kappa_2^2 - \kappa_4 - 2\kappa_1\kappa_3, \\ \vartheta_3 &= \kappa_3^2 - \kappa_5^2. \end{aligned} \tag{5.6}$$

Let $z = \omega^2$. Then (5.5) becomes

$$z^3 + \vartheta_1 z^2 + \vartheta_2 z + \vartheta_3 = 0. \tag{5.7}$$

In order to consider the existence of positive zeros of the above third degree polynomials, we need the following lemma.

Denote $\Delta = \vartheta_1^2 - 3\vartheta_2$ and $z_1 = \frac{-\vartheta_1+\sqrt{\Delta}}{3}$.

Lemma 5.1 ([26]). *Let $g(z) = z^3 + \vartheta_1 z^2 + \vartheta_2 z + \vartheta_3$.*

(1) If $\vartheta_3 < 0$, then equation (5.7) has at least one positive root.

(2) If $\vartheta_3 \geq 0$ and $\Delta < 0$, then equation (5.7) has no positive root.

(3) If $\vartheta_3 \geq 0$, then equation (5.7) has positive roots if and only if $z_1 > 0$ and $g(z_1) \leq 0$.

From Lemma 5.1, we can conclude that if $\vartheta_3 \geq 0$ and $\Delta < 0$, the positive equilibrium E^* of system (1.1) is locally stable. If (i) $\vartheta_3 < 0$, or (ii) $\vartheta_3 \geq 0$ and $g(z_1) \leq 0$ where $z_1 = \frac{-\vartheta_1+\sqrt{\Delta}}{3} > 0$, then stability switches may occur. In order to find the τ values of stability switches, for each positive root $\omega(\tau)$ of equation (5.5), we define the angle $\theta(\tau) \in (\pi, 2\pi)$ as a solution of

$$\sin\theta(\tau) = -\frac{\kappa_3\kappa_4\omega - \kappa_2\kappa_5\omega + \kappa_5\omega^3}{(\kappa_4\omega)^2 + \kappa_5^2},$$

$$\cos\theta(\tau) = -\frac{\kappa_3\kappa_5 + \kappa_2\kappa_4\omega^2 - \kappa_4\omega^4}{(\kappa_4\omega)^2 + \kappa_5^2}.$$

For each $\omega(\tau)$ satisfying (5.5), we define

$$\mathcal{S}_n(\tau) = \tau - \frac{\theta(\tau) + 2n\pi}{\omega(\tau)}, \ n = 0, \pm 1, \pm 2, \cdots. \tag{5.8}$$

According to the well known work on characteristic equation of delay differential equations with delay-dependent parameters developed by Beretta and Kuang [**?**], we have the following results.

Theorem 5.1. *Suppose $\mathcal{R}_0 > 1$ and $\vartheta_1, \vartheta_2, \vartheta_3$ are defined in (5.6). For system (1.1), we have*

(1) If $\vartheta_3 \geq 0$ and $\Delta < 0$, the endemic equilibrium E^* is locally asymptotically stable for all $\tau \geq 0$.

(2) Let $\vartheta_3 < 0$, or $\vartheta_3 \geq 0$ and $g(z_1) \leq 0$ where $z_1 = \frac{-\vartheta_1 + \sqrt{\Delta}}{3} > 0$. Assume that there is a $\tau_1^* > 0$ satisfying $\mathcal{S}_n(\tau_1^*) = 0$ for some $n \in \mathbb{N}_0$ and that (5.5) has a pair of simple and conjugate pure imaginary roots $\lambda = \pm i\omega(\tau_1)$ with $\omega(\tau_1^*) > 0$. Then a pair of simple conjugate pure imaginary roots $\lambda = \pm i\omega$ exists at $\tau = \tau_1^*$ which crosses the imaginary axis from left to right if $\delta(\tau_1^*) > 0$ and crosses the imaginary axis from right to left if $\delta(\tau_1^*) < 0$, where

$$\delta(\tau_1^*) = sign\{\frac{d(Re\lambda)}{d\tau}|_{\lambda=i\omega(\tau_1^*)}\} = sign\{\frac{d\mathcal{S}_n(\tau)}{d\tau}|_{\tau=\tau_1^*}\}.$$

Theorem 5.2. *If $\mathcal{R}_0 > 1$ and $\mu_1 > \beta$ hold true, then the endemic equilibrium E^* is globally attractive.*

Proof. Let $(S(t), I(t), R(t), B(t))$ be any positive solution of system (1.1) with initial conditions (1.5). Let

$$\begin{aligned}
\overline{S} = \limsup_{t\to+\infty} S(t), \overline{I} = \limsup_{t\to+\infty} I(t), \overline{R} = \limsup_{t\to+\infty} R(t), \overline{B} = \limsup_{t\to+\infty} B(t),\\
\underline{S} = \liminf_{t\to+\infty} S(t), \underline{I} = \liminf_{t\to+\infty} I(t), \underline{R} = \liminf_{t\to+\infty} R(t), \underline{B} = \liminf_{t\to+\infty} B(t).
\end{aligned}$$

In the following we claim that $\overline{S} = \underline{S} = S^*$, $\overline{I} = \underline{I} = I^*$, $\overline{R} = \underline{R} = R^*$, $\overline{B} = \underline{B} = B^*$.

It follows from the first equation of system (1.1) that

$$\frac{dS}{dt} \leq A - \mu_1 S,$$

which yields

$$\limsup_{t\to+\infty} S(t) \leq \frac{A}{\mu_1} \triangleq M_1^S.$$

Hence, for $\varepsilon > 0$ sufficiently small there is a $T_1 > 0$ such that if $t > T_1$, $S(t) \leq M_1^S + \varepsilon$.

It follows from the second equation of system (1.1) that

$$I(t) = \int_{t-\tau}^{t} \frac{\beta S(u)B(u)}{K + B(u)} e^{-(\mu_1+\delta)(t-u)} du = \int_0^{\tau} \frac{\beta S(t-\varsigma)B(t-\varsigma)}{K + B(t-\varsigma)} e^{-(\mu_1+\delta)\varsigma} d\varsigma. \tag{5.9}$$

We derive from (5.9) that

$$\begin{aligned}
\limsup_{t\to+\infty} I(t) &= \limsup_{t\to+\infty} \int_0^\tau \frac{\beta S(t-\varsigma)B(t-\varsigma)}{K+B(t-\varsigma)} e^{-(\mu_1+\delta)\varsigma} d\varsigma \\
&\le \int_0^\tau \frac{\beta \limsup_{t\to+\infty} B(t-\varsigma)}{K+\limsup_{t\to+\infty} B(t-\varsigma)} \limsup_{t\to+\infty} S(t-\varsigma) e^{-(\mu_1+\delta)\varsigma} d\varsigma \\
&\le \int_0^\tau \frac{\beta \limsup_{t\to+\infty} B(t)}{K+\limsup_{t\to+\infty} B(t)} \limsup_{t\to+\infty} S(t) e^{-(\mu_1+\delta)\varsigma} d\varsigma \\
&\le \frac{\beta \limsup_{t\to+\infty} B(t)}{K+\limsup_{t\to+\infty} B(t)} M_1^S \int_0^\tau e^{-(\mu_1+\delta)\varsigma} d\varsigma \\
&= \frac{\beta M_1^S}{\mu_1+\delta}(1-e^{-(\mu_1+\delta)\tau}) \frac{\limsup_{t\to+\infty} B(t)}{K+\limsup_{t\to+\infty} B(t)} \\
&\le \frac{\beta M_1^S}{\mu_1+\delta}(1-e^{-(\mu_1+\delta)\tau}) \frac{\frac{\eta}{\mu_2}\limsup_{t\to+\infty} I(t)}{K+\frac{\eta}{\mu_2}\limsup_{t\to+\infty} I(t)}.
\end{aligned} \tag{5.10}$$

From (5.10), we can obtain

$$\limsup_{t\to+\infty} I(t) \le \frac{\beta M_1^S(1-e^{-(\mu_1+\delta)\tau})}{\mu_1+\delta} - \frac{K\mu_2}{\eta} \triangleq M_1^I.$$

Hence, for $\varepsilon > 0$ sufficiently small there is a $T_2 > T_1 > 0$ such that if $t > T_2$, $I(t) \le M_1^I + \varepsilon$.

From the last equation of (1.1), we can get

$$\limsup_{t\to+\infty} B(t) \le \frac{\mu_2}{\eta} \limsup_{t\to+\infty} I(t) \le \frac{\mu_2}{\eta} M_1^I \triangleq M_1^B.$$

Hence, for $\varepsilon > 0$ sufficiently small there is a $T_3 > T_2 > 0$ such that if $t > T_3$, $B(t) \le M_1^B + \varepsilon$.

It follows from the third equation of system (1.1) that, for $t > T_2 + \tau$,

$$\frac{dR}{dt} \le \frac{\beta e^{-(\mu_1+\delta)\tau}(M_1^S+\varepsilon)(M_1^B+\varepsilon)}{K+M_1^B+\varepsilon} - \mu_1 R(t).$$

Hence,

$$\limsup_{t\to+\infty} R(t) \le \frac{1}{\mu_1} \frac{\beta e^{-(\mu_1+\delta)\tau}(M_1^S+\varepsilon)(M_1^B+\varepsilon)}{K+M_1^B+\varepsilon}.$$

Since this inequality holds true for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that

$$\limsup_{t\to+\infty} R(t) \le \frac{1}{\mu_1} \frac{\beta e^{-(\mu_1+\delta)\tau} M_1^S M_1^B}{K+M_1^B} \triangleq M_1^R.$$

Therefore, for $\varepsilon > 0$ sufficiently small there is a $T_4 > T_3 + \tau > 0$ such that if $t > T_4$, $R(t) \le M_1^R + \varepsilon$.

It follows from the first equation of system (1.1) that, for $t > T_4$

$$\frac{dS(t)}{dt} \geq A - \frac{\beta(M_1^S + \varepsilon)(M_1^B + \varepsilon)}{K + M_1^B + \varepsilon} - \mu_1 S(t).$$

By comparison we derive that

$$\liminf_{t\to+\infty} S(t) \geq \frac{1}{\mu_1}[A - \frac{\beta(M_1^S + \varepsilon)(M_1^B + \varepsilon)}{K + M_1^B + \varepsilon}].$$

Since this inequality holds true for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $\liminf\limits_{t\to+\infty} S(t) \geq N_1^S$, where

$$N_1^S = \frac{1}{\mu_1}[A - \frac{\beta M_1^S M_1^B}{K + M_1^B}].$$

Hence, for $\varepsilon > 0$ sufficiently small there is a $T_5 > T_4$ such that if $t > T_5$, $S(t) \geq N_1^S - \varepsilon$. It follows from the second equation of system (1.1) that for $t > T_4$

$$\begin{aligned}
\liminf_{t\to+\infty} I(t) &= \liminf_{t\to+\infty} \int_0^\tau \frac{\beta S(t-\varsigma)B(t-\varsigma)}{K + B(t-\varsigma)} e^{-(\mu_1+\delta)\varsigma} d\varsigma \\
&\geq \int_0^\tau \frac{\beta \liminf\limits_{t\to+\infty} B(t-\varsigma)}{K + \liminf\limits_{t\to+\infty} B(t-\varsigma)} \liminf_{t\to+\infty} S(t-\varsigma) e^{-(\mu_1+\delta)\varsigma} d\varsigma \\
&\geq \int_0^\tau \frac{\beta \liminf\limits_{t\to+\infty} B(t)}{K + \liminf\limits_{t\to+\infty} B(t)} \liminf_{t\to+\infty} S(t) e^{-(\mu_1+\delta)\varsigma} d\varsigma \\
&\geq \frac{\beta \liminf\limits_{t\to+\infty} B(t)}{K + \liminf\limits_{t\to+\infty} B(t)} N_1^S \int_0^\tau e^{-(\mu_1+\delta)\varsigma} d\varsigma \\
&= \frac{\beta N_1^S}{\mu_1 + \delta} (1 - e^{-(\mu_1+\delta)\tau}) \frac{\liminf\limits_{t\to+\infty} B(t)}{K + \liminf\limits_{t\to+\infty} B(t)} \\
&\geq \frac{\beta N_1^S}{\mu_1 + \delta} (1 - e^{-(\mu_1+\delta)\tau}) \frac{\frac{\eta}{\mu_2} \liminf\limits_{t\to+\infty} I(t)}{K + \frac{\eta}{\mu_2} \liminf\limits_{t\to+\infty} I(t)}.
\end{aligned} \tag{5.11}$$

By Theorem 4.1, we see that if $\mathcal{R}_0 > 1$, $\liminf\limits_{t\to+\infty} I(t) > 0$. Therefore, we derive from (5.11) that

$$\liminf_{t\to+\infty} I(t) \geq \frac{\beta N_1^S (1 - e^{-(\mu_1+\delta)\tau})}{\mu_1 + \delta} - \frac{K\mu_2}{\eta} \triangleq N_1^I.$$

Hence, for $\varepsilon > 0$ sufficiently small there is a $T_6 > T_5$ such that if $t > T_6$, $I(t) \geq N_1^I - \varepsilon$.

From the last equation of (1.1), we can get

$$\liminf_{t\to+\infty} B(t) \geq \frac{\mu_2}{\eta} \liminf_{t\to+\infty} I(t) \geq \frac{\mu_2}{\eta} N_1^I \triangleq N_1^B.$$

Hence, for $\varepsilon > 0$ sufficiently small there is a $T_7 > T_6 > 0$ such that if $t > T_7$, $B(t) \leq N_1^B - \varepsilon$.

It follows from the third equation of system (1.1) that, for $t > T_7 + \tau$,

$$\frac{dR}{dt} \geq \frac{\beta e^{-(\mu_1+\delta)\tau}(N_1^S - \varepsilon)(N_1^B - \varepsilon)}{K + N_1^B - \varepsilon} - \mu_1 R(t).$$

Hence,

$$\liminf_{t\to+\infty} R(t) \geq \frac{1}{\mu_1}\frac{\beta e^{-(\mu_1+\delta)\tau}(N_1^S - \varepsilon)(N_1^B - \varepsilon)}{K + N_1^B - \varepsilon}.$$

Since this inequality holds true for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that

$$\liminf_{t\to+\infty} R(t) \geq \frac{1}{\mu_1}\frac{\beta e^{-(\mu_1+\delta)\tau}N_1^S N_1^B}{K + N_1^B} \triangleq N_1^R.$$

Therefore, for $\varepsilon > 0$ sufficiently small there is a $T_8 > T_7 + \tau > 0$ such that if $t > T_8$, $R(t) \geq N_1^R - \varepsilon$.

Again, it follows from the first equation of system (1.1) that, for $t > T_8$

$$\frac{dS(t)}{dt} \leq A - \frac{\beta(N_1^S - \varepsilon)(N_1^B - \varepsilon)}{K + N_1^B - \varepsilon} - \mu_1 S(t).$$

By comparison we derive that

$$\liminf_{t\to+\infty} S(t) \leq \frac{1}{\mu_1}[A - \frac{\beta(N_1^S - \varepsilon)(N_1^B - \varepsilon)}{K + N_1^B - \varepsilon}].$$

Since this inequality holds true for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $\liminf_{t\to+\infty} S(t) \leq M_2^S$, where

$$M_2^S = \frac{1}{\mu_1}[A - \frac{\beta N_1^S N_1^B}{K + N_1^B}].$$

Hence, for $\varepsilon > 0$ sufficiently small there is a $T_9 > T_8$ such that if $t > T_9$, $S(t) \leq M_2^S + \varepsilon$.

Repeating the above arguments, we obtain eight sequences M_n^S, M_n^I, M_n^R, M_n^B, N_n^S, N_n^I, N_n^R, N_n^B $(n = 1, 2, 3, \cdots)$ such that, for $n \geq 2$,

$$\begin{aligned}
M_n^S &= \frac{1}{\mu_1}[A - \frac{\beta N_{n-1}^S N_{n-1}^B}{K + N_{n-1}^B}],\\
N_n^S &= \frac{1}{\mu_1}[A - \frac{\beta M_n^S M_n^B}{K + M_n^B}],\\
M_n^I &= \frac{\beta M_n^S(1 - e^{-(\mu_1+\delta)\tau})}{\mu_1 + \delta} - \frac{K\mu_2}{\eta},\\
N_n^I &= \frac{\beta N_n^S(1 - e^{-(\mu_1+\delta)\tau})}{\mu_1 + \delta} - \frac{K\mu_2}{\eta},\\
M_n^B &= \frac{\eta}{\mu_2} M_n^I,\\
N_n^B &= \frac{\eta}{\mu_2} N_n^I,\\
M_n^R &= \frac{1}{\mu_1}\frac{\beta e^{-(\mu_1+\delta)\tau} M_n^S M_n^B}{K + M_n^B},\\
N_n^R &= \frac{1}{\mu_1}\frac{\beta e^{-(\mu_1+\delta)\tau} N_n^S N_n^B}{K + N_n^B}.
\end{aligned} \tag{5.12}$$

Clearly, we have that

$$N_n^S \le \underline{S} \le \overline{S} \le M_n^S, N_n^I \le \underline{I} \le \overline{I} \le M_n^I,$$
$$N_n^R \le \underline{R} \le \overline{R} \le M_n^R, N_n^B \le \underline{B} \le \overline{B} \le M_n^B.$$

It follows from (5.12) that

$$M_{n+1}^S = \frac{1}{\mu_1}[(1-\frac{\beta}{\mu_1})A + \frac{(\mu_1-\beta)K(\mu_1+\delta)\mu_2}{\eta\mu_1(1-e^{-(\mu_1+\delta)\tau})} + \frac{\beta^2}{\mu_1}M_n^S]. \tag{5.13}$$

Noting that $M_n^S > S^*$ and $\mu_1 > \beta$, it follows from (5.13) that

$$\begin{aligned} M_{n+1}^S - M_n^S &= \frac{1}{\mu_1}(1-\frac{\beta}{\mu_1})[A + \frac{(\mu_1+\delta)}{1-e^{-(\mu_1+\delta)\tau}}\frac{K\mu_2}{\eta} - (\beta+\mu_1)M_n^S] \\ &\le \frac{1}{\mu_1}(1-\frac{\beta}{\mu_1})[A + \frac{(\mu_1+\delta)}{1-e^{-(\mu_1+\delta)\tau}}\frac{K\mu_2}{\eta} - (\beta+\mu_1)S^*] \\ &= 0. \end{aligned}$$

Therefore, the sequence M_n^S is monotonically non-increasing. Hence, $\lim_{n\to+\infty} M_n^S$ exists. Taking $n \to +\infty$, it follows from (5.13) that

$$\lim_{n\to+\infty} M_n^S = S^*.$$

We obtain from (5.11) and (5.13) that

$$\lim_{n\to+\infty} N_n^S = S^*, \lim_{n\to+\infty} M_n^I = I^*, \lim_{n\to+\infty} M_n^R = R^*, \lim_{n\to+\infty} M_n^B = B^*,$$
$$\lim_{n\to+\infty} N_n^I = I^*, \lim_{n\to+\infty} N_n^R = R^*, \lim_{n\to+\infty} N_n^B = B^*. \tag{5.14}$$

It therefore follows from (5.13), (5.14) and (4.15) that

$$\lim_{t\to+\infty} S(t) = S^*, \lim_{t\to+\infty} I(t) = I^*, \lim_{t\to+\infty} R(t) = R^*, \lim_{t\to+\infty} B(t) = B^*.$$

Hence, the endemic equilibrium E^* is globally attractive.

From Theorems 5.1 and 5.2, we can get the following result.

Theorem 5.3. *If $\mathcal{R}_0 > 1$, $\mu_1 > \beta$, $\vartheta_3 \ge 0$ and $\Delta < 0$ hold true, then the endemic equilibrium E^* is globally asymptotically stable.*

6. Numerical simulations

In the previous sections, we introduced the analytical tools proposed and used them for a qualitative analysis of the system obtaining some results about the dynamics of the system. In this section, we perform a numerical analysis of the model based on the previous results. In order to illustrate feasibility of the main results of Theorems 5.2 and 5.3, we perform some numerical simulations by using the software Matlab 7.0.

Our model involves 8 parameters, including the delay τ. We choose a set of parameters which are listed in Table 1. In order to support our results about instability switches, we computed the numerical solution of system (1.1) for different

Table 1. Estimation of parameters.

Parameters	Meaning	Values	Reference
A	Recruitment rate of susceptible population	4.2/day	Assumed
β	Exposure rate to contaminated water	0.2143/day	[12]
η	Contribution of infected individuals to the population of V. cholera	100 cells/L-per day	[6]
μ_1	Natural death rate of human	5.48×10^{-5}/day	[21]
$\check{\mu}$	Rate of loss of V. cholera	1.06 /day	[6]
$\hat{\mu}$	Growth rate of V. cholera	0.73/day	[6]
δ	Disease-induced death rate	0.015/day	[12]
K	Concentration of V. cholera in water	9.5×10^6cells/L	[12]
τ	Infectious period	Varied	Assumed

values of τ. Since the zeros of τ_0 occur at $\tau_{01} = 9.1974$ and $\tau_{02} = 11.0892$, we considered the values $\tau = 9$ in the stability region, $\tau = 9.5$ in the instability region and $\tau = 12$ again in the stability region. In the first and third cases (Fig. 1 and Fig. 3), the solution shows dumped oscillations revealing the asymptotic stability of equilibrium E^*, whereas in the second case (Fig. 2) the oscillations are sustained, thus confirming that E^* is unstable.

Although the conditions of Theorem 5.3 (especially, $\mu_1 > \beta$) are not satisfied, the endemic equilibrium E^* will be asymptotically stable by numerical simulations (Fig. 1 and Fig. 3). Therefore, we can affirm that the conditions of Theorem 5.3 have room for improvement.

7. Discussion

In this paper, we formulate a delay cholera epidemic model with a constant infectious period. The model equations are delay differential equations with delay dependent parameters. We discuss the global attractivity of the disease-free equilibrium and the endemic equilibrium of system (1.1) by using iterative schemes and comparison principles, respectively. We also present the permanence of system (1.1). By using the geometric stability switch criteria in delay differential systems with delay dependent parameters, we obtain that there could exist stability switch about the endemic equilibrium. And we have confirmed it via the numerical simulations. We also find that the endemic equilibrium E^* will be asymptotically stable by numerical simulations although the conditions of Theorem 5.2 are not satisfied. Perhaps, we may prove the globally asymptotical stability of the endemic equilibrium E^* by using the method of constructing the appropriate Lyapunov function. We leave it in the future.

In order to consider the effects of infectious period, we differentiate $\mathcal{R}_0$ and I^* with respect to τ. We can obtain $\frac{\partial \mathcal{R}_0}{\partial \tau} = \frac{A\beta\eta(\mu_1+\delta)e^{-(\mu_1+\delta)\tau}}{K\mu_1\mu_2(\mu_1+\delta)} > 0$ and $\frac{\partial I^*}{\partial \tau} = \frac{A\beta\eta(\mu_1+\delta)e^{-(\mu_1+\delta)\tau}}{\mu_1 r+\beta r+\mu_1\delta+\beta\mu_1+\mu_1^2+\beta\delta} > 0$. Therefore, the number of secondary infections will increase when the infectious period increases. And the number of the infectives will increase when the infectious period increases. We can conclude that prolonging infectious period by medical interventions will have negative effect. The infectious period, plays a significant role in cholera surveillance, prevention, and control [1].

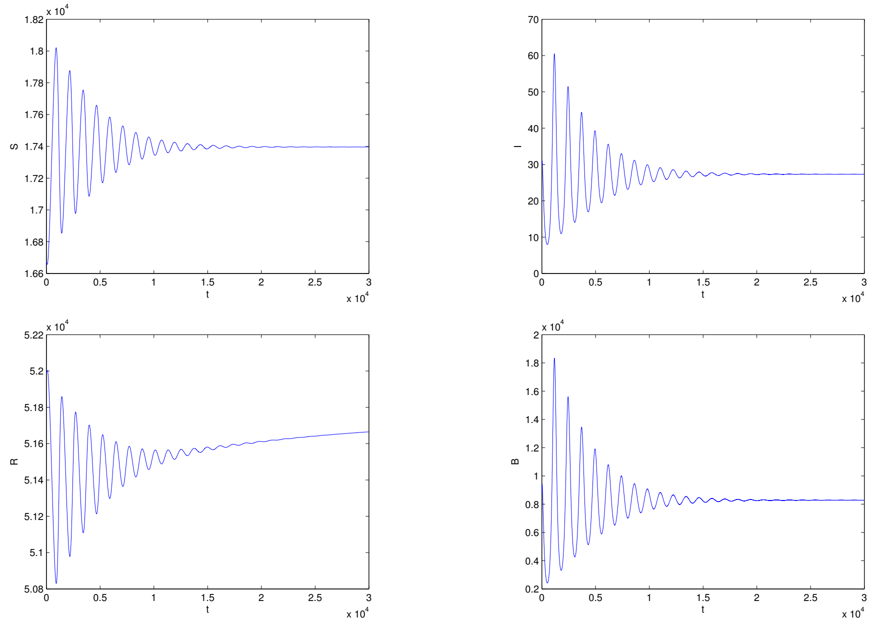

Figure 1. Time evolution of all the population for the model (1.1) with $\tau = 9$ and initial value (15920,30,52313,9274).

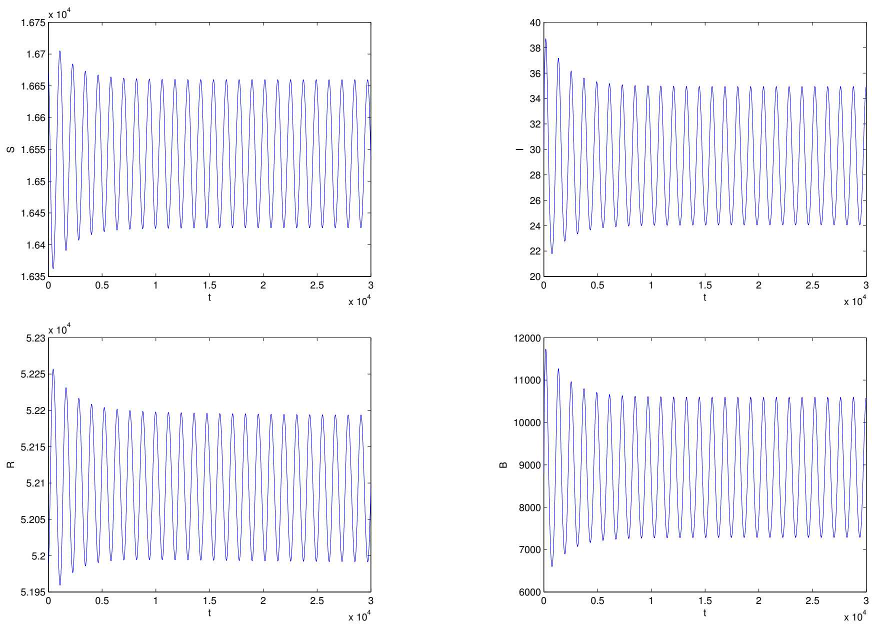

Figure 2. Time evolution of all the population for the model (1.1) with $\tau = 9.5$ and initial value (15920,30,52313,9274).

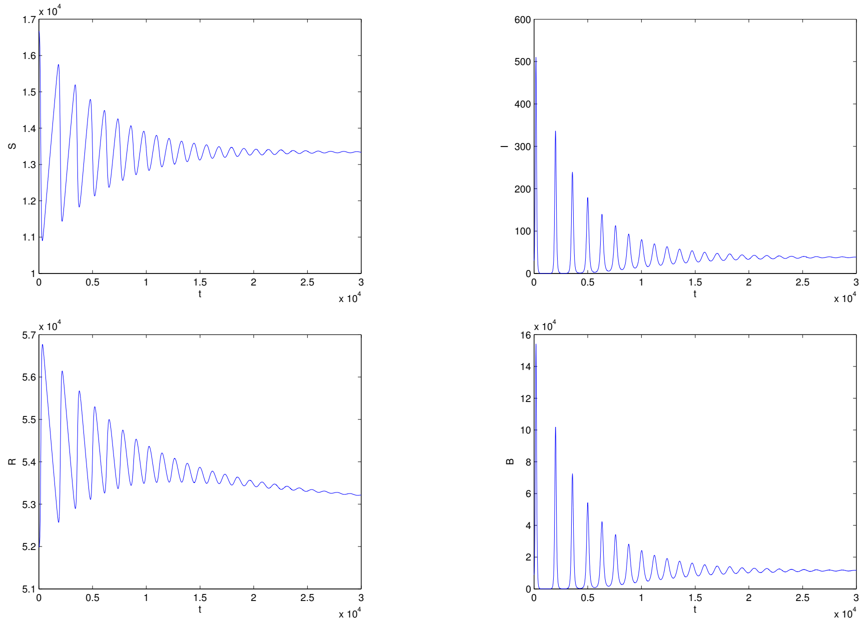

Figure 3. Time evolution of all the population for the model (1.1) with $\tau = 12$ and initial value (15920,30,52313,9274).

The long infectious period for diseases can give individuals a false sense of security. Cholera with long infectious periods are more likely to spread extensively. Hence, we should shorten infectious periods to intervene cholera.

In [4], the authors considered an age-of-infection cholera model. Under some assumptions, the global dynamics of a PDE cholera model was shown to be determined completely by the basic reproduction number $\mathcal{R}_0$. The disease died out if $\mathcal{R}_0$ was below or at the threshold value 1 and otherwise the disease persists. The global stability of the disease-free and endemic equilibria was proved by the construction of Lyapunov functionals. Our model is different from the one proposed in [4], which incorporates simultaneously the age-of-infection structure of individuals and the age structure of pathogen with infectivities given by kernel functions.

Lastly, we can improve the cholera model by several ways. For example, we can consider a cholera model with both constant latency time and constant infectious period. In this case, we may add an equation for the dynamics of the latented populations. We may also consider the vaccination effort of the cholera. And we can add an equation for the dynamics of the vaccinated populations. All of them will be left in the future.

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