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LOCAL BIFURCATION OF CRITICAL PERIODS IN QUADRATIC-LIKE CUBIC SYSTEMS*

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Abstract In this paper, we investigate quadratic-like cubic systems having a center at O for the local bifurcation of critical periods. We provide an inductive algorithm to compute polynomials of periodic coefficients, find structures of solutions for systems of algebraic equations corresponding to weak centers of finite order, and derive conditions on parameters under which the considered equilibrium is a weak center of order k, $k = 0, 1, 2, 3, 4$. Furthermore, we show that with appropriate perturbations, at most four critical periods bifurcate from the weak center of finite order, and we give conditions under which exactly k critical periods bifurcate from the center O for each integer $k = 1, 2, 3, 4$.

Keywords Quadratic-like cubic systems, critical period bifurcation, pseudo division, variety decomposition.

MSC(2010) 34C07, 34C23.

1. Introduction

It has been an interesting problem to determine the number of critical periods bifurcating from a weak center of finite order or an isochronous center since great attention was paid to discuss monotonicity of the period function of dynamical systems ([3–5]). In 1989, Chicone and Jacobs [6] introduced the theory of weak centers and discussed the problem of local bifurcation of critical periods for quadratic Bautin's systems and planar Hamiltonian systems of Newton's type. In 1993, Rousseau and Toni [15] investigated such a bifurcation for a nondegenerate center with homogeneous cubic nonlinearities and proved that at most three local critical periods bifurcate from a weak linear center of finite order or from the linear isochronous center, and at most two local critical periods from the nonlinear isochronous center. Later, efforts were made to nonhomogeneous ones, e.g., reduced cubic Kukles systems (Rousseau and Toni [16]), reversible cubic perturbations of quadratic isochronous centers (Zhang, Hou, and Zeng [21]), reversible cubic systems (Chen and Zhang [2]), cubic Liénard equations with cubic damping (Zou, Chen, and Zhang [22]), planar cubic Hamiltonian systems (Yu, Han, and Zhang [20]), generalized Loud systems

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with degree ≥ 3 (Villadelprat [18]), reversible rigidly isochronous centers (Chen, Romanovski, and Zhang [1], Liu and Han [11], Li and Han [10]), quartic rigidly isochronous centers under any small quartic homogeneous perturbations (Peng and Feng [14]), generalized Lotka-Volterra systems of unspecific degree (Wang, Chen, and Zhang [19]), and several families of complexification cubic systems (Ferčec *et al* [7]).

The quadratic-like cubic system

$$\dot{x} = \mu x + y + p(x,y) + xf(x,y), \quad \dot{y} = -x + \mu y + q(x,y) + yf(x,y), \tag{1.1}$$

where p, q, and f are quadratic homogeneous polynomials, is another interesting class of nonhomogeneous cubic differential systems. This system, also called a cubic system with degenerate infinity, was studied early in 1981 [17] for the significance that in the Poincaré compactification, the equator of S^2 (i.e., the circle at infinity) entirely consists of singular points. Gasull and Prohens [8] exhibited at least three limit cycles for such a system. Lloyd *et al.* [12] proved that at most five limit cycles bifurcate from the weak focus at the origin. Moreover, conditions for the origin to be a center or an isochronous center have been given [12,13]. In this paper, we consider the local bifurcation of critical periods for the quadratic-like cubic system (1.1) with $\mu = 0$. The forms of both the linear and cubic terms in (1.1) are unchanged by rotation of coordinates [12], and they use a rotation transformation to simplify (1.1) to the form

$$\begin{cases} \dot{x} = y + a_1x^2 + (a_2 + 2b_1)xy + (a_3 - a_1)y^2 + xf(x,y), \\ \dot{y} = -x + b_1x^2 + (b_2 - 2a_1)xy - b_1y^2 + yf(x,y), \end{cases} \tag{1.2}$$

where a_i and b_i are real constants and

$$f(x,y) = a_4x^2 + a_5xy + (a_6 - a_4)y^2.$$

Theorem 4.3 of [13] indicates that the origin is a center of this system if and only if $\lambda = (a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2)$ lies in one of the sets:

$$\begin{aligned}
C_1 &:= \{\lambda | a_3 = a_6 = 0\}, \\
C_2 &:= \{\lambda | a_2 = -4b_1, a_4 = -a_1b_1, a_6 = -a_3b_1\}, \\
C_3 &:= \{\lambda | a_4 = -(a_2b_2 + 4b_1b_2 - 5a_3a_2 - 20a_3b_1 - 4a_2a_1)/16, \\
&\qquad a_5 = (25a_3^2 - 40a_1a_3 + 10b_2a_3 + 3a_2^2 - 3b_2^2 + 8b_2a_1 + 8b_1a_2)/16, a_6 = a_2a_3/4, \\
&\qquad b_2^2 + 2b_2a_3 - 8b_2a_1 - 3a_3^2 - 8a_3a_1 + a_2^2 + 8a_2b_1 + 16b_1^2 + 16a_1^2 = 0\}, \\
C_4 &:= \{\lambda | a_4 = -(a_2b_2 + 4b_1b_2 - 4a_1a_2)/16, \\
&\qquad a_5 = -(3b_2^2 - 8b_2a_1 - 3a_2^2 - 8a_2b_1)/16, a_6 = a_2a_3/4\}.
\end{aligned}$$

Theorem 5.3 of [12] indicates that the origin is an isochronous center if and only if λ lies in the cone $\mathcal{IC}$, which is defined as the union of the sets

$$\begin{aligned}
IC_1 &:= \{\lambda | a_3 = a_6 = 0, a_2 = -4b_1, b_2 = 4a_1\}, \\
IC_2 &:= \{\lambda | a_1 = a_3, b_2 = 3a_3, a_2 = -4b_1, a_4 = a_6 = -a_3b_1\}, \\
IC_3 &:= \{\lambda | a_1 = 4a_3/3, b_2 = 10a_3/3, a_2 = -4b_1, a_4 = -4a_3b_1/3, a_5 = b_1^2, a_6 = -a_3b_1\}, \\
IC_4 &:= \{\lambda | a_1 = a_3, b_2 = 6a_3, a_2 = -4b_1, a_4 = a_6 = -a_3b_1, a_5 = b_1^2\}.
\end{aligned}$$

In this paper, we discuss the local bifurcation of critical periods for the quadratic-like cubic system (1.2) from a weak center of finite order. We prove that the origin is either a weak center of at most order 4, or an isochronous center. We further prove that at most four critical periods bifurcate for each integer $k = 1, 2, 3, 4$, and give conditions under which exactly k critical periods bifurcate from the center O.

2. Orders of the Center

Let $P(r,\lambda)$ denote the minimum period of the periodic orbit around the origin through a point $(r,0)$. By Lemma 2.1 in [6], $P(r,\lambda)$ is analytic locally and can be represented as its Taylor series, $P(r,\lambda) = 2\pi + \sum_{k=2}^{\infty} p_k(\lambda)r^k$. If there exists $\lambda^* = (a_1,a_2,a_3,a_4,a_5,a_6,b_1,b_2)$ such that $p_2(\lambda^*) = \cdots = p_{2k+1}(\lambda^*) = 0$ and $p_{2k+2}(\lambda^*) \neq 0$ for any integer k, then (1.2) has a weak center of order k at O.

Lemma 2.1. *The origin of system* (1.2) *is a weak center of order* 0 *when* $\lambda \in C_1 \setminus \mathcal{IC}$.

Proof. When $\lambda \in C_1$, we use the computer algebra system Maple to calculate

$$p_2(\lambda) = \frac{\pi}{12}(4b_1 + a_2)^2 + \frac{\pi}{12}(b_2 - 4a_1)^2. \tag{2.1}$$

Then one can check that the variety $V(p_2) = IC_1$. Thus this lemma is proved. □

Lemma 2.2. *The origin of system* (1.2) *is a weak center of order at most* 3 *when* $\lambda \in C_2 \setminus \mathcal{IC}$. *More concretely, the center is of order* k $(k = 0,1,2,3)$ *if and only if* $\lambda \in C_2 \cap \Lambda_{II}^k$, *where*

$$\begin{aligned}
\Lambda_{II}^0 :=& \{\lambda |\, a_3(a_3 - \tfrac{1}{3}b_2) > 0\} \cup \{\lambda |\, a_3(a_3 - \tfrac{1}{3}b_2) \le 0, a_1 \neq \delta_1 \text{ and } a_1 \neq \delta_2\},\\
\Lambda_{II}^1 :=& \{\lambda |\, a_1 = \tfrac{11}{6}a_3, b_2 = \tfrac{13}{3}a_3, a_3 \neq 0, a_5 \neq -\tfrac{65}{288}a_3^2 + b_1^2\}\\
&\cup \{\lambda |\, a_1 = \tfrac{5}{6}a_3, b_2 = \tfrac{13}{3}a_3, a_3 \neq 0\}\\
&\cup \{\lambda |\, a_1 = \delta_1 \text{ (or } \delta_2), a_5 \neq \delta_3 \text{ (or } \delta_4), b_2 \neq \tfrac{13}{3}a_3, a_3(a_3 - \tfrac{1}{3}b_2) < 0\},\\
\Lambda_{II}^2 :=& \{\lambda |\, a_1 = \tfrac{11}{6}a_3, a_5 = -\tfrac{65}{288}a_3^2 + b_1^2, b_2 = \tfrac{13}{3}a_3, a_3 \neq 0\}\\
&\cup \{\lambda |\, a_1 = \delta_1, a_5 = \delta_3, b_2 \neq \tfrac{13}{3}a_3, \tfrac{10}{3}a_3, \ell_2 a_3, \ell_4 a_3, 0 < 3a_3 < b_2\}\\
&\cup \{\lambda |\, a_1 = \delta_2, a_5 = \delta_4, b_2 \neq \tfrac{13}{3}a_3, 6a_3, \ell_1 a_3, \ell_3 a_3, 0 < 3a_3 < b_2\}\\
&\cup \{\lambda |\, a_1 = \delta_1, a_5 = \delta_3, b_2 \neq \tfrac{13}{3}a_3, 6a_3, \ell_1 a_3, \ell_3 a_3, b_2 < 3a_3 < 0\}\\
&\cup \{\lambda |\, a_1 = \delta_2, a_5 = \delta_4, b_2 \neq \tfrac{13}{3}a_3, \tfrac{10}{3}a_3, \ell_2 a_3, \ell_4 a_3, b_2 < 3a_3 < 0\},\\
\Lambda_{II}^3 :=& \{\lambda |\, a_1 = \delta_1, a_5 = \delta_3, b_2 = \ell_2 a_3 \text{ or } \ell_4 a_3, a_3 > 0\}\\
&\cup \{\lambda |\, a_1 = \delta_2, a_5 = \delta_4, b_2 = \ell_1 a_3 \text{ or } \ell_3 a_3, a_3 > 0\}\\
&\cup \{\lambda |\, a_1 = \delta_1, a_5 = \delta_3, b_2 = \ell_1 a_3 \text{ or } \ell_3 a_3, a_3 < 0\}\\
&\cup \{\lambda |\, a_1 = \delta_2, a_5 = \delta_4, b_2 = \ell_2 a_3 \text{ or } \ell_4 a_3, a_3 < 0\}
\end{aligned}$$

and

$$\delta_{1,2} := \frac{1}{4}\left(b_2 + a_3 \pm \sqrt{-3a_3(3a_3 - b_2)}\right), \quad \delta_{3,4} := -\frac{s_1 \pm s_2\sqrt{-3a_3(3a_3 - b_2)}}{16(13a_3 - 3b_2)},$$

$s_1 := 300a_3^3 + 49b_2^2a_3 - 236b_2a_3^2 - 3b_2^3 + 48b_1^2b_2 - 208a_3b_1^2,$
$s_2 := 2(6a_3 - b_2)(20a_3 - 3b_2),$
$\ell_i :=$ RootsOf$(3x^4 - 151x^3 + 2442x^2 - 13716x + 2473)$ *such that* $\ell_1 < \ell_2 < \ell_3 < \ell_4$.

Proof. When $\lambda \in C_2$, we use the computer algebra system Maple to calculate the period coefficients up to a nonzero factor:

$$\begin{aligned} p_2(\lambda) = {}& 16a_1^2 - 8a_1a_3 + b_2^2 - b_2a_3 - 8b_2a_1 + 10a_3^2, \\ p_4(\lambda) = {}& b_2^4 + 1540a_3^4 + 2560a_1^4 - 2176a_1^3b_2 - 64b_2^3a_1 + 6720a_3^2a_1^2 + 21a_3^2b_2^2 - 2b_2^3a_3 \\ & +700a_3^3b_2 - 4480a_3^3a_1 + 624a_1^2b_2^2 - 2560a_1^3a_3 + 336b_1^2b_2^2 - 768a_1^2a_5 \\ & -1008a_3^2a_5 + 5376a_1^2b_1^2 + 3888a_3^2b_1^2 - 48a_5b_2^2 + 576a_1^2a_3b_2 + 96a_1b_2^2a_3 \\ & -2016a_1a_3^2b_2 + 96a_5b_2a_3 + 384a_5b_2a_1 + 768a_3a_5a_1 - 3072a_1a_3b_1^2 \\ & -384b_1^2a_3b_2 - 2688b_1^2a_1b_2, \\ & \vdots \end{aligned} \tag{2.2}$$

We omit the expressions of $p_6(\lambda)$ and $p_8(\lambda)$, with 76 and 170 terms, respectively. From (2.2), we get

$$V(p_2) = \{\lambda |\ a_1 = \delta_1 \text{ or } \delta_2,\ a_3(a_3 - \frac{1}{3}b_2) \le 0\}. \tag{2.3}$$

Since the origin is a weak center of order 0 if and only if $p_2 \neq 0$, it follows from (2.3) that $\lambda \notin V(p_2)$, i.e.,

$$a_3(a_3 - \frac{1}{3}b_2) > 0 \quad \text{or} \quad a_3(a_3 - \frac{1}{3}b_2) \le 0, \quad a_1 \neq \delta_1, \delta_2,$$

which means that $\lambda \in \Lambda_{II}^0 \cap C_2$.

When $\lambda \in V(p_2)$, we further identify the center of order $k \ge 1$. Consider p_2, p_4 as polynomials in a single variable a_1, and let lcoeff$(\cdot, a_1)$ denote the leading coefficients. In this case, lcoeff$(p_2, a_1) = 16$. Using the method given in [9, pp. 368-369], we see that $p_2 = p_4 = 0$ if and only if $p_2 = 0$ and prem$(p_4, p_2, a_1) = 0$, where prem(p_4, p_2, a_1), called the pseudo-remainder of p_4 divided by p_2, is defined by

$$\text{prem}(p_4, p_2, a_1) := (\text{lcoeff}(p_2, a_1))^k \text{rem}(p_4, p_2, a_1), \ k = \deg(p_4) - \deg(p_2) + 1, \tag{2.4}$$

rem(p_4, p_2, a_1) denotes the remainder of p_4 divided by p_2, and deg(p_2), deg(p_4) denote the orders of a_1 in p_2, p_4, respectively. This implies that

$$V(p_2, p_4) = V(p_2, \text{prem}(p_4, p_2, a_1)) = V(p_2, m_1, m_2) \cup V(\frac{p_2, \text{prem}(p_4, p_2, a_1)}{m_1}), \tag{2.5}$$

where prem(p_4, p_2, a_1) is a polynomial of λ and can be calculated by (2.4), i.e.,

$$\begin{aligned} \text{prem}(p_4, p_2, a_1) = {}& m_1a_5 + m_2, \\ m_1 := {}& -196608a_3(11a_3 - 8a_1 - b_2), \\ m_2 := {}& -12288a_3(480a_1a_3^2 - 176a_1a_3b_2 + 128a_1b_1^2 + 16a_1b_2^2 + 420a_3^3 \\ & -400a_3^2b_2 - 176a_3b_1^2 + 97a_3b_2^2 + 16b_1^2b_2 - 7b_2^3). \end{aligned} \tag{2.6}$$

Thus $\lambda \in V(p_2, p_4)$ if and only if one of the following conditions holds:

$$\text{I}: \begin{cases} p_2 = 0, \\ m_1 = 0, \\ m_2 = 0. \end{cases} \qquad \text{II}: \begin{cases} p_2 = 0, \\ \text{prem}(p_4, p_2, a_1) = 0, \\ m_1 \neq 0. \end{cases}$$

For case I, one can check that

$$V(p_2, m_1, m_2) = IC_1 \cup IC_2. \tag{2.7}$$

For case II, when $b_2 \neq 13a_3/3$, from $p_2 = 0$, we get $a_3(a_3 - \frac{1}{3}b_2) > 0$ and $a_1 = \delta_{1,2}$. Substituting a_1 into m_1, since $m_1 \neq 0$, we know that $b_2 \neq 3a_3$ and $a_3 \neq 0$. Moreover, we get $a_5 = -m_2/m_1 = \delta_{3,4}$ from $\text{prem}(p_4, p_2, a_1) = 0$ and $m_1 \neq 0$. When $b_2 = 13a_3/3$, we solve $p_2 = 0$ to obtain $a_1 = 11a_3/6$ or $a_1 = 5a_3/6$. Furthermore, from $p_4 = 0$, we get:

$$a_1 = \frac{11}{6}a_3, \qquad a_5 = -\frac{65}{288}a_3^2 + b_1^2.$$

It follows that $m_1 = 8a_3 \neq 0$. Thus

$$\begin{aligned} V(\frac{p_2, \text{prem}(p_4, p_2, a_1)}{m_1}) = &\{\lambda |\, a_1 = \frac{11}{6}a_3, a_5 = -\frac{65}{288}a_3^2 + b_1^2, b_2 = \frac{13}{3}a_3, a_3 \neq 0\} \\ &\cup \{\lambda |\, a_1 = \delta_1, a_5 = \delta_3, b_2 \neq \frac{13}{3}a_3, a_3(a_3 - \frac{1}{3}b_2) < 0\}) \\ &\cup \{\lambda |\, a_1 = \delta_2, a_5 = \delta_4, b_2 \neq \frac{13}{3}a_3, a_3(a_3 - \frac{1}{3}b_2) < 0\}). \end{aligned} \tag{2.8}$$

Thus, from (2.5) we obtain that

$$\begin{aligned} V(p_2, p_4) = IC_1 \cup IC_2 \cup &\{\lambda |\, a_1 = \frac{11}{6}a_3, a_5 = -\frac{65}{288}a_3^2 + b_1^2, b_2 = \frac{13}{3}a_3, a_3 \neq 0\} \\ &\cup \{\lambda |\, a_1 = \delta_1, a_5 = \delta_3, b_2 \neq \frac{13}{3}a_3, a_3(a_3 - \frac{1}{3}b_2) < 0\} \\ &\cup \{\lambda |\, a_1 = \delta_2, a_5 = \delta_4, b_2 \neq \frac{13}{3}a_3, a_3(a_3 - \frac{1}{3}b_2) < 0\}. \end{aligned} \tag{2.9}$$

It follows from (2.3) and (2.9) that the origin is a weak center of order 1 if and only if $\lambda \in V(p_2)/V(p_2, p_4)$, i.e., one of the following condition holds:

(i) $a_1 = 11a_3/6,\ b_2 = 13a_3/3,\ a_3 \neq 0,\ a_5 \neq -65a_3^2/288 + b_1^2$,

(ii) $a_1 = 5a_3/6,\ b_2 = 13a_3/3,\ a_3 \neq 0$,

(iii) $a_1 = \delta_1$ (or δ_2), $a_5 \neq \delta_3$ (or δ_4), $b_2 \neq 13a_3/3,\ a_3(a_3 - b_2/3) < 0$,

implying that $\lambda \in \Lambda_{II}^1 \cap C_2$.

From (2.5) and (2.7), we further compute

$$\begin{aligned} V(p_2, p_4, p_6) = &(V(p_2, m_1, m_2) \cap V(p_6)) \cup (V(\frac{p_2, \text{prem}(p_4, p_2, a_1)}{m_1}) \cap V(p_6)) \\ = &IC_1 \cup IC_2 \cup V(\frac{p_2, \text{prem}(p_4, p_2, a_1), \text{prem}(p_6, p_2, a_1)}{m_1}). \end{aligned} \tag{2.10}$$

Notice that $\text{lcoeff}(\text{prem}(p_4, p_2, a_1), a_5) = m_1$. It follows from (2.10) that

$$V(p_2, p_4, p_6) = IC_1 \cup IC_2 \cup V(\frac{p_2, \text{prem}(p_4, p_2, a_1), m_3}{m_1})$$

$$= IC_1 \cup IC_2 \cup V(\frac{p_2, \text{prem}(p_4, p_2, a_1), \text{prem}(m_3, p_2, a_1)}{m_1}), \tag{2.11}$$

where

$$m_3 := \text{prem}(\text{prem}(p_6, p_2, a_1), \text{prem}(p_4, p_2, a_1), a_5), \tag{2.12}$$

and $\text{prem}(m_3, p_2, a_1)$ can be calculated by

$$\begin{aligned}\text{prem}(m_3, p_2, a_1) &= 14708396095021850296320 0(3a_3 - b_2)a_3^5(m_4 a_1 + m_5),\\ m_4 &:= 6192a_3^3 - 3336a_3^2 b_2 + 504a_3 b_2^2 - 16b_2^3,\\ m_5 &:= -3816a_3^4 + 1536a_3^3 b_2 - 48a_3^2 b_2^2 - 27a_3 b_2^3 + b_2^4.\end{aligned}$$

From (2.8), we know $b_2 \neq 3a_3$ and $a_3 \neq 0$ when $\lambda \in V((p_2, \text{prem}(p_4, p_2, a_1))/m_1)$. It follows from (2.11) that

$$\begin{aligned}V(p_2, p_4, p_6) &= IC_1 \cup IC_2 \cup V(\frac{p_2, \text{prem}(p_4, p_2, a_1), m_4, m_5}{m_1})\\ &= \cup V(\frac{p_2, \text{prem}(p_4, p_2, a_1), \text{prem}(m_3, p_2, a_1)}{m_1, m_4}).\end{aligned}$$

One can check that $\lambda \in V((p_2, \text{prem}(p_4, p_2, a_1), m_4, m_5)/m_1)$ if and only if $a_1 = a_3 = a_5 = b_2 = 0$, which contradicts $a_3 \neq 0$. When $m_4 \neq 0$, we can solve $a_1 = -m_4/m_5$ from $\text{prem}(m_3,p_2,a_1) = 0$. Substituting it in p_2, we get

$$p_2 = (3a_3 - b_2)(10a_3 - 3b_2)(6a_3 - b_2)^2 m_6 m_4^{-2},$$

where

$$m_6 := 3b_2^4 - 151b_2^3 a_3 + 2442a_3^2 b_2^2 - 13716a_3^3 b_2 + 24732a_3^4. \tag{2.13}$$

Notice that $b_2 \neq 3a_3$ when $\lambda \in V((p_2, \text{prem}(p_4,p_2,a_1), \text{prem}(m_3,p_2,a_1))/(m_1, m_4))$. When $b_2 = 10a_3/3$, from $a_1 = -m_4/m_5$ and $\text{prem}(p_4, p_2, a_1) = 0$, we get $a_1 = 4a_3/3$ and $a_5 = b_1^2$, which implies $\lambda \in IC_3$. When $b_2 = 6a_3$, from $a_1 = -m_4/m_5$ and $\text{prem}(p_4, p_2, a_1) = 0$, we get $a_1 = a_3/3$ and $a_5 = b_1^2$, which implies $\lambda \in IC_4$. Consider the case $m_6 = 0$. One can check that there are four real roots ℓ_i, $i = 1, 2, 3, 4$, for the equation $3x^4 - 151x^3 + 2442x^2 - 13716x + 2473 = 0$. Assume that $\ell_1 < \ell_2 < \ell_3 < \ell_4$. Then $\ell_1 \in [4, 4.5]$, $\ell_2 \in [4.5, 5]$, $\ell_3 \in [19, 20]$, and $\ell_4 \in [22, 23]$. Thus one can solve $b_2 = \ell_i a_3$, $i = 1, 2, 3, 4$, from $m_6 = 0$. Moreover, when $a_3 > 0$ (or < 0) and $b_2 = \ell_i a_3$, $i = 1, 3$, from $a_1 = -m_4/m_5$, we get $a_3(a_3 - b_2/3) < 0$ and $a_1 = \delta_2$ (or δ_1). Solving a_5 from $\text{prem}(p_4, p_2, a_1) = 0$ in this case, we get $a_5 = \delta_4$ (or δ_3). When $a_3 > 0$ (or < 0), and $b_2 = \ell_i a_3$, $i = 2, 4$, from $a_1 = -m_4/m_5$, we get $a_3(a_3 - b_2/3) < 0$ and $a_1 = \delta_1$ (or δ_2). Solving a_5 from $\text{prem}(p_4, p_2, a_1) = 0$ in this case, we get $a_5 = \delta_3$ (or δ_4). Hence we get

$$\begin{aligned}V(p_2, p_4, p_6) = IC_1 \cup IC_2 \cup IC_3 \cup IC_4 &\\ \cup\{\lambda \mid b_2 = \ell_i a_3, a_1 = \delta_2, a_5 = \delta_4, i = 1, 3,\ a_3 > 0\}&\\ \cup\{\lambda \mid b_2 = \ell_i a_3, a_1 = \delta_1, a_5 = \delta_3, i = 2, 4,\ a_3 > 0\}&\\ \cup\{\lambda \mid b_2 = \ell_i a_3, a_1 = \delta_1, a_5 = \delta_3, i = 1, 3,\ a_3 < 0\}&\\ \cup\{\lambda \mid b_2 = \ell_i a_3, a_1 = \delta_1, a_5 = \delta_3, i = 2, 4,\ a_3 < 0\}&.\end{aligned} \tag{2.14}$$

It follows from (2.9) and (2.14) that the origin is a weak center of order 2 if and only if $\lambda \in V(p_2, p_4)/V(p_2, p_4, p_6)$, i.e., one of the following conditions holds:

(i) $a_1 = 11a_3/6, a_5 = -65a_3^2/288 + b_1^2, b_2 = 13a_3/3,$

(ii) $a_1 = \delta_1, a_5 = \delta_3, b_2 \neq 13a_3/3, 10a_3/3, \ell_2 a_3, \ell_4 a_3, 0 < 3a_3 < b_2$,

(iii) $a_1 = \delta_2, a_5 = \delta_4, b_2 \neq 13a_3/3, 6a_3, \ell_1 a_3, \ell_3 a_3, 0 < 3a_3 < b_2$,

(iv) $a_1 = \delta_1, a_5 = \delta_3, b_2 \neq 13a_3/3, 6a_3, \ell_1 a_3, \ell_3 a_3, b_2 < 3a_3 < 0$,

(v) $a_1 = \delta_2, a_5 = \delta_4, b_2 \neq 13a_3/3, 10a_3/3, \ell_2 a_3, \ell_4 a_3, b_2 < 3a_3 < 0$,

implying that $\lambda \in \Lambda^2_{II} \cap C_2$.

Similarly, from (2.11), we have

$$\begin{aligned} V(p_2,p_4,p_6,p_8) &= IC_1 \cup IC_2 \cup (V(\frac{p_2, \mathrm{prem}(p_4,p_2,a_1), \mathrm{prem}(m_3,p_2,a_1)}{m_1}) \cap V(p_8)) \\ &= IC_1 \cup IC_2 \cup V(\frac{p_2, \mathrm{prem}(p_4,p_2,a_1), \mathrm{prem}(m_3,p_2,a_1), \mathrm{prem}(p_8,p_2,a_1)}{m_1}). \end{aligned}$$

Notice that $\mathrm{lcoeff}(\mathrm{prem}(p_4,p_2,a_1),a_5) = m_1$. Then

$$\begin{aligned} V(p_2,p_4,p_6,p_8) &= IC_1 \cup IC_2 \cup V(\frac{p_2, \mathrm{prem}(p_4,p_2,a_1), \mathrm{prem}(m_3,p_2,a_1), m_7}{m_1}) \\ &= IC_1 \cup IC_2 \cup V(\frac{p_2, \mathrm{prem}(p_4,p_2,a_1), \mathrm{prem}(m_3,p_2,a_1), \mathrm{prem}(m_7,p_2,a_1)}{m_1}), \end{aligned}$$

where

$$m_7 := \mathrm{prem}(\mathrm{prem}(p_8,p_2,a_1), \mathrm{prem}(p_4,p_2,a_1), a_5), \tag{2.15}$$

and $\mathrm{prem}(m_7,p_2,a_1)$ can be calculated by

$$\begin{aligned} \mathrm{prem}(m_7,p_2,a_1) = {} & 426411541381947449139690586767360\,0a_3^6(3a_3-b2)^2(663264a_1a_3^5 \\ & -121344a_1a_3^4b_2 + 5515776a_1a_3^3b_1^2 - 75912a_1a_3^3b_2^2 - 3134208a_1a_3^2b_1^2b_2 \\ & +18472a_1a_3^2b_2^3 + 492800a_1a_3b_1^2b_2^2 - 720a_1a_3b_2^4 - 16128a_1b_1^2b_2^3 - 1480392a_3^6 \\ & +1067964a_3^5b_2 - 822528a_3^4b_1^2 - 331302a_3^4b_2^2 - 204288a_3^3b_1^2b_2 + 61217a_3^3b_2^3 \\ & +321664a_3^2b_1^2b_2^2 - 6136a_3^2b_2^4 - 60704a_3b_1^2b_2^3 + 189a_3b_2^5 + 2016b_1^2b_2^4). \end{aligned}$$

Solving $p_2 = \mathrm{prem}(p_4,p_2,a_1) = \mathrm{prem}(m_3,p_2,a_1) = \mathrm{prem}(m_7,p_2,a_1) = 0$ with $m_1 \neq 0$, we get $a_1 = 4a_3/3$, $a_5 = b_1^2$, and $b_2 = 10a_3/3$, which implies $\lambda \in IC_3$, or $a_1 = a_3/3$, $a_5 = b_1^2$, and $b_2 = 6a_3$, which implies $\lambda \in IC_4$. Thus we obtain that

$$V(p_2,p_4,p_6,p_8) = IC_1 \cup IC_2 \cup IC_3 \cup IC_4, \tag{2.16}$$

implying that the origin is a weak center of order at most 3 when $\lambda \in C_2 \setminus \mathcal{IC}$. It follows from (2.14) and (2.16) that the origin is a weak center of order 3 if and only if $\lambda \in V(p_2,p_4,p_6)/V(p_2,p_4,p_6,p_8)$, i.e., one of the following conditions holds:

(i) $a_1 = \delta_1, a_5 = \delta_3, b_2 = \ell_2 a_3$ or $\ell_4 a_3, a_3 > 0$,

(ii) $a_1 = \delta_2, a_5 = \delta_4, b_2 = \ell_1 a_3$ or $\ell_3 a_3, a_3 > 0$,

(iii) $a_1 = \delta_1, a_5 = \delta_3, b_2 = \ell_1 a_3$ or $\ell_3 a_3, a_3 < 0$,

(iv) $a_1 = \delta_2, a_5 = \delta_4, b_2 = \ell_2 a_3$ or $\ell_4 a_3, a_3 < 0$,

implying that $\lambda \in \Lambda^3_{II} \cap C_2$. Thus this lemma is proved. □

Lemma 2.3. *The origin of system* (1.2) *is a weak center of order at most* 1 *when* $\lambda \in C_3 \setminus \mathcal{IC}$, *and the center is of order* k ($k = 0, 1$) *if and only if* $\lambda \in C_3 \cap \Lambda_{III}^k$, *where*

$$\begin{aligned}\Lambda_{III}^0 :=& \{\lambda|\ b_2 \neq \frac{13}{3}a_3, a_3 \neq 0\} \cup \{\lambda|\ b_2 = \frac{13}{3}a_3, a_3 \neq 0, 4a_3^2 - (a_2+4b_1)^2 < 0\}\\ &\cup\{\lambda|\ b_2 = \frac{13}{3}a_3, a_1 \neq \frac{4}{3}a_3 \pm \frac{\sqrt{4a_3^2-(a_2+4b_1)^2}}{4}, a_3 \neq 0, 4a_3^2 - (a_2+4b_1)^2 \geq 0\},\\ \Lambda_{III}^1 :=& \{\lambda|\ a_1 = \frac{4}{3}a_3 \pm \frac{\sqrt{4a_3^2-(a_2+4b_1)^2}}{4}, b_2 = \frac{13}{3}a_3, a_3 \neq 0, 4a_3^2 - (a_2+4b_1)^2 \geq 0\}.\end{aligned}$$

Proof. When $\lambda \in C_3$, we use the computer algebra system Maple to calculate the period coefficients up to a nonzero factor:

$$\begin{aligned} p_2(\lambda) =& b_2^2 + 16b_1^2 + 10a_3^2 + a_2^2 + 16a_1^2 - 8a_1a_3 - b_2a_3 + 8a_2b_1 - 8b_2a_1,\\ p_4(\lambda) =& -5120b_1^2a_1^2 + 14b_2^3a_3 + 16b_2^3a_1 - 672b_2^2a_1^2 + 2560b_2a_1^3 - 3920b_1^2a_3^2\\ &-800a_1^2a_2^2 - 3600a_3^2a_1^2 - 3328b_1^3a_2 - 32a_2^2b_2^2 + 760a_3^3a_1 - 220a_3^3b_2\\ &+183b_2^2a_3^2 - 545a_3^2a_2^2 - 448a_2^3b_1 - 1824a_2^2b_1^2 + 640a_1^3a_3 - 32b_1^2b_2^2\\ &+80a_2^2a_3b_2 + 280a_2^2a_3a_1 + 2560b_1^2b_2a_1 - 2440b_1a_3^2a_2 - 648b_2^2a_3a_1\\ &-64b_1a_2b_2^2 + 480a_1^2a_3b_2 + 1680a_3^2b_2a_1 + 1664b_1b_2a_1a_2 + 448b_1b_2a_3a_2\\ &+704a_1a_3b_1a_2 + 8b_2^4 - 2560b_1^4 - 2560a_1^4 - 40a_2^4 + 3a_3^4 - 3328b_1a_1^2a_2\\ &+800b_1^2a_3b_2 + 400a_1a_2^2b_2 + 640b_1^2a_3a_1. \end{aligned} \tag{2.17}$$

Let p_0 denote the left side of the last equality in C_3, i.e.,

$$p_0 = b_2^2 + 2b_2a_3 - 8b_2a_1 - 3a_3^2 - 8a_3a_1 + a_2^2 + 8a_2b_1 + 16b_1^2 + 16a_1^2.$$

Since $p_0 = 0$, we obtain from (2.17) that $p_2(\lambda) = 0$ if and only if:

$$p_2 - p_0 = a_3(13a_3 - 3b_2) = 0.$$

When $a_3 = 0$, we get $a_2 = 4b_1$ and $b_2 = 4a_1$ from $p_0 = 0$, which implies that $\lambda \in IC_1$ and the origin is an isochronous center. When $b_2 = 13a_3/3$, we get

$$p_2 = 220a_3^2/9 + 16b_1^2 + a_2^2 + 16a_1^2 - 128a_1a_3/3 + 8a_2b_1.$$

Consider $p_2 = 0$ as a quadratic equation that regards a_1 as variable. Its discriminant is $\Delta = 256a_3^2 - 4(a_2+4b_1)^2$. Then we get $a_1 = 4a_3/3 \pm \sqrt{4a_3^2-(a_2+4b_1)^2}/4$ if $\Delta \geq 0$. Thus

$$V(p_2) = IC_1 \cup \{\lambda|\ a_1 = \frac{4}{3}a_3 \pm \frac{\sqrt{4a_3^2-(a_2+4b_1)^2}}{4}, b_2 = \frac{13}{3}a_3, 4a_3^2-(a_2+4b_1)^2 \geq 0\}. \tag{2.18}$$

Since the origin is a weak center of order 0 if and only if $p_2 \neq 0$, it follows from (2.18) that $\lambda \notin V(p_2)$, i.e., one of the following conditions holds:

(i) $b_2 \neq 13a_3/3$, $a_3 \neq 0$,

(ii) $b_2 = 13a_3/3$, $a_3 \neq 0$, $4a_3^2 - (a_2+4b_1)^2 < 0$,

(iii) $b_2 = 13a_3/3$, $a_3 \neq 0$, $4a_3^2-(a_2+4b_1)^2 \geq 0$, $a_1 \neq 4a_3/3 \pm \sqrt{4a_3^2-(a_2+4b_1)^2}/4$,

which means $\lambda \in \Lambda_{III}^0 \cap C_3$.

When $\lambda \in V(p_2)$, we further identify the center of order 1. Under the condition $b_2 = 13a_3/3$, one can calculate that $\mathrm{lcoeff}(p_0, a_1) = 16$ and $\mathrm{prem}(p_4, p_0, a_1) =$

$573440a_3^4$. Moreover, $a_3 = 0$ implies that $\lambda \in IC_1$ and the origin is an isochronous center. Thus

$$V(p_2, p_4) = IC_1. \tag{2.19}$$

It follows from (2.18) and (2.19) that the origin is a weak center of order 1 if and only if $\lambda \in V(p_2)/V(p_2, p_4)$, i.e.,

$$a_1 = 4a_3/3 \pm \sqrt{4a_3^2 - (a_2 + 4b_1)^2}/4, \quad b_2 = 13a_3/3, a_3 \neq 0, \; b_2 = 13a_3/3,$$
$$4a_3^2 - (a_2 + 4b_1)^2 \geq 0,$$

implying that $\lambda \in \Lambda_{III}^1 \cap C_3$. Therefore, this lemma is proved. □

Lemma 2.4. *The origin of system* (1.2) *is a weak center of order at most* 4 *when* $\lambda \in C_4 \setminus \mathcal{IC}$*, and the center is of order* k ($k = 0, 1, 2, 3, 4$) *if and only if* $\lambda \in C_4 \cap \Lambda_{IV}^k$*, where*

$$\Lambda_{IV}^0 := \{\lambda | \Delta > 0\} \cup \{\lambda | a_1 \neq \sigma_1 \text{ and } a_1 \neq \sigma_2, \; \Delta \leq 0\},$$

$$\begin{aligned}
\Lambda_{IV}^1 := &\{\lambda | a_1 = \sigma_1 \text{ or } \sigma_2, b_2 = 4a_3, a_2 = 0, a_3 \neq 0, \; \Delta \leq 0\} \\
&\cup \{\lambda | a_1 = \sigma_1 \text{ or } \sigma_2, b_2 = 4a_3, a_2 \neq 0, a_3 \neq 0, b_1 \neq -\frac{a_2^2 + 2a_3^2}{4a_2}, \; \Delta \leq 0\} \\
&\cup \{\lambda | a_1 = \sigma_1 \text{ or } \sigma_2, a_3 \neq 0, b_2 \neq 4a_3, \; \Delta \leq 0, \Gamma > 0\} \\
&\cup \{\lambda | a_1 = \sigma_1, a_3 \neq 0, b_2 \neq 4a_3, \; \Delta \leq 0, \Gamma \leq 0, E_1 > 0\} \\
&\cup \{\lambda | a_1 = \sigma_1, b_1 \neq \varsigma_1, a_3 \neq 0, b_2 \neq 4a_3, \; \Delta \leq 0, \Gamma \leq 0, E_1 \leq 0\} \\
&\cup \{\lambda | a_1 = \sigma_2, a_3 \neq 0, b_2 \neq 4a_3, \; \Delta \leq 0, \Gamma \leq 0, E_2 < 0\} \\
&\cup \{\lambda | a_1 = \sigma_2, b_1 \neq \varsigma_2, a_3 \neq 0, b_2 \neq 4a_3, \; \Delta \leq 0, \Gamma \leq 0, E_2 \geq 0\},
\end{aligned}$$

$$\begin{aligned}
\Lambda_{IV}^2 := &\{\lambda | a_1 = \frac{(5a_2 + \sqrt{3a_2^2 - 4a_3^2})a_3}{4a_2}, b_1 = -\frac{a_2^2 + 2a_3^2}{4a_2}, b_2 = 4a_3, a_2 \neq 0, \frac{3\sqrt{26}}{13}a_3, \\
&3a_2^2 - 4a_3^2 \geq 0, a_3 > 0\} \cup \{\lambda | a_1 = \frac{(5a_2 + \sqrt{3a_2^2 - 4a_3^2})a_3}{4a_2}, b_1 = -\frac{a_2^2 + 2a_3^2}{4a_2}, b_2 = 4a_3, \\
&a_2 \neq 0, -\frac{3\sqrt{26}}{13}a_3, 3a_2^2 - 4a_3^2 \geq 0, a_3 < 0\} \cup \{\lambda | a_1 = \frac{(5a_2 - \sqrt{3a_2^2 - 4a_3^2})a_3}{4a_2}, \\
&b_1 = -\frac{a_2^2 + 2a_3^2}{4a_2}, b_2 = 4a_3, a_2 \neq 0, -\frac{3\sqrt{26}}{13}a_3, 3a_2^2 - 4a_3^2 \geq 0, a_3 > 0\} \\
&\cup \{\lambda | a_1 = \frac{(5a_2 - \sqrt{3a_2^2 - 4a_3^2})a_3}{4a_2}, b_1 = -\frac{a_2^2 + 2a_3^2}{4a_2}, b_2 = 4a_3, a_2 \neq 0, \frac{3\sqrt{26}}{13}a_3, \\
&3a_2^2 - 4a_3^2 \geq 0, a_3 < 0\} \cup \{\lambda | \; a_1 = \sigma_3, a_2 = 0, a_3 \neq 0, b_1 = \varsigma_1, b_2 \neq \varrho_i, i = 1, 2, 3, 4, \\
&\Gamma \leq 0\} \cup \{\lambda | a_1 = \sigma_4, a_2 = 0, b_1 = \varsigma_2, a_3 \neq 0, b_2 \neq \varrho_i, i = 1, 2, 3, 4, \; \Gamma \leq 0\} \\
&\cup \{\lambda | a_1 = \sigma_3, b_1 = \varsigma_1, a_2 \neq 0, a_3 \neq 0, b_2 \neq \varrho_i, i = 1, 2, 3, 4, \Gamma \leq 0, \Omega \geq 0\} \\
&\cup \{\lambda | a_1 = \sigma_4, b_1 = \varsigma_2, a_2 \neq 0, a_3 \neq 0, b_2 \neq \varrho_i, i = 1, 2, 3, 4, \Gamma \leq 0, \Omega \geq 0\} \\
&\cup \{\lambda | a_1 = \sigma_3, b_1 = \varsigma_1, a_2 \neq 0, \tau_1, a_3 \neq 0, b_2 \neq \varrho_i, i = 1, 2, 3, 4, \Gamma \leq 0, \Omega < 0, E_3 \geq 0\} \\
&\cup \{\lambda | a_1 = \sigma_4, b_1 = \varsigma_2, a_2 \neq 0, a_3 \neq 0, b_2 \neq \varrho_i, i = 1, 2, 3, 4, \Gamma \leq 0, \Omega < 0, E_3 < 0\} \\
&\cup \{\lambda | a_1 = \sigma_4, b_1 = \varsigma_2, a_2 \neq 0, \tau_2, a_3 \neq 0, b_2 \neq \varrho_i, i = 1, 2, 3, 4, \Gamma \leq 0, \Omega < 0, E_3 \geq 0\} \\
&\cup \{\lambda | a_1 = \sigma_3, b_1 = \varsigma_1, a_2 \neq 0, a_3 \neq 0, b_2 \neq \varrho_i, i = 1, 2, 3, 4, \Gamma \leq 0, \Omega < 0, E_3 < 0\},
\end{aligned}$$

$$\Lambda_{IV}^3 := \{\lambda | \; a_1 = \frac{4}{3}a_3, a_2 = \pm\frac{3\sqrt{26}}{13}a_3, b_1 = \mp\frac{11\sqrt{26}}{78}a_3, b_2 = 4a_3, a_3 \neq 0\}$$

$$\cup\{\lambda | a_1 = \frac{7}{4}a_3, a_2 = 0, b_1 = \pm\frac{3}{4}a_3, b_2 = 6a_3, a_3 \neq 0\}$$

$$\cup\{\lambda | a_1 = \frac{26+\sqrt{97}}{12}a_3, a_2 = 0, b_1 = \pm\frac{\sqrt{562+62\sqrt{97}}}{48}a_3, b_2 = \frac{23+\sqrt{97}}{4}a_3, a_3 \neq 0\}$$

$$\cup\{\lambda | a_1 = \frac{26-\sqrt{97}}{12}a_3, a_2 = 0, b_1 = \pm\frac{\sqrt{562-62\sqrt{97}}}{48}a_3, b_2 = \frac{23-\sqrt{97}}{4}a_3, a_3 \neq 0\}$$

$$\cup\{\lambda | a_1 = \frac{60a_3^2 - 15a_3b_2 + 2b_2^2}{24a_3}, b_1 = \varsigma_{3,4}, a_2 = \tau_{1,2}, a_3 \neq 0, b_2 \neq \varrho_i, \ i = 1,2,3,4, \kappa_2 a_3, \kappa_3 a_3, \ \Omega < 0\}.$$

$$\Lambda_{IV}^4 := \{\lambda | a_1 = \frac{60a_3^2 - 15a_3b_2 + 2b_2^2}{24a_3}, a_2 = \tau_{3,4}, b_1 = \varsigma_{5,6}, b_2 = \kappa_2 a_3, a_3 \neq 0\}$$

$$\cup\{\lambda | a_1 = \frac{60a_3^2 - 15a_3b_2 + 2b_2^2}{24a_3}, a_2 = \tau_{3,4}, b_1 = \varsigma_{5,6}, b_2 = \kappa_3 a_3, a_3 \neq 0\}$$

and

$$\sigma_{1,2} := \frac{a_3 + b_2 \pm \sqrt{-\Delta}}{4},$$

$$\sigma_{3,4} := \frac{a_2^2 a_3 + a_2^2 b_2 - 56a_3^3 + 62a_3^2 b_2 - 20a_3 b_2^2 + 2b_2^3 \pm a_2\sqrt{-\Gamma}}{4(a_2^2 + 16a_3^2 + b_2^2 - 8a_3b_2)},$$

$$\varsigma_{1,2} := \frac{a_2(-a_2^2 + 2a_3^2 - a_3b_2) \pm (4a_3 - b_2)\sqrt{-\Gamma}}{4(a_2^2 + 16a_3^2 + b_2^2 - 8a_3b_2)},$$

$$\varsigma_{3,4} := \mp\frac{5184a_3^4 - 3528a_3^3 b_2 + 867a_3^2 b_2^2 - 96a_3 b_2^3 + 4b_2^4}{24a_3\sqrt{-\Omega}},$$

$$\varsigma_{5,6} := \mp\frac{(409\kappa_i^2 - 4416\kappa_i + 10368)|a_3|}{24\sqrt{-113\kappa_i^2 + 1224\kappa_i - 2880}}, \ i = 2,3,$$

$$\tau_{1,2} := \pm\frac{(54a_3^2 - 23a_3b_2 + 2b_2^2)(6a_3 - b_2)}{\sqrt{-\Omega}},$$

$$\tau_{3,4} := \pm\frac{(36 - 5\kappa_i)(24 - 7\kappa_i)|a_3|}{3\sqrt{-113\kappa_i^2 + 1224\kappa_i - 2880}}, \ i = 2,3,$$

$$\Delta := 3a_3(3a_3 - b_2) + (4b_1 + a_2)^2,$$

$$\Gamma := (3a_3 - b_2)(156a_3^3 - 96a_3^2 b_2 + 3a_3 a_2^2 + 18a_3 b_2^2 - b_2^3),$$

$$\Omega := 3240a_3^4 - 2376a_3^3 b_2 + 657a_3^2 b_2^2 - 84a_3 b_2^3 + 4b_2^4,$$

$$E_1 := (6a_3 - b_2)(4a_3 - b_2)(3a_3 - b_2) + a_2\sqrt{-\Gamma},$$

$$E_2 := (6a_3 - b_2)(4a_3 - b_2)(3a_3 - b_2) - a_2\sqrt{-\Gamma},$$

$$E_3 := (3a_3 - b_2)(252a_3^3 - 162a_3^2 b_2 + 33a_3 b_2^2 - 2b_2^3),$$

$$\varrho_1 := 3a_3, \ \varrho_2 := 4a_3, \ \varrho_3 := 6a_3, \ \varrho_4 := \frac{23 \pm \sqrt{97}}{4}a_3,$$

$$\kappa_i := \mathrm{RootsOf}(x^3 - 90x^2 + 270) \ \textit{such that} \quad \kappa_1 < \kappa_2 < \kappa_3.$$

Proof. When $\lambda \in C_4$, we use the computer algebra system Maple to calculate the period coefficients up to a nonzero factor:

$$\begin{aligned}
p_2 &= b_2^2 + 16b_1^2 + 16a_1^2 + 10a_3^2 + a_2^2 + 8a_2b_1 - 8a_3a_1 - 8b_2a_1 - b_2a_3,\\
p_4 &= 1940a_2a_3^2b_1 - 25a_2^2b_2a_3 - 1260a_3^2b_2a_1 + 5b_2^4 + 1280b_1^4 + 20a_2^4 + 1280a_1^4\\
&\quad +770a_3^4 - 832b_1b_2a_1a_2 - 832b_1a_1a_3a_2 - 104b_1a_3b_2a_2 - 200a_2^2a_3a_1\\
&\quad +1664b_1a_1^2a_2 + 104b_1b_2^2a_2 - 1280b_1^2a_1a_3 - 200a_2^2b_2a_1 + 480b_2a_1^2a_3\\
&\quad -1280b_1^2b_2a_1 - 160b_1^2a_3b_2 - 10b_2^3a_3 + 480b_2^2a_1^2 + 385a_2^2a_3^2 - 80b_2^3a_1 + 912b_1^2a_2^2
\end{aligned} \tag{2.20}$$

$$+400a_2^2a_1^2 - 1280b_2a_1^3 + 3040a_3^2b_1^2 + 160b_1^2b_2^2 - 1280a_1^3a_3 + 25b_2^2a_2^2 + 105a_3^2b_2^2$$
$$+1664b_1^3a_2 + 224b_1a_2^3 + 3360a_3^2a_1^2 - 2240a_1a_3^3 + 350a_3^3b_2 + 2560b_1^2a_1^2,$$
$$\vdots$$

We omit the expressions of $p_6(\lambda)$ and $p_8(\lambda)$, which have more than 107 terms. From (2.20), we get

$$V(p_2) = \{\lambda |\ a_1 = \sigma_1 \text{ or } \sigma_2,\ \Delta \leq 0\}. \tag{2.21}$$

Since the origin is a weak center of order 0 if and only if $p_2 \neq 0$, it follows from (2.21) that $\lambda \notin V(p_2)$, i.e.,

$$a_1 \neq \sigma_1, \sigma_2 \quad \text{or} \quad \Delta > 0,$$

from which we know that $\lambda \in \Lambda_{IV}^0 \cap C_4$.

When $\lambda \in V(p_2)$, we further identify the center of order $k \geq 1$. Consider p_2, p_4 as polynomials in a single variable a_1. Note that $\text{lcoeff}(p_2, a_1) = 16$, and we have

$$V(p_2, p_4) = V(p_2, \text{prem}(p_4, p_2, a_1)) = V(p_2, w_1, w_2) \cup V(\frac{p_2, \text{prem}(p_4, p_2, a_1)}{w_1}), \tag{2.22}$$

where

$$\begin{aligned} \text{prem}(p_4, p_2, a_1) &= w_1a_1 + w_2, \\ w_1 &:= -737280a_3^2(4a_3 - b_2), \\ w_2 &:= 184320a_3^2(a_2^2 + 4a_2b_1 - 14a_3^2 + 12b_2a_3 - 2b_2^2). \end{aligned} \tag{2.23}$$

Thus $\lambda \in V(p_2, p_4)$ if and only if one of the following conditions holds:

$$\text{I}: \begin{cases} p_2 = 0, \\ w_1 = 0, \\ w_2 = 0. \end{cases} \qquad \text{II}: \begin{cases} p_2 = 0, \\ \text{prem}(p_4, p_2, a_1) = 0. \\ w_1 \neq 0. \end{cases}$$

For case I, we get $a_3 = 0$ or $b_2 = 4a_3$ from $w_1 = 0$. When $a_3 = 0$, we get $a_6 = a_2a_3/4 = 0$ from $\lambda \in C_4$. Substituting $a_3 = 0$ into p_2, we get

$$p_2 = (4a_1 - b_2)^2 + (a_2 + 4b_1)^2.$$

Then, from $p_2 = 0$, it follows that $b_2 = 4a_1$ and $a_2 = -4b_1$, which implies $\lambda \in IC_1$. When $a_3 \neq 0$, $b_2 = 4a_3$, and $a_2 = 0$, we get $w_2 = 368640a_3^4$. Then it comes in conflict with $a_3 = 0$. When $a_3 \neq 0$, $b_2 = 4a_3$, and $a_2 \neq 0$, we get $b_1 = -(2a_3^2 + a_2^2)/(4a_2)$ from $w_2 = 0$. Then $p_2 = 16a_1^2 - 40a_3a_1 + 22a_3^2 + 4a_3^4/a_2^2$. Solving for a_1 from $p_2 = 0$, we know

$$a_1 = \frac{(5a_2 \pm \sqrt{3a_2^2 - 4a_3^2})a_3}{4a_2} \quad \text{if} \quad 3a_2^2 - 4a_3^2 \geq 0.$$

Thus

$$V(p_2, w_1, w_2) = IC_1 \cup \{\lambda |\ a_1 = \frac{(5a_2 \pm \sqrt{3a_2^2 - 4a_3^2})a_3}{4a_2},\ b_1 = -\frac{2a_3^2 + a_2^2}{4a_2}, b_2 = 4a_3,$$

$$a_2 \neq 0, a_3 \neq 0, 3a_2^2 - 4a_3^2 \geq 0\}. \tag{2.24}$$

For case II, since $w_1 \neq 0$, we get $a_1 = -w_2/w_1$ from $\mathrm{prem}(p_4, p_2, a_1) = 0$. Substituting a_1 into p_2, we get $p_2 = (4a_3 - b_2)^{-2} w_3$, where

$$w_3 = (16a_2^2 + 256a_3^2 + 16b_2^2 - 128b_2a_3)b_1^2 + (8a_2^3 + 8b_2a_3a_2 - 16a_2a_3^2)b_1 - 21b_2^3a_3 \\ -20a_2^2a_3^2 + 150a_3^2b_2^2 - 444a_3^3b_2 - b_2^2a_2^2 + 468a_3^4 + b_2^4 + 10a_2^2b_2a_3 + a_2^4.$$

Notice that $\mathrm{loceff}(w_3, b_1) = 16(4a_3 - b_2)^2 + 16a_2^2 > 0$, since $w_1 \neq 0$, and the discriminant for $w_3 = 0$ is

$$\Delta^* = -64(4a_3 - b_2)^2(3a_3 - b_2)(3a_3a_2^2 + 156a_3^3 - 96a_3^2b_2 + 18a_3b_2^2 - b_2^3).$$

Solving b_1 from $w_3 = 0$, we get $b_1 = \varsigma_1, \varsigma_2$ when $\Delta^* \geq 0$. Then, from $a_1 = -w_2/w_1$, we get $a_1 = \sigma_3$ when $b_1 = \varsigma_1$ or $a_1 = \sigma_4$ when $b_1 = \varsigma_2$. Comparing the values of $a_1 = \sigma_1, \sigma_2$ in $V(p_2)$, one can check that when substituting $b_1 = \varsigma_1, \varsigma_2$ in $a_1 = \sigma_1, \sigma_2$, they are coincident with one of the values σ_3, σ_4. Moreover, since

$$\sigma_3 - \frac{1}{4}(a_3 + b_2) = -\frac{(6a_3 - b_2)(4a_3 - b_2)(3a_3 - b_2) + a_2\sqrt{-\Gamma}}{4(a_2^2 + 16a_3^2 + b_2^2 - 8a_3b_2)} = -\frac{E_1}{4(a_2^2 + 16a_3^2 + b_2^2 - 8a_3b_2)},$$

then $a_1 = \sigma_3$ is coincident with σ_1 if $E_1 \leq 0$ and $a_1 = \sigma_3$ is coincident with σ_2 if $E_1 > 0$. Similarly, since

$$\sigma_4 - \frac{1}{4}(a_3 + b_2) = -\frac{(6a_3 - b_2)(4a_3 - b_2)(3a_3 - b_2) - a_2\sqrt{-\Gamma}}{4(a_2^2 + 16a_3^2 + b_2^2 - 8a_3b_2)} = -\frac{E_2}{4(a_2^2 + 16a_3^2 + b_2^2 - 8a_3b_2)},$$

then $a_1 = \sigma_4$ is coincident with σ_1 if $E_2 < 0$, and $a_1 = \sigma_4$ is coincident with σ_2 if $E_2 \geq 0$. Thus

$$V\left(\frac{p_2, \mathrm{prem}(p_4, p_2, a_1)}{w_1}\right) = \{\lambda |\ a_1 = \sigma_3,\ b_1 = \varsigma_1,\ a_3 \neq 0,\ b_2 \neq 4a_3,\ \Gamma \leq 0\} \\ \cup \{\lambda |\ a_1 = \sigma_4,\ b_1 = \varsigma_2,\ a_3 \neq 0,\ b_2 \neq 4a_3,\ \Gamma \leq 0\}.$$

From (2.22), we can obtain that

$$V(p_2, p_4) = IC_1 \cup \{\lambda |\ 3a_2^2 - 4a_3^2 \geq 0, a_2 \neq 0, a_3 \neq 0, a_1 = \frac{(5a_2 \pm \sqrt{3a_2^2 - 4a_3^2})a_3}{4a_2}, \\ b_1 = -\frac{a_2^2 + 2a_3^2}{4a_2}, b_2 = 4a_3\} \cup \{\lambda |\ \Gamma \leq 0, b_2 \neq 4a_3, a_3 \neq 0, a_1 = \sigma_3, b_1 = \varsigma_1\} \\ \cup \{\lambda |\ \Gamma \leq 0, b_2 \neq 4a_3, a_3 \neq 0, a_1 = \sigma_4, b_1 = \varsigma_2\}. \tag{2.25}$$

It follows from (2.21) and (2.25) that the origin is a weak center of order 1 if and only if $\lambda \in V(p_2)/V(p_2, p_4)$, i.e., one of the following condition holds:

(i) $\Delta \leq 0$, $a_1 = \sigma_1$ (or σ_2), $b_2 = 4a_3$, $a_2 = 0$, $a_3 \neq 0$,

(ii) $\Delta \leq 0$, $a_1 = \sigma_1$ (or σ_2), $b_2 = 4a_3$, $a_2 \neq 0$, $a_3 \neq 0$, $b_1 \neq -(a_2^2 + 2a_3^2)/(4a_2)$,

(iii) $\Delta \leq 0$, $\Gamma > 0$, $a_1 = \sigma_1$ (or σ_2), $a_3 \neq 0$, $b_2 \neq 4a_3$,

(iv) $\Delta \leq 0$, $\Gamma \leq 0$, $E_1 > 0$, $a_1 = \sigma_1$, $a_3 \neq 0$, $b_2 \neq 4a_3$,

(v) $\Delta \leq 0$, $\Gamma \leq 0$, $E_1 \leq 0$, $a_1 = \sigma_1$, $b_1 \neq \varsigma_1$, $a_3 \neq 0$, $b_2 \neq 4a_3$,

(vi) $\Delta \leq 0$, $\Gamma \leq 0$, $E_2 < 0$, $a_1 = \sigma_2$, $a_3 \neq 0$, $b_2 \neq 4a_3$,

(vii) $\Delta \leq 0$, $\Gamma \leq 0$, $E_2 \geq 0$, $a_1 = \sigma_2$, $b_1 \neq \varsigma_2$, $a_3 \neq 0$, $b_2 \neq 4a_3$, which implies that $\lambda \in \Lambda_{IV}^1 \cap C_4$.

When $\lambda \in V(p_2, p_4)$, we compute

$$V(p_2, p_4, p_6) = V(p_2, w_1, w_2, p_6) \cup V(\frac{p_2, \text{prem}(p_4, p_2, a_1), p_6}{w_1}). \tag{2.26}$$

Since $\text{lcoeff}(p_2, a_1) = 16$, we get

$$V(p_2, w_1, w_2, p_6) = V(p_2, w_1, w_2, \text{prem}(p_6, p_2, a_1)).$$

From (2.24), when $\lambda \in V(p_2, w_1, w_2)/IC_1$, we get $b_1 = -(2a_3^2 + a_2^2)/(4a_2)$, $b_2 = 4a_3$ and $a_3 \neq 0$. Then

$$\text{prem}(p_6, p_2, a_1) = -6341787648a_3^5(3a_1 - 4a_3)$$

in this case. Since $a_3 \neq 0$, solving $\text{prem}(p_6, p_2, a_1) = 0$, we get $a_1 = 4a_3/3$. Combining this with $p_2 = 0$, we get $a_2 = \pm 3\sqrt{26}a_3/13$. It follows that

$$V(p_2,w_1,w_2,p_6) = IC_1 \cup \{\lambda | a_1 = \frac{4}{3}a_3, a_2 = \pm\frac{3\sqrt{26}}{13}a_3, b_1 = \mp\frac{11\sqrt{26}}{78}a_3, b_2 = 4a_3, a_3 \neq 0\}. \tag{2.27}$$

Furthermore, we consider the second part of the right side in equation (2.26). Since $\text{lcoeff}(p_2, a_1) = 16$, we get

$$V(\frac{p_2, \text{prem}(p_4, p_2, a_1), p_6}{w_1}) = V(\frac{p_2, \text{prem}(p_4, p_2, a_1), \text{prem}(p_6, p_2, a_1)}{w_1}).$$

Since $\text{prem}(p_4, p_2, a_1) = 0$ and $w_1 \neq 0$, we get $a_1 = -w_2/w_1$. Then

$$\begin{aligned}
\text{prem}(p_6, p_2, a_1) &= w_4 b_1 + w_5,\\
w_4 &= \frac{19025362944a_3^4(3a_3 - b_2)a_2}{4a_3 - b_2},\\
w_5 &= \frac{792723456a_3^3(3a_3 - b_2)(6a_2^2a_3 - 324a_3^3 + 192a_3^2b_2 - 35a_3b_2^2 + 2b_2^3)}{4a_3 - b_2}.
\end{aligned}$$

Thus

$$V(\frac{p_2, \text{prem}(p_4, p_2, a_1), p_6}{w_1}) = V_{11} \cup V_{12}, \tag{2.28}$$

where

$$\begin{aligned}
V_{11} &:= V(\frac{p_2, \text{prem}(p_4, p_2, a_1), w_4, w_5}{w_1}),\\
V_{12} &:= V(\frac{p_2, \text{prem}(p_4, p_2, a_1), \text{prem}(p_6, p_2, a_1)}{w_1, w_4}).
\end{aligned}$$

Moreover, $\lambda \in V((p_2, \text{prem}(p_4, p_2, a_1), \text{prem}(p_6, p_2, a_1))/w_1)$ if and only if one of the following conditions holds:

$$\text{I}: \begin{cases} p_2 = 0, \\ \text{prem}(p_4, p_2, a_1) = 0, \\ w_4 = 0, \\ w_5 = 0, \\ w_1 \neq 0. \end{cases} \qquad \text{II}: \begin{cases} p_2 = 0, \\ \text{prem}(p_4, p_2, a_1) = 0, \\ \text{prem}(p_6, p_2, a_1) = 0, \\ w_4 \neq 0, \\ w_1 \neq 0. \end{cases}$$

For case I, from $w_1 \neq 0$, we know that $a_3 \neq 0$. Then $w_4 = 0$ implies $b_2 = 3a_3$ or $a_2 = 0$. When $b_2 = 3a_3$, solving for a_2 from $p_2 = 0$, we get $a_2 = -4b_1$. Then one can calculate $a_1 = a_3$ and $a_4 = a_6 = -a_3b_1$ from $a_1 = -w_2/w_1$ and $\lambda \in C_4$. Thus $\lambda \in IC_2$ in this case. When $b_2 \neq 3a_3$ and $a_2 = 0$, solving for b_2 from $w_5 = 0$, we get $b_2 = 6a_3$ or $b_2 = (23/4 \pm \sqrt{97}/4)a_3$. Solving for b_1 from $p_2 = 0$ when $b_2 = 6a_3$, we get $b_1 = \pm 3a_3/4$. Moreover, one can calculate $a_1 = 7a_3/4$. Solving for b_1 from $p_2 = 0$ when $b_2 = (23/4 + \sqrt{97}/4)a_3$, we get $b_1 = \pm\sqrt{562 + 62\sqrt{97}}a_3/48$. Furthermore, one can calculate $a_1 = (26 + \sqrt{97})a_3/12$. Solving for b_1 from $p_2 = 0$ when $b_2 = (23/4 - \sqrt{97}/4)a_3$, we get $b_1 = \pm\sqrt{562 - 62\sqrt{97}}a_3/48$. Moreover, one can calculate $a_1 = (26 - \sqrt{97})a_3/12$. Therefore,

$$\begin{aligned} V_{11} = {} & IC_2 \cup \{\lambda |\ a_1 = \frac{7}{4}a_3, a_2 = 0, b_1 = \pm\frac{3}{4}a_3, b_2 = 6a_3, a_3 \neq 0\} \\ & \cup \{\lambda |\ a_1 = \frac{26+\sqrt{97}}{12}a_3, a_2 = 0, b_1 = \pm\frac{\sqrt{562+62\sqrt{97}}}{48}a_3, b_2 = \frac{23+\sqrt{97}}{4}a_3, \\ & a_3 \neq 0\} \cup \{\lambda |\ a_1 = \frac{26-\sqrt{97}}{12}a_3, a_2 = 0, b_1 = \pm\frac{\sqrt{562-62\sqrt{97}}}{48}a_3, \\ & b_2 = \frac{23-\sqrt{97}}{4}a_3, a_3 \neq 0\}. \end{aligned} \tag{2.29}$$

For case II, from $\text{prem}(p_6, p_2, a_1) = 0$ and $w_4 \neq 0$, we get $b_1 = -w_5/w_4$. Then

$$p_2 = (-3240a_3^4 + 2376a_3^3b_2 - 657a_3^2b_2^2 + 84a_3b_2^3 - 4b_2^4)a_2^2 - (54a_3^2 - 23a_3b_2 + 2b_2^2)^2(-b_2 + 6a_3)^2.$$

Solving for a_2 from $p_2 = 0$, we get $a_2 = \tau_{1,2}$ if $\Omega < 0$. Then from $a_1 = -w_2/w_1$ and $b_1 = -w_5/w_4$, one can check that $a_1 = (60a_3^2 - 15a_3b_2 + 2b_2^2)/(24a_3)$ and $b_1 = \varsigma_{3,4}$. Comparing the values of $b_1 = \varsigma_{1,2}$ in $V(p_2, p_4)$, one can check that substituting $a_2 = \tau_1, \tau_2$ in $b_1 = \varsigma_1, \varsigma_2$, they are coincident with one of the values ς_3, ς_4. Moreover, since

$$\begin{aligned} \frac{4\varsigma_3(\tau_1^2 + 16a_3^2 - 8a_3b_2 + b_2^2) - \tau_1(-\tau_1^2 + 2a_3^2 - a_3b_2)}{4a_3 - b_2} &= \frac{(3a_3 - b_2)(252a_3^3 - 162a_3^2b_2 + 33a_3b_2^2 - 2b_2^3)}{\sqrt{-\Omega}} \\ &= \frac{E_3}{\sqrt{-\Omega}}, \end{aligned}$$

then when $a_2 = \tau_1$, $b_1 = \varsigma_3$ is coincident with ς_1 if $E_3 \geq 0$ and $b_1 = \varsigma_3$ is coincident with ς_2 if $E_3 < 0$. Furthermore, when $a_2 = \tau_1$, $a_1 = (60a_3^2 - 15a_3b_2 + 2b_2^2)/(24a_3)$ is coincident with σ_3 if $E_3 \geq 0$, and $a_1 = (60a_3^2 - 15a_3b_2 + 2b_2^2)/(24a_3)$ is coincident with σ_4 if $E_3 < 0$. Similarly, since

$$\frac{4\varsigma_4(\tau_2^2 + 16a_3^2 - 8a_3b_2 + b_2^2) - \tau_2(-\tau_2^2 + 2a_3^2 - a_3b_2)}{4a_3 - b_2} = -\frac{E_3}{\sqrt{-\Omega}},$$

then when $a_2 = \tau_2$, $b_1 = \varsigma_4$ is coincident with ς_2 if $E_3 \geq 0$, and $b_1 = \varsigma_4$ is coincident with ς_1 if $E_3 < 0$. When $a_2 = \tau_2$, $a_1 = (60a_3^2 - 15a_3b_2 + 2b_2^2)/(24a_3)$ is coincident with σ_4 if $E_3 \geq 0$, and $a_1 = (60a_3^2 - 15a_3b_2 + 2b_2^2)/(24a_3)$ is coincident with σ_3 if $E_3 < 0$. Therefore,

$$\begin{aligned} V_{12} = \{\lambda |\ & \Omega < 0, a_1 = \frac{60a_3^2 - 15a_3b_2 + 2b_2^2}{24a_3}, b_1 = \varsigma_{3,4}, a_2 = \tau_{1,2}, a_3 \neq 0, \\ & b_2 \neq 3a_3, 4a_3, 6a_3, \frac{23 \pm \sqrt{97}}{4}a_3\}. \end{aligned}$$

From (2.27), (2.29), and (2.30),

$$
\begin{aligned}
V(p_2,p_4,p_6) = {} & IC_1 \cup IC_2 \cup \{\lambda|\ a_1=\frac{4}{3}a_3, a_2=\pm\frac{3\sqrt{26}}{13}a_3, b_1=\mp\frac{11\sqrt{26}}{78}a_3, b_2=4a_3,\\
& a_3\neq 0\} \cup \{\lambda|\ a_1=\frac{7}{4}a_3, a_2=0, b_1=\pm\frac{3}{4}a_3, b_2=6a_3, a_3\neq 0\}\\
& \cup\{\lambda|\ a_1=\frac{26+\sqrt{97}}{12}a_3, a_2=0, b_1=\pm\frac{\sqrt{562+62\sqrt{97}}}{48}a_3,\\
& b_2=\frac{23+\sqrt{97}}{4}a_3, a_3\neq 0\} \cup \{\lambda|\ a_1=\frac{26-\sqrt{97}}{12}a_3,\\
& a_2=0, b_1=\pm\frac{\sqrt{562-62\sqrt{97}}}{48}a_3, b_2=\frac{23-\sqrt{97}}{4}a_3, a_3\neq 0\}\\
& \cup\{\lambda|\ \Omega<0, a_1=\frac{60a_3^2-15a_3b_2+2b_2^2}{24a_3}, b_1=\varsigma_{3,4}, a_2=\tau_{1,2},\\
& a_3\neq 0, b_2\neq 3a_3, 4a_3, 6a_3, \frac{23\pm\sqrt{97}}{4}a_3\}.
\end{aligned} \tag{2.30}
$$

It follows from (2.25) and (2.30) that the origin is a weak center of order 2 if and only if $\lambda\in V(p_2,p_4)/V(p_2,p_4,p_6)$, i.e., one of the following conditions holds:

(i) $3a_2^2-4a_3^2\geq 0$, $a_3>0$, $a_1=(5a_2+\sqrt{3a_2^2-4a_3^2})a_3/(4a_2)$, $b_1=-(a_2^2+2a_3^2)/(4a_2)$, $b_2=4a_3$, $a_2\neq 0, 3\sqrt{26}a_3/13$,

(ii) $3a_2^2-4a_3^2\geq 0$, $a_3<0$, $a_1=(5a_2+\sqrt{3a_2^2-4a_3^2})a_3/(4a_2)$, $b_1=-(a_2^2+2a_3^2)/(4a_2)$, $b_2=4a_3$, $a_2\neq 0, -3\sqrt{26}a_3/13$,

(iii) $3a_2^2-4a_3^2\geq 0$, $a_3>0$, $a_1=(5a_2-\sqrt{3a_2^2-4a_3^2})a_3/(4a_2)$, $b_1=-(a_2^2+2a_3^2)/(4a_2)$, $b_2=4a_3$, $a_2\neq 0, -3\sqrt{26}a_3/13$,

(iv) $3a_2^2-4a_3^2\geq 0$, $a_3<0$, $a_1=(5a_2-\sqrt{3a_2^2-4a_3^2})a_3/(4a_2)$, $b_1=-(a_2^2+2a_3^2)/(4a_2)$, $b_2=4a_3$, $a_2\neq 0, 3\sqrt{26}a_3/13$,

(v) $\Gamma\leq 0$, $a_1=\sigma_3$, $a_2=0$, $b_1=\varsigma_1$, $a_3\neq 0$, $b_2\neq 3a_3, 4a_3, 6a_3, (23\pm\sqrt{97})a_3/4$,

(vi) $\Gamma\leq 0$, $a_1=\sigma_4$, $a_2=0$, $b_1=\varsigma_2$, $a_3\neq 0$, $b_2\neq 3a_3, 4a_3, 6a_3, (23\pm\sqrt{97})a_3/4$,

(vii) $\Gamma\leq 0$, $\Omega\geq 0$, $a_1=\sigma_3$, $b_1=\varsigma_1$, $a_2\neq 0$, $a_3\neq 0$, $b_2\neq 3a_3, 4a_3, 6a_3, (23\pm\sqrt{97})a_3/4$,

(viii) $\Gamma\leq 0$, $\Omega\geq 0$, $a_1=\sigma_4$, $b_1=\varsigma_2$, $a_2\neq 0$, $a_3\neq 0$, $b_2\neq 3a_3, 4a_3, 6a_3, (23\pm\sqrt{97})a_3/4$,

(ix) $\Gamma\leq 0$, $\Omega<0$, $E_3\geq 0$, $a_1=\sigma_3$, $b_1=\varsigma_1$, $a_2\neq 0, \tau_1$, $a_3\neq 0$, $b_2\neq 3a_3, 4a_3, 6a_3, (23\pm\sqrt{97})a_3/4$,

(x) $\Gamma\leq 0$, $\Omega<0$, $E_3<0$, $a_1=\sigma_4$, $b_1=\varsigma_2$, $a_2\neq 0$, $a_3\neq 0$, $b_2\neq 3a_3, 4a_3, 6a_3, (23\pm\sqrt{97})a_3/4$,

(xi) $\Gamma\leq 0$, $\Omega<0$, $E_3\geq 0$, $a_1=\sigma_4$, $b_1=\varsigma_2$, $a_2\neq 0, \tau_2$, $a_3\neq 0$, $b_2\neq 3a_3, 4a_3, 6a_3, (23\pm\sqrt{97})a_3/4$,

(xii) $\Gamma\leq 0$, $\Omega<0$, $E_3<0$, $a_1=\sigma_3$, $b_1=\varsigma_1$, $a_2\neq 0$, $a_3\neq 0$, $b_2\neq 3a_3, 4a_3, 6a_3, (23\pm\sqrt{97})a_3/4$,

which implies that $\lambda\in\Lambda_{IV}^2\cap C_4$.

When $\lambda\in V(p_2,p_4,p_6)$, from (2.26) and (2.28), we get

$$
V(p_2,p_4,p_6,p_8)=V_{21}\cup V_{22}\cup V_{23}, \tag{2.31}
$$

where

$$\begin{aligned} V_{21} &:= V(p_2, w_1, w_2, p_6, p_8) \cap V(p_8) = V(p_2, w_1, w_2, p_6, p_8), \\ V_{22} &:= V_{11} \cap V(p_8) = V(\frac{p_2, \mathrm{prem}(p_4, p_2, a_1), w_4, w_5, p_8}{w_1}), \\ V_{23} &:= V_{12} \cap V(p_8) = V(\frac{p_2, \mathrm{prem}(p_4, p_2, a_1), \mathrm{prem}(p_6, p_2, a_1), p_8}{w_1, w_4}). \end{aligned}$$

From (2.27), it is easy to check that

$$V_{21} = V(p_2, w_1, w_2, \mathrm{prem}(p_6, p_2, a_1), \mathrm{prem}(p_8, p_2, a_1)) = IC_1, \tag{2.32}$$

$$V_{22} = V(\frac{p_2, \mathrm{prem}(p_4, p_2, a_1), w_4, w_5, \mathrm{prem}(p_8, p_2, a_1)}{w_1}) = IC_2. \tag{2.33}$$

To compute V_{23}, we must substitute $a_1 = -w_2/w_1$ and $b_1 = -w_5/w_4$ in p_2 and $\mathrm{prem}(p_8, p_2, a_1)$. Then

$$p_2 = w_6 a_2^{-2} a_3^{-2}/36, \quad \text{and} \quad \mathrm{prem}(p_8, p_2, a_1) = -597939978240 w_7 a_2^{-2} a_3^2,$$

where

$$\begin{aligned} w_6 :=\ & 3240a_2^2a_3^4 - 2376a_2^2a_3^3b_2 + 657a_2^2a_3^2b_2^2 - 84a_2^2a_3b_2^3 + 4a_2^2b_2^4 + 104976a_3^6 \\ & -124416a_3^5b_2 + 59544a_3^4b_2^2 - 14736a_3^3b_2^3 + 1993a_3^2b_2^4 - 140a_3b_2^5 + 4b_2^6, \\ w_7 :=\ & 11340a_2^2a_3^5 - 11664a_2^2a_3^4b_2 + 4824a_2^2a_3^3b_2^2 - 999a_2^2a_3^2b_2^3 + 102a_2^2a_3b_2^4 \\ & -4a_2^2b_2^5 + 419904a_3^7 - 602640a_3^6b_2 + 362592a_3^5b_2^2 - 118488a_3^4b_2^3 \\ & +22708a_3^3b_2^4 - 2553a_3^2b_2^5 + 156a_3b_2^6 - 4b_2^7. \end{aligned}$$

Notice that $\mathrm{lcoeff}(w_6, b_2) = 4$ and $a_2a_3 \neq 0$ when $\lambda \in V_{23}$. Then

$$V_{23} = V(\frac{p_2, \mathrm{prem}(p_4, p_2, a_1), \mathrm{prem}(p_6, p_2, a_1), \mathrm{prem}(w_7, w_6, b_2)}{w_1, w_4}),$$

where

$$\mathrm{prem}(w_7, w_6, b_2) = -32a_2^2a_3(3a_3 - b_2)(270a_3^3 - 90a_3^2b_2 + b_2^3).$$

Since $b_2 \neq 3a_3$ and $a_2a_3 \neq 0$ when $\lambda \in V_{23}$, we must consider $270a_3^3 - 90a_3^2b_2 + b_2^3 = 0$. One can check that there are three real roots κ_i, $i = 1, 2, 3$, for the equation $x^3 - 90x^2 + 270 = 0$. Assume $\kappa_1 < \kappa_2 < \kappa_3$. Then $\kappa_1 \in [-11, -10.731]$, $\ell_2 \in [3.455, 3, 461]$, $\ell_3 \in [7.2, 7.3]$. Thus one can solve $b_2 = \kappa_i a_3$, $i = 1, 2, 3$, from $270a_3^3 - 90a_3^2b_2 + b_2^3 = 0$. We can solve $a_1 = (60a_3^2 - 15a_3b_2 + 2b_2^2)/(24a_3)$, $a_2 = \tau_{1,2}$, and $b_1 = \varsigma_{3,4}$ from $p_2 = 0$, $a_1 = -w_2/w_1$, and $b_1 = -w_5/w_4$. Moreover, when $270a_3^3 - 90a_3^2b_2 + b_2^3 = 0$, one can check that $\tau_{3,4}$ and $\varsigma_{5,6}$ are coincident with $\tau_{1,2}$ and $\varsigma_{3,4}$ when $a_3 > 0$, and $\tau_{3,4}$ and $\varsigma_{5,6}$ are coincident with $\tau_{2,1}$ and $\varsigma_{4,3}$ when $a_3 < 0$. Furthermore, for the polynomial under the square root sign in $\tau_{3,4}$ and $\varsigma_{5,6}$, it is easy to check that $-2880a_3^2 + 1224a_3b_2 - 113b_2^2 < 0$ when $b_2 = \kappa_1 a_3$, and $-2880a_3^2 + 1224a_3b_2 - 113b_2^2 > 0$ when $b_2 = \kappa_i a_3$, $i = 1, 2$. Therefore,

$$\begin{aligned} V_{23} = & \{\lambda |\ a_1 = \frac{60a_3^2 - 15a_3b_2 + 2b_2^2}{24a_3}, a_2 = \tau_{3,4}, b_1 = \varsigma_{5,6}, b_2 = \kappa_2 a_3, a_3 \neq 0\} \\ & \cup \{\lambda |\ a_1 = \frac{60a_3^2 - 15a_3b_2 + 2b_2^2}{24a_3}, a_2 = \tau_{3,4}, b_1 = \varsigma_{5,6}, b_2 = \kappa_2 a_3, a_3 \neq 0\}. \end{aligned} \tag{2.34}$$

From (2.32), (2.33), and (2.34),

$$
\begin{aligned}
V(p_2,p_4,p_6,p_8) = IC_1 \cup IC_2 \cup \{\lambda|\ a_1 = \frac{60a_3^2 - 15a_3b_2 + 2b_2^2}{24a_3}, a_2 = \tau_{3,4}, b_1 = \varsigma_{5,6},\\
b_2 = \kappa_2 a_3, a_3 \neq 0\} \cup \{\lambda|\ a_1 = \frac{60a_3^2 - 15a_3b_2 + 2b_2^2}{24a_3},\\
a_2 = \tau_{3,4}, b_1 = \varsigma_{5,6}, b_2 = \kappa_2 a_3, a_3 \neq 0\}.
\end{aligned}
\tag{2.35}
$$

It follows from (2.30) and (2.35) that the origin is a weak center of order 3 if and only if $\lambda \in V(p_2,p_4,p_6)/V(p_2,p_4,p_6,p_8)$, i.e., one of the following conditions holds:

(i) $a_1 = 4a_3/3$, $a_2 = \pm 3\sqrt{26}a_3/13$, $b_1 = \mp 11\sqrt{26}a_3/78$, $b_2 = 4a_3$, $a_3 \neq 0$,

(ii) $a_1 = 7a_3/4$, $a_2 = 0$, $b_1 = \pm 3a_3/4$, $b_2 = 6a_3$, $a_3 \neq 0$,

(iii) $a_1 = (26+\sqrt{97})a_3/12$, $a_2 = 0$, $b_1 = \pm\sqrt{562+62\sqrt{97}}a_3/48$, $b_2 = (23+\sqrt{97})a_3/4$, $a_3 \neq 0$,

(iv) $a_1 = (26-\sqrt{97})a_3/12$, $a_2 = 0$, $b_1 = \pm(\sqrt{562-62\sqrt{97}})a_3/48$, $b_2 = (23-\sqrt{97})a_3/4$, $a_3 \neq 0$,

(v) $\Omega < 0$, $a_1 = (60a_3^2 - 15a_3b_2 + 2b_2^2)/(24a_3)$, $b_1 = \varsigma_{3,4}$, $a_2 = \tau_{1,2}$, $a_3 \neq 0$, $b_2 \neq 3a_3, 4a_3, 6a_3, (23\pm\sqrt{97})a_3/4, \kappa_2 a_3, \kappa_3 a_3$,

which implies that $\lambda \in \Lambda_{IV}^3 \cap C_4$.

When $\lambda \in V(p_2,p_4,p_6,p_8)$, from (2.31), (2.32), and (2.33), we get

$$V(p_2,p_4,p_6,p_8,p_{10}) = IC_1 \cup IC_2 \cup (V_{23} \cap V(p_{10})).$$

We know that $a_1 = -w_2/w_1$, $b_1 = -w_5/w_4$, and $270a_3^3 - 90a_3^2b_2 + b_2^3 = 0$ when $\lambda \in V_{23}$. Thus

$$a_1 = \frac{60a_3^2 - 15a_3b_2 + 2b_2^2}{24a_3}, \quad b_1 = -\frac{6a_2^2 - 864a_3^2 + 372a_3b_2 - 35b_2^2}{24a_2}. \tag{2.36}$$

Under (2.36), we calculate p_{10}, which is a polynomial of a_2, a_3, b_2, and the number of terms is 170. Denote $270a_3^3 - 90a_3^2b_2 + b_2^3 = 0$ by w_8. Substituting $a_2 = \tau_{3,4}$ in $\mathrm{prem}(p_{10}, w_8, b_2)$, we get $\mathrm{prem}(p_{10}, w_8, b_2) = w_9(2880a_3^2 - 1224a_3b_2 + 113b_2^2)^{-8}$. Since $\mathrm{resultant}(w_8, w_9, b_2) \neq 0$, we get

$$V(p_2,p_4,p_6,p_8,p_{10}) = IC_1 \cup IC_2. \tag{2.37}$$

It follows from (2.35) and (2.37) that the origin is a weak center of order 4 if and only if $\lambda \in V(p_2,p_4,p_6,p_8)/V(p_2,p_4,p_6,p_8,p_{10})$, i.e., one of the following condition holds:

(i) $a_1 = (60a_3^2 - 15a_3b_2 + 2b_2^2)/(24a_3)$, $a_2 = \tau_{3,4}$, $b_1 = \varsigma_{5,6}$, $b_2 = \kappa_2 a_3$, $a_3 \neq 0$,

(ii) $a_1 = (60a_3^2 - 15a_3b_2 + 2b_2^2)/(24a_3)$, $a_2 = \tau_{3,4}$, $b_1 = \varsigma_{5,6}$, $b_2 = \kappa_2 a_3$, $a_3 \neq 0$,

which implies that $\lambda \in \Lambda_{IV}^4 \cap C_4$. This completes the proof. □

3. Local Bifurcation of Critical Periods

In this section, we investigate how many local critical periods can be produced from a perturbed weak center O. For a weak center of finite order, by Lemma 2.2 in [6], if the weak center corresponding to a parameter value λ^* has order k, no more than k local critical periods bifurcate from this weak center at the parameter value λ^*. It suggests identifying whether exactly n critical periods bifurcate from this weak center at the parameter value λ^* for any $n \le k$. The independence of the period coefficients $p_2, p_4, \dots p_{2k}$ with respect to p_{2k+2} at λ^* gives a sufficient condition for that. As defined in [6], polynomials $f_i : \mathbb{R}^N \to \mathbb{R}$, $i = 1, ..., l$, are said to be independent with respect to the polynomial $f : \mathbb{R}^N \to \mathbb{R}$, at $\lambda^* \in V(f_1, f_2, \dots, f_l)$ if the following three conditions are satisfied:

(i) Every neighborhood of λ^* in $\mathbb{R}^N$ contains a point $\lambda^o \in V(f_1, f_2, \dots, f_{l-1})$ such that $f_l(\lambda^o) \cdot f(\lambda^o) < 0$.

(ii) If $\lambda^* \in V(f_1, f_2, \dots, f_j)$ and $f_{j+1}(\lambda^*) \ne 0$, $2 \le j \le l-1$, then every neighborhood W of λ contains a point $\lambda^o \in V(f_1, f_2, \dots, f_{j-1})$ such that $f_j(\lambda^o) \cdot f_{j+1}(\lambda^*) < 0$.

(iii) If $\lambda^* \in V(f_1)$ and $f_2(\lambda^*) \ne 0$, then every neighborhood of λ^* contains a point λ^o such that $f_1(\lambda^o) \cdot f_2(\lambda^*) < 0$.

It is easy to see that if $f_1, \dots, f_l$ are independent with respect to f_{l+1} at $\lambda_* \in V(f_1, \dots, f_l)$, then for each $k = 2, \dots, l$, $f_1, \dots, f_{k-1}$ are independent with respect to f_k at every $\lambda \in V(f_1, \dots, f_{k-1})$ such that $f_k(\lambda) \ne 0$.

We will now discuss how many local critical periods can be produced from a perturbed system of (1.2) near O. For the first case, a direct result is the following theorem.

Theorem 3.1. *In the case that $C_1 \setminus \mathcal{IC}$, no local critical periods occur in a perturbed system of* (1.2).

Furthermore, combining the above independent conditions with the results obtained in the last section, we have the following theorem.

Theorem 3.2. *In the case that $C_2 \setminus \mathcal{IC}$, for each $k = 1, 2, 3$, at most k local critical periods occur in a perturbed system of* (1.2) *for $\lambda \in \Lambda^k_{II}$. Furthermore, there are perturbations of* (1.2) *where $\lambda \in \Lambda^k_{II}$, with exactly k critical periods.*

Proof. We obtain the first assertion directly by Lemma 2.2 in [6]. To prove the second part, by Theorem 2.1 in [6], we must prove that p_2, p_4, and p_6 are independent with respect to p_8.

For any $\lambda^* = (a_1^*, a_3^*, a_5^*, b_1^*, b_2^*) \in \Lambda^3_{II}$, we have $p_2(\lambda^*) = p_4(\lambda^*) = p_6(\lambda^*) = 0$ and $p_8(\lambda^*) \ne 0$. From the proof of Lemma 2, we know that

$$a_5^* = -\frac{m_2(a_1^*, a_3^*, b_1^*, b_2^*)}{m_1(a_1^*, a_3^*, b_1^*, b_2^*)} \quad \text{and} \quad m_1(a_1^*, a_3^*, b_1^*, b_2^*) \ne 0,$$

where m_1 and m_2 are defined in (2.6) and are considered to be functions of a_1, a_3, b_1, b_2. Then every neighborhood of λ^* contains a point $\lambda^o = (a_1^o, a_3^o, a_5^o, b_1^o, b_2^o)$, where

$$a_1^o = a_1^* - \frac{1}{4}(3a_3^* - b_2^*)\mathrm{sgn}(p_8(\lambda^*)s_1(\lambda^*))\epsilon + \frac{1}{4}(4a_1^* - a_3^* - b_2^*)(\sqrt{1 + \mathrm{sgn}(p_8(\lambda^*)s_1(\lambda^*))\epsilon} - 1),$$

$$a_3^o = a_3^*, \quad a_5^o = -\frac{m_2(a_1^o, a_3^o, b_1^o, b_2^o)}{m_1(a_1^o, a_3^o, b_1^o, b_2^o)}, \quad b_1^o = b_1^*, \quad b_2^o = b_2^* - (3a_3^* - b_2^*)\mathrm{sgn}(p_8(\lambda^*)s_1(\lambda^*))\epsilon,$$

$$s_1(\lambda) = 147083960950218502963200a_3^5(b_2 - 3a_3)(72a_1b_2^3 - 1908a_1b_2^2a_3 + 11364a_1b_2a_3^2 - 19296a_1a_3^3 - 3b_2^4 + 43b_2^3a_3 + 759b_2^2a_3^2 - 7053b_2a_3^3 + 13842a_3^4),$$

and $\epsilon > 0$ is sufficiently small. We can check that

$$\begin{aligned} p_2(\lambda^o) &= p_2(\lambda^*)(1+\epsilon) = 0, \\ p_4(\lambda^o) &= \text{prem}(p_4, p_2, a_1)(\lambda^o) = a_5^o m_1(a_1^o, a_3^o, b_1^o, b_2^o) + m_2(a_1^o, a_3^o, b_1^o, b_2^o) = 0, \\ p_6(\lambda^o) &= \text{prem}(m_3, p_2, a_1)(\lambda^o) \\ &= \text{prem}(m_3, p_2, a_1)(\lambda^*) - s_1(\lambda^*)\text{sgn}(p_8(\lambda^*)s_1(\lambda^*))\epsilon + O(\epsilon^2) \\ &= -s_1(\lambda^*)\text{sgn}(p_8(\lambda^*)s_1(\lambda^*))\epsilon + O(\epsilon^2), \\ p_8(\lambda^o) &= \text{prem}(m_7, p_2, a_1)(\lambda^o) = \text{prem}(m_7, p_2, a_1)(\lambda^*) + O(\epsilon) = p_8(\lambda^*) + O(\epsilon), \end{aligned}$$

where m_3 and m_7 are defined in (2.12) and (2.15). Then $\lambda^o \in V(p_2, p_4)$ and

$$p_6(\lambda^o)p_8(\lambda^o) = -p_8(\lambda^*)s_1(\lambda^*)\text{sgn}(p_8(\lambda^*)s_1(\lambda^*))\epsilon + O(\epsilon^2). \tag{3.1}$$

Moreover, we claim that $s_1(\lambda^*) \neq 0$. In fact, with $p_2(\lambda^*) = 0$ and $m_6(a_3^*, b_2^*) = 0$, which is obtained from the proof of Lemma 2 and defined in (2.13), $s_1(\lambda^*) = 0$ implies $a_1^* = a_3^* = b_2^* = 0$ and $\lambda^* \notin \Lambda_{II}^3$. It follows from (3.1) that $p_6(\lambda^o)p_8(\lambda^o) < 0$.

For any $\lambda^* = (a_1^*, a_3^*, a_5^*, b_1^*, b_2^*) \in \Lambda_{II}^2$, the discussion is divided into two parts. If λ^* is in the first set of Λ_{II}^2, i.e.,

$$a_1^* = \frac{11}{6}a_3^*, \quad b_2^* = \frac{13}{3}a_3^*, \quad a_5^* = -\frac{65}{288}{a_3^*}^2 + {b_1^*}^2 \quad \text{and} \quad a_3^* \neq 0,$$

then every neighborhood of λ^* contains a point $\lambda^o = (a_1^*, a_3^*, a_5^* + \epsilon, b_1^*, b_2^*)$, where $\epsilon > 0$ is sufficiently small. We can verify that $p_2(\lambda^o) = 0$,

$$p_4(\lambda^o) = 384{a_3^*}^2\epsilon \quad \text{and} \quad p_6(\lambda^*) = -\frac{109375}{4}{a_3^*}^6.$$

Thus $p_4(\lambda^o)p_6(\lambda^*) = -10500000a_3^8\epsilon < 0$. If λ^* is in the remainder sets of Λ_{II}^2, then from the proof of Lemma 2, we know:

$$a_5^* = -\frac{m_2(a_1^*, a_3^*, b_1^*, b_2^*)}{m_1(a_1^*, a_3^*, b_1^*, b_2^*)} \quad \text{and} \quad m_1(a_1^*, a_3^*, b_1^*, b_2^*) \neq 0,$$

where m_1 and m_2 are defined in (2.6) and are considered to be functions of a_1, a_3, b_1, b_2. Thus every neighborhood of λ^* contains a point $\lambda^o = (a_1^o, a_3^o, a_5^o, b_1^o, b_2^o)$, where

$$\begin{aligned} &a_1^o = a_1^*, \quad a_3^o = a_3^*, \quad a_5^o = -\frac{m_2(a_1^*, a_3^*, b_1^*, b_2^*)}{m_1(a_1^*, a_3^*, b_1^*, b_2^*)} - \text{sgn}(a_3^*(8a_1^* - 11a_3^* + b_2^*)p_6(\lambda^*))\epsilon, \\ &b_1^o = b_1^*, \quad b_2^o = b_2^*, \end{aligned}$$

and $\epsilon > 0$ is sufficiently small. We can check that

$$\begin{aligned} p_2(\lambda^o) &= p_2(\lambda^*) = 0, \\ p_4(\lambda^o) &= \text{prem}(p_4, p_2, a_1)(\lambda^o) = -196608a_3^*(8a_1^* - 11a_3^* + b_2^*)\text{sgn}(a_3^*(8a_1^* - 11a_3^* + b_2^*)p_6(\lambda^*))\epsilon. \end{aligned}$$

Moreover, we claim that $8a_1^* - 11a_3^* + b_2^* \neq 0$. In fact, if $8a_1^* - 11a_3^* + b_2^* = 0$, then from $p_2(\lambda^*) = 0$, we get $b_2 = 13a_3/3$ or $b_2 = 3a_3$, which are contradictory with the last four sets of Λ_{II}^2. Otherwise, $a_3^* \neq 0$ and $p_6(\lambda^*) \neq 0$, as $\lambda^* \in \Lambda_{II}^2$. Therefore,

$$p_4(\lambda^o)p_6(\lambda^*) = -196608a_3^*(8a_1^* - 11a_3^* + b_2^*)p_6(\lambda^*)\text{sgn}(a_3^*(8a_1^* - 11a_3^* + b_2^*)p_6(\lambda^*))\epsilon < 0.$$

For any $\lambda^* = (a_1^*, a_3^*, a_5^*, b_1^*, b_2^*) \in \Lambda_{II}^1$, the discussion is divided into three parts. If λ^* is in the first set of Λ_{II}^1, i.e.,

$$a_1^* = \frac{11}{6}a_3^*, \quad b_2^* = \frac{13}{3}a_3^*, \quad a_5^* \neq -\frac{65}{288}{a_3^*}^2 + {b_1^*}^2 \quad \text{and} \quad a_3^* \neq 0,$$

then every neighborhood of λ^* contains a point $\lambda^o = (a_1^* - \text{sgn}(a_3^*(65{a_3^*}^2 - 288{b_1^*}^2 + 288a_5^*))\epsilon, a_3^*, a_5^*, b_1^*, b_2^*)$, where $\epsilon > 0$ is sufficiently small. We can verify that

$$\begin{aligned} p_2(\lambda^o) &= -16a_3\text{sgn}(a_3^*(65{a_3^*}^2 - 288{b_1^*}^2 + 288a_5^*))\epsilon + O(\epsilon^2), \\ p_4(\lambda^*) &= \frac{4}{3}{a_3^*}^2(65{a_3^*}^2 - 288{b_1^*}^2 + 288a_5^*). \end{aligned}$$

Notice that $a_3^*(65{a_3^*}^2 - 288{b_1^*}^2 + 288a_5^*) \neq 0$ when λ^* is in the first set of Λ_{II}^1. Thus

$$p_2(\lambda^o)p_4(\lambda^*) = -\frac{64}{3}{a_3^*}^3(65{a_3^*}^2 - 288{b_1^*}^2 + 288a_5^*)\text{sgn}(a_3^*(65{a_3^*}^2 - 288{b_1^*}^2 + 288a_5^*))\epsilon + O(\epsilon^2) < 0.$$

If λ^* is in the second set of Λ_{II}^1, i.e.,

$$a_1^* = \frac{5}{6}a_3^*, \quad b_2^* = \frac{13}{3}a_3^* \quad \text{and} \quad a_3^* \neq 0,$$

then every neighborhood of λ^* contains a point $\lambda^o = (a_1^* + \text{sgn}(a_3^*)\epsilon, a_3^*, a_5^*, b_1^*, b_2^*)$, where $\epsilon > 0$ is sufficiently small. We can check that

$$p_2(\lambda^o) = -16a_3\text{sgn}(a_3^*)\epsilon + O(\epsilon^2) \quad \text{and} \quad p_4(\lambda^*) = 140{a_3^*}^4.$$

Notice that $a_3^* \neq 0$ when λ^* is in the second set of Λ_{II}^1. Thus

$$p_2(\lambda^o)p_4(\lambda^*) = -2240{a_3^*}^5\text{sgn}(a_3^*)\epsilon + O(\epsilon^2) < 0.$$

If λ^* is in the third set of Λ_{II}^1, i.e., $p_2(\lambda^*) = 0$, $p_4(\lambda^*) \neq 0$ and $b_2^* \neq 13a_3^*/3$, then every neighborhood of λ^* contains a point $\lambda^o = (a_1^* + \text{sgn}((b_2^* + a_3^* - 4a_1^*)p_4(\lambda^*))\epsilon, a_3^*, a_5^*, b_1^*, b_2^*)$, where $\epsilon > 0$ is sufficiently small. Hence

$$\begin{aligned} p_2(\lambda^o) &= p_2(\lambda^*) - 8(b_2^* + a_3^* - 4a_1^*)\text{sgn}((b_2^* + a_3^* - 4a_1^*)p_4(\lambda^*))\epsilon + O(\epsilon^2) \\ &= -8(b_2^* + a_3^* - 4a_1^*)\text{sgn}((b_2^* + a_3^* - 4a_1^*)p_4(\lambda^*))\epsilon + O(\epsilon^2). \end{aligned}$$

Notice that $b_2^* + a_3^* - 4a_1^* \neq 0$ when $\lambda^* \in \Lambda_{II}^1$. In fact, from $b_2^* + a_3^* - 4a_1^* \neq 0$ and $p_2(\lambda^*) = 0$, we get $a_1^* = a_3^*$ and $b_2^* = 3a_3^*$, which implies $\lambda^* \in IC_2$. Therefore,

$$p_2(\lambda^o)p_4(\lambda^*) = -8(b_2^* + a_3^* - 4a_1^*)p_4(\lambda^*)\text{sgn}((b_2^* + a_3^* - 4a_1^*)p_4(\lambda^*))\epsilon + O(\epsilon^2) < 0.$$

This completes the proof. □

Theorem 3.3. *In case $C_3 \setminus \mathcal{IC}$, at most one local critical period occurs in a perturbed system of* (1.2) *for $\lambda \in \Lambda_{III}^1$. Furthermore, there are perturbations of* (1.2), *where $\lambda \in \Lambda_{III}^1$, with exactly one critical period.*

Proof. We can obtain the first assertion directly by Lemma 2.2 in [6]. To prove the second part, by Theorem 2.1 in [6], we must prove that p_2 is independent with respect to p_4. For any $\lambda^* = (a_1^*, a_2^*, a_3^*, b_1^*, b_2^*) \in \Lambda_{III}^1$, i.e.,

$$a_3^* \neq 0, \quad b_2^* = \frac{13}{3}a_3^*, \quad 4{a_3^*}^2 - (a_2^* + 4b_1^*)^2 \geq 0, \quad a_1^* = \frac{4}{3}a_3^* \pm \frac{\sqrt{4{a_3^*}^2 - (a_2^* + 4b_1^*)^2}}{4},$$

every neighborhood of λ^* contains a point $\lambda^o = (a_1^* + \mathrm{sgn}(a_3^*)\epsilon/4, a_2^*, a_3^*, b_1^*, b_2^* + \mathrm{sgn}(a_3^*)\epsilon)$, where $\epsilon > 0$ is sufficiently small. It is easy to check that $p_0(\lambda^o) = p_0(\lambda^*) = 0$, which implies $\lambda^o \in C_3$. Moreover,

$$\begin{aligned} p_2(\lambda^o) &= \mathrm{prem}(p_2, p_0, a_1)(\lambda^o) = -48a_3^*\mathrm{sgn}(a_3^*)\epsilon, \\ p_4(\lambda^o) &= \mathrm{prem}(p_4, p_0, a_1)(\lambda^o) \\ &= 573440{a_3^*}^4 - 8192a_3^*(180a_1^*a_3^* + 45{a_2^*}^2 + 72a_2^*b_1^* - 120{a_3^*}^2)\mathrm{sgn}(a_3^*)\epsilon. \end{aligned}$$

Thus

$$p_2(\lambda^o)p_4(\lambda^o) = -27525120{a_3^*}^5\mathrm{sgn}(a_3^*)\epsilon < 0.$$

This completes the proof. □

Theorem 3.4. *In case $C_4 \setminus \mathcal{IC}$, for each $k = 1, 2, 3, 4$, at most k local critical periods occur in a perturbed system of* (1.2) *for $\lambda \in \Lambda_{IV}^k$. Furthermore, there are perturbations of* (1.2) *where $\lambda \in \Lambda_{IV}^k$, with exactly k critical periods.*

Proof. The first assertion can be obtained directly by Lemma 2.2 in [6]. Regarding the second part, by Theorem 2.1 in [6], we must prove that p_2, p_4, p_6, and p_8 are independent with respect to p_{10}.

For any $\lambda^* = (a_1^*, a_2^*, a_3^*, b_1^*, b_2^*) \in \Lambda_{IV}^4$, we have $p_2(\lambda^*) = p_4(\lambda^*) = p_6(\lambda^*) = p_8(\lambda^*) = 0$ and $p_{10}(\lambda^*) \neq 0$. From the proof of Lemma 4, we know

$$\begin{aligned} a_1^* &= \frac{60{a_3^*}^2 - 15a_3^*b_2^* + 2{b_2^*}^2}{24a_3^*}, \qquad a_2^* = \tau_{3,4}(a_3^*, b_2^*), \\ b_1^* &= \varsigma_{5,6}(a_3^*, b_2^*)\ , \qquad 270{a_3^*}^3 - 90{a_3^*}^2b_2^* + {b_2^*}^3 = 0, \quad a_3^* \neq 0, \end{aligned}$$

where $\tau_{3,4}$ and $\varsigma_{5,6}$ are defined in Lemma 4 and considered to be functions of a_3 and b_2. Choose $\lambda^o = (a_1^o, a_2^o, a_3^o, b_1^o, b_2^o)$, where

$$a_1^o = \frac{60{a_3^*}^2 - 15a_3^*(b_2^* + \mathrm{sgn}(\phi_1(\lambda^*)p_{10}(\lambda^*))\epsilon) + 2(b_2^* + \mathrm{sgn}(\phi_1(\lambda^*)p_{10}(\lambda^*))\epsilon)^2}{24a_3^*},$$

$$a_2^o = \begin{cases} \tau_{1,2}(a_3^*, b_2^* + \mathrm{sgn}(\phi_1(\lambda^*)p_{10}(\lambda^*))\epsilon), & a_3^* > 0, \\ \tau_{2,1}(a_3^*, b_2^* + \mathrm{sgn}(\phi_1(\lambda^*)p_{10}(\lambda^*))\epsilon), & a_3^* < 0, \end{cases}$$

$$a_3^o = a_3^*,$$

$$b_1^o = \begin{cases} \varsigma_{3,4}(a_3^*, b_2^* + \mathrm{sgn}(\phi_1(\lambda^*)p_{10}(\lambda^*))\epsilon), & a_3^* > 0, \\ \varsigma_{4,3}(a_3^*, b_2^* + \mathrm{sgn}(\phi_1(\lambda^*)p_{10}(\lambda^*))\epsilon), & a_3^* < 0, \end{cases}$$

$$b_2^o = b_2^* + \mathrm{sgn}(\phi_1(\lambda^*)p_{10}(\lambda^*))\epsilon,$$

$\phi_1(\lambda) = (6a_3 - b_2)(90{a_3}^2 - 15a_3b_2 - 4{b_2}^2)$, and $\epsilon > 0$ is sufficiently small. Since under the condition $270a_3^3 - 90a_3^2b_2 + b_2^3 = 0$, $\tau_{3,4}$ and $\varsigma_{5,6}$ are coincident with $\tau_{1,2}$ and $\varsigma_{3,4}$ when $a_3 > 0$, and $\tau_{3,4}$ and $\varsigma_{5,6}$ are coincident with $\tau_{2,1}$ and $\varsigma_{4,3}$ when $a_3 < 0$, then $\lambda^o = (a_1^o, a_2^o, a_3^o, b_1^o, b_2^o)$ lies in any neighborhood of λ^*. Moreover, one can check that $p_2(\lambda^o) = p_4(\lambda^o) = p_6(\lambda^o) = 0$ and

$$p_8(\lambda^o) = -4455{a_3^*}^4\phi_1(\lambda^*)\mathrm{sgn}(\phi_1(\lambda^*)p_{10}(\lambda^*))\epsilon + O(\epsilon^2).$$

One can also check that $\phi_1(\lambda^*) \neq 0$ under $270{a_3^*}^3 - 90{a_3^*}^2b_2^* + {b_2^*}^3 = 0$. From Lemma 2.1 in [6], $p_{10}(\lambda)$ is an analytic function at λ^*. Then $p_{10}(\lambda^o) = p_{10}(\lambda^*) + O(\epsilon)$. Thus

$$p_8(\lambda^o)p_{10}(\lambda^o) = -4455{a_3^*}^4\phi_1(\lambda^*)p_{10}(\lambda^*)\mathrm{sgn}(\phi_1(\lambda^*)p_{10}(\lambda^*))\epsilon + O(\epsilon^2) < 0.$$

For any $\lambda^* = (a_1^*, a_2^*, a_3^*, b_1^*, b_2^*) \in \Lambda_{IV}^3$, the discussion is divided into three parts. If λ^* is in the first set of Λ_{IV}^3, then

$$a_1^* = \frac{4}{3}a_3^*, \quad a_2^* = \pm\frac{3\sqrt{26}}{13}a_3^*, \quad b_1^* = \mp\frac{11\sqrt{26}}{78}a_3^*, \quad b_2^* = 4a_3^*, \quad a_3^* \neq 0.$$

Choose $\lambda^o = (a_1^o, a_2^o, a_3^o, b_1^o, b_2^o)$, where

$$\begin{aligned}
a_1^o &= \frac{4}{3}a_3^* + \mathrm{sgn}(a_3^*)\epsilon,\\
a_2^o &= \begin{cases} \pm 6{a_3^*}^2/\sqrt{26{a_3^*}^2 - 24a_3^*\mathrm{sgn}(a_3^*)\epsilon - 144\epsilon^2}, & a_3^* > 0,\\ \mp 6{a_3^*}^2/\sqrt{26{a_3^*}^2 - 24a_3^*\mathrm{sgn}(a_3^*)\epsilon - 144\epsilon^2}, & a_3^* < 0,\end{cases}\\
a_3^o &= a_3^*,\\
b_1^o &= \begin{cases} \mp(11{a_3^*}^2 - 6a_3^*\mathrm{sgn}(a_3^*)\epsilon - 36\epsilon^2)/3\sqrt{26{a_3^*}^2 - 24a_3^*\mathrm{sgn}(a_3^*)\epsilon - 144\epsilon^2}, & a_3^* > 0,\\ \pm(11{a_3^*}^2 - 6a_3^*\mathrm{sgn}(a_3^*)\epsilon - 36\epsilon^2)/3\sqrt{26{a_3^*}^2 - 24a_3^*\mathrm{sgn}(a_3^*)\epsilon - 144\epsilon^2}, & a_3^* < 0,\end{cases}\\
b_2^o &= 4a_3^*,
\end{aligned}$$

and $\epsilon > 0$ is sufficiently small. One can check that $\lambda^o = (a_1^o, a_2^o, a_3^o, b_1^o, b_2^o)$ lies in any neighborhood of λ^*. Moreover, we calculate $p_2(\lambda^o) = p_4(\lambda^o) = 0$,

$$p_6(\lambda^o) = -18144{a_3^*}^5\mathrm{sgn}(a_3)\epsilon + O(\epsilon^2), \quad \text{and} \quad p_8(\lambda^*) = 115830{a_3^*}^8.$$

Since $a_3^* \neq 0$, then $p_6(\lambda^o)p_8(\lambda^*) < 0$. If λ^* is in the second, third, or fourth set of Λ_{IV}^3, then $b_2^* \neq 4a_3^*$, $a_2^* = 0$, $p_2(\lambda^*) = p_4(\lambda^*) = p_6(\lambda^*) = 0$, and $p_8(\lambda^*) \neq 0$. From the proof of Lemma 4, we know

$$\begin{aligned}
&a_1^* = -\frac{w_2(0, a_3^*, \varsigma_{1,2}(0, a_3^*, b_2^*), b_2^*)}{w_1(0, a_3^*, \varsigma_{1,2}(0, a_3^*, b_2^*), b_2^*)}, \qquad a_2^* = 0, \qquad b_1^* = \varsigma_{1,2}(0, a_3^*, b_2^*),\\
&b_2^* = 6a_3^*, \quad \frac{23+\sqrt{97}}{4}a_3^* \ \text{ or } \ \frac{23-\sqrt{97}}{4}a_3^*, \qquad a_3^* \neq 0,
\end{aligned}$$

where $\varsigma_{1,2}$ are considered to be functions of a_2, a_3, b_2, and w_1, w_2 are defined in (2.23) and are considered to be functions of a_2, a_3, b_1, b_2. Choose $\lambda^o = (a_1^o, a_2^o, a_3^o, b_1^o, b_2^o)$, where

$$\begin{aligned}
a_1^o &= -\frac{w_2(0, a_3^*, \varsigma_{1,2}(0, a_3^*, b_2^* - \mathrm{sgn}(a_3^*\phi_2(\lambda^*)p_8(\lambda^*))\epsilon), b_2^* - \mathrm{sgn}(a_3^*\phi_2(\lambda^*)p_8(\lambda^*))\epsilon)}{w_1(0, a_3^*, \varsigma_{1,2}(0, a_3^*, b_2^* - \mathrm{sgn}(a_3^*\phi_2(\lambda^*)p_8(\lambda^*))\epsilon), b_2^* - \mathrm{sgn}(a_3^*\phi_2(\lambda^*)p_8(\lambda^*))\epsilon)},\\
a_2^o &= 0,\\
a_3^o &= a_3^*,\\
b_1^o &= \varsigma_{1,2}(0, a_3^*, b_2^* - \mathrm{sgn}(a_3^*\phi_2(\lambda^*)p_8(\lambda^*))\epsilon),\\
b_2^o &= b_2^* - \mathrm{sgn}(a_3^*\phi_2(\lambda^*)p_8(\lambda^*))\epsilon,
\end{aligned}$$

$\phi_2(\lambda) = 876a_3^4 - 792a_3^3b_2 + 263a_3^2b_2^2 - 38a_3b_2^3 + 2b_2^4$, and $\epsilon > 0$ is sufficiently small. One can check that $\lambda^o = (a_1^o, a_2^o, a_3^o, b_1^o, b_2^o)$ lies in any neighborhood of λ^*. Moreover, a short calculation reveals that $p_2(\lambda^o) = p_4(\lambda^o) = 0$, and

$$p_6(\lambda^o) = -\frac{2268a_3^3\phi_2(\lambda^*)}{(4a_3 - b_2)^2}\mathrm{sgn}(a_3^*\phi_2(\lambda^*)p_8(\lambda^*))\epsilon + O(\epsilon^2).$$

It is easy to verify that $\phi_2(\lambda^*) \neq 0$, as $b_2^* = 6a_3^*$ or $(23/4 \pm \sqrt{97}/4)a_3^*$ and $a_3^* \neq 0$. Thus $p_6(\lambda^o)p_8(\lambda^*) < 0$. If λ^* is in the last set of Λ_{IV}^3, then $b_2^* \neq 4a_3^*$, $a_2^* \neq 0$, $p_2(\lambda^*) = p_4(\lambda^*) = p_6(\lambda^*) = 0$, and $p_8(\lambda^*) \neq 0$. From the proof of Lemma 4, we know that

$$\Omega(a_3^*, b_2^*) < 0, \qquad a_1^* = -\frac{w_2(\tau_{1,2}(a_3^*, b_2^*), a_3^*, \varsigma_{3,4}(a_3^*, b_2^*), b_2^*)}{w_1(\tau_{1,2}(a_3^*, b_2^*), a_3^*, \varsigma_{3,4}(a_3^*, b_2^*), b_2^*)}, \qquad a_2^* = \tau_{1,2}(a_3^*, b_2^*),$$

$$b_1^* = \varsigma_{3,4}(a_3^*, b_2^*), \qquad b_2^* \neq 3a_3^*,\ 4a_3^*,\ 6a_3^*,\ \frac{23 \pm \sqrt{97}}{4}a_3^*, \qquad a_3^* \neq 0,$$

where Ω, $\tau_{1,2}$, and $\varsigma_{3,4}$ are considered to be functions of a_3, b_2, and w_1, w_2 are defined in (2.23) and considered to be functions of a_2, a_3, b_1, b_2. Choose $\lambda^o = (a_1^o, a_2^o, a_3^o, b_1^o, b_2^o)$, where

$$a_1^o = \begin{cases} -(w_2(\varrho_{1,2}, a_3^*, \varsigma_{1,2}(\varrho_{1,2}, a_3^*, b_2^*), b_2^*))/(w_1(\varrho_{1,2}, a_3^*, \varsigma_{1,2}(\varrho_{1,2}, a_3^*, b_2^*), b_2^*)), & E_3(a_3^*, b_2^*) \geq 0, \\ -(w_2(\varrho_{1,2}, a_3^*, \varsigma_{2,1}(\varrho_{1,2}, a_3^*, b_2^*), b_2^*))/(w_1(\varrho_{1,2}, a_3^*, \varsigma_{2,1}(\varrho_{1,2}, a_3^*, b_2^*), b_2^*)), & E_3(a_3^*, b_2^*) < 0, \end{cases}$$

$$a_2^o = \varrho_{1,2},$$
$$a_3^o = a_3^*,$$
$$b_1^o = \begin{cases} \varsigma_{1,2}(\varrho_{1,2}, a_3^*, b_2^*), & E_3(a_3^*, b_2^*) \geq 0, \\ \varsigma_{2,1}(\varrho_{1,2}, a_3^*, b_2^*), & E_3(a_3^*, b_2^*) < 0, \end{cases}$$
$$b_2^o = b_2^*,$$
$$\varrho_{1,2} = \pm\frac{(54{a_3^*}^2 - 23a_3^*b_2^* + 2{b_2^*}^2)(6a_3^* - b_2^*) + (4a_3^* - b_2^*)\mathrm{sgn}(\phi_3(\lambda^*)p_8(\lambda^*))\epsilon}{\sqrt{-\Omega(a_3^*, b_2^*) - 2(6a_3^* - b_2^*)(9a_3^* - 2b_2^*)\mathrm{sgn}(\phi_3(\lambda^*)p_8(\lambda^*))\epsilon - \epsilon^2}},$$

$\phi_3(\lambda) = a_3(3a_3 - b_2)(4a_3 - b_2)$, and $\epsilon > 0$ is sufficiently small. Since $\varsigma_{5,6}$ are coincident with $\varsigma_{3,4}$ when $E_3 \geq 0$, and $\varsigma_{5,6}$ are coincident with $\varsigma_{4,3}$ when $E_3 < 0$, then $\lambda^o = (a_1^o, a_2^o, a_3^o, b_1^o, b_2^o)$ lies in any neighborhood of λ^*. Moreover, we calculate $p_2(\lambda^o) = p_4(\lambda^o) = 0$. Since under $p_2 = p_4 = 0$ and $\mathrm{lcoeff}(\mathrm{prem}(p_6, p_2a_1), a_1) = -737280a_3^2(4a_3 - b_2) \neq 0$, we get

$$\begin{aligned} p_6 &= \mathrm{prem}(\mathrm{prem}(p_6, p_2, a_1), \mathrm{prem}(p_6, p_2, a_1), a_1) \\ &= -584459149639680a_3^5(3a_3 - b_2)\omega(a_2, a_3, b_1, b_2), \end{aligned}$$

where $\omega(a_2, a_3, b_1, b_2) = 6a_2^2a_3 + 24a_2a_3b_1 - 324a_3^3 + 192a_3^2b_2 - 35a_3b_2^2 + 2b_2^3$. Consider $\omega(a_2, a_3^*, \varsigma_{1,2}(a_2, a_3^*, b_2^*), b_2^*) = (4a_3^* - b_2^*)\epsilon$ as the equation of a_2. Solving for a_2 from this equation, we get $a_2 = \varrho_{1,2}$. Then

$$\begin{aligned} p_6(\lambda^o) &= -584459149639680{a_3^o}^5(3a_3^o - b_2^o)\omega(a_2^o, a_3^o, b_1^o, b_2^o) \\ &= -584459149639680{a_3^*}^4\phi_3(\lambda^*)\mathrm{sgn}(\phi_3(\lambda^*)p_8(\lambda^*))\epsilon. \end{aligned}$$

It is easy to check that $\phi_3(\lambda^*) \neq 0$, since $b_2^* \neq 3a_3^*$, $4a_3^*$, and $a_3^* \neq 0$. Thus $p_6(\lambda^o)p_8(\lambda^*) < 0$.

For any $\lambda^* = (a_1^*, a_2^*, a_3^*, b_1^*, b_2^*) \in \Lambda_{IV}^2$, we have $p_2(\lambda^*) = p_4(\lambda^*) = 0$ and $p_6(\lambda^*) \neq 0$. From Theorem 4, we know

$$\Delta(a_2^*, a_3^*, b_1^*, b_2^*) \leq 0, \quad a_1^* = \sigma_1(a_2^*, a_3^*, b_1^*, b_2^*) \ \text{ or } \ \sigma_2(a_2^*, a_3^*, b_1^*, b_2^*), \quad a_3^* \neq 0.$$

We prove the independence in two cases: $a_2^* \neq -4b_1^*$ and $a_2^* = -4b_1^*$. If $a_2^* \neq -4b_1^*$, then every neighborhood of λ^* contains a point $\lambda^o = (a_1^*, a_2^* - 4\mathrm{sgn}((a_2^* +$

$4b_1^*)p_6(\lambda^*))\epsilon, a_3^o, b_1^o + \text{sgn}((a_2^* + 4b_1^*)p_6(\lambda^*))\epsilon, b_2^o)$, where $\epsilon > 0$ is sufficiently small. One can check that $p_2(\lambda^o) = 0$ and

$$p_4(\lambda^o) = -737280{a_3^*}^2(a_2^* + 4b_1^*)\text{sgn}((a_2^* + 4b_1^*)p_6(\lambda^*))\epsilon.$$

Thus $p_4(\lambda^0)p_6(\lambda^*) < 0$. If $a_2^* = -4b_1^*$, then every neighborhood of λ^* contains a point $\lambda^o = (a_1^o, a_2^o, a_3^o, b_1^o, b_2^o)$, where

$$\begin{aligned} a_1^o &= a_1^* + \frac{\text{sgn}(\phi_4(\lambda^*)p_6(\lambda^*))\epsilon}{4} + \frac{6(a_3^* - b_2^*)\text{sgn}(\phi_4(\lambda^*)p_6(\lambda^*))\epsilon + 3\epsilon^2}{4(a_3^* + \text{sgn}(\phi_4(\lambda^*)p_6(\lambda^*))\epsilon)}, \\ a_2^o &= a_2^*, \\ a_3^o &= a_3^* + \text{sgn}(\phi_4(\lambda^*)p_6(\lambda^*))\epsilon, \\ b_1^o &= b_1^*, \\ b_2^o &= b_2^* + \frac{6(a_3^* - b_2^*)\text{sgn}(\phi_4(\lambda^*)p_6(\lambda^*))\epsilon + 3\epsilon^2}{a_3^* + \text{sgn}(\phi_4(\lambda^*)p_6(\lambda^*))\epsilon}, \end{aligned}$$

$$\phi_4(\lambda) = \begin{cases} a_3(6a_3(3a_3 - b_2) + (6a_3 - b_2)\sqrt{-3a_3(3a_3 - b_2)}), & a_1 = \sigma_1, \\ a_3(6a_3(3a_3 - b_2) - (6a_3 - b_2)\sqrt{-3a_3(3a_3 - b_2)}), & a_1 = \sigma_2, \end{cases}$$

and $\epsilon > 0$ is sufficiently small. One can check that $p_2(\lambda^o) = 0$ and

$$p_4(\lambda^o) = -184320\phi_4(\lambda^*)\text{sgn}(\phi_4(\lambda^*)p_6(\lambda^*))\epsilon.$$

Moreover, it is easy to find that $\phi_4(\lambda^*) = 0$ and $a_3^* \neq 0$ only if $b_2^* = 3a_3^*$ or $b_2^* = 6(2 \pm \sqrt{2})a_3^*$. When $b_2^* = 3a_3^*$, under $a_2^* = -4b_1^*$ and $p_2(\lambda^*) = 0$, we get $a_1^* = a^*$, which implies $\lambda^* \in IC_2$ and is contradictory with $p_6(\lambda^*) \neq 0$. When $b_2^* = 6(2 \pm \sqrt{2})a_3^*$, under $a_2^* = -4b_1^*$, $a_3 \neq 0$, and $p_2(\lambda^*) = 0$ we get $p_4(\lambda^*) \neq 0$. Then we know that $\phi_4(\lambda^*) \neq 0$ and $p_4(\lambda^o)p_6(\lambda^*) < 0$.

For any $\lambda^* = (a_1^*, a_2^*, a_3^*, b_1^*, b_2^*) \in \Lambda_{IV}^1$, we have $p_2(\lambda^*) = 0$ and $p_4(\lambda^*) \neq 0$. We also consider the cases $a_2^* \neq -4b_1^*$ and $a_2^* = -4b_1^*$. If $a_2^* \neq -4b_1^*$, then every neighborhood of λ^* contains a point $\lambda^o = (a_1^*, a_2^* - \text{sgn}((a_2^* + 4b_1^*)p_4(\lambda^*))\epsilon, a_3^*, b_1^*, b_2^*)$, where $\epsilon > 0$ is sufficiently small. One can check that

$$p_2(\lambda^o) = -2(a_2^* + 4b_1^*)\text{sgn}((a_2^* + 4b_1^*)p_4(\lambda^*))\epsilon + \epsilon^2.$$

Then $p_2(\lambda^o)p_4(\lambda^*) < 0$. If $a_2^* = -4b_1^*$, then every neighborhood of λ^* contains a point $\lambda^o = (a_1^* - \text{sgn}(\phi_5(\lambda^*)p_4(\lambda^*))\epsilon, a_2^*, a_3^*, b_1^*, b_2^*)$, where $\phi_5(\lambda) = 4a_1 - b_2 - a_3$ and $\epsilon > 0$ is sufficiently small. One can check that

$$p_2(\lambda^o) = -8\phi_5(\lambda^*)\text{sgn}(\phi_5(\lambda^*)p_4(\lambda^*))\epsilon + 16\epsilon^2.$$

Moreover, solving $\phi_5(\lambda^*) = 0$, $a_2^* = -4b_1^*$, $p_2(\lambda^*) = 0$, and $a_3 \neq 0$, we get $\lambda^* \in IC_2$, which is contradictory with $p_4(\lambda^*) \neq 0$. Then $\phi_5(\lambda^*) \neq 0$ and $p_2(\lambda^o)p_4(\lambda^*) < 0$. This completes the proof. □

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References

[1] X. Chen, V.G. Romanovski and W. Zhang, *Critical periods of perturbations of reversible rigidly isochronous centers*, J. Differential Equations, 2011, 251, 1505–1525.

[2] X. Chen and W. Zhang, *Decomposition of algebraic sets and applications to weak centers of cubic systems*, J. Comput. Appl. Math., 2009, 232, 565–581.

[3] C. Chicone, *The monotonicity of the period function for planar Hamiltonian vector fields*, J. Differential Equations, 1987, 69 (3), 310–321.

[4] C. Chicone and F. Dumortier, *A quadratic system with a nonmonotonic period function*, Proc. Amer. Math. Soc., 1988, 102 (3), 706–710.

[5] S.-N. Chow and D. Wang, *On the monotonicity of the period function of some second order equations*, Časopis Pěst. Mat., 1986, 111 (1), 14–25.

[6] C. Chicone and M. Jacobs, *Bifurcation of critical periods for plane vector fields*, Trans. Amer. Math. Soc., 1989, 312, 433–486.

[7] B. Ferčec, V. Levandovskyyb, V. G. Romanovski and D. S. Shafer, *Bifurcation of critical periods of polynomial systems*, J. Differential Equations, 2015, 259, 3825–3853.

[8] A. Gasull and R. Prohens, *Quadratic and cubic systems with degenerate infinity*, J. Math. Anal. Appl., 1996, 198, 25–34.

[9] D. E. Knuth, *The Art of Computer Programming, Vol. 2/ Seminumerical Algorithms*, Addison-Wesley, Reading-London-Amsterdam, 1969.

[10] N. Li and M. Han, *Critical period bifurcation by perturbing a reversible rigidly isochronous center with multiple parameters*, Internat. J. Bifur. Chaos, 2015, 25 (5), 11 pages (1550070).

[11] C. Liu and M. Han, *Bifurcation of critical periods from the reversible rigidly isochronous centers*, Nonlinear Anal., 2014, 95, 388–403.

[12] N. G. Lloyd, C. J. Christopher, J. Devlin, J. M. Pearson and N. Yasmin, *Quadratic-like cubic systems*, Diff. Eqn. Dynam. Systems, 1997, 5, 329–345.

[13] J. M. Pearson, N. G. Lloyd and C. J. Christopher, *Algorithmic derivation of centre conditions*, SIAM Rev., 1996, 38, 619–636.

[14] L. Peng and Z. Feng, *Bifurcation of critical periods from a quartic isochronous center*, Internat. J. Bifur. Chaos, 2014, 24 (6), 16 pages (1450089).

[15] C. Rousseau and B. Toni, *Local bifurcation of critical periods in vector fields with homogeneous nonlinearities of the third degree*, Canad. Math. Bull., 1993, 36, 473–484.

[16] C. Rousseau and B. Toni, *Local bifurcation of critical periods in the reduced Kukles system*, Canad. Math. Bull., 1997, 49, 338–358.

[17] J. Sotomayor, *Curvas Definidas por Equacoes Diferenciais no Plano*, IMPA, Rio de Janeiro., 1981.

[18] J. Villadelprat, *Bifurcation of local critical periods in the generalized Loud's system*, Appl. Math. Comput., 2012, 218, 6803–6813.

[19] Z. Wang, X. Chen and W. Zhang, *Local Bifurcations of Critical Periods in a Generalized 2-D LV System*, Appl. Math. Comput., 2009, 214, 17–25.

[20] P. Yu, M. Han and J. Zhang, *Critical periods of third-order planar Hamiltonian systems*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 2010, 20, 2213–2224.

[21] W. Zhang, X. Hou and Z. Zeng, *Weak center and bifurcation of critical periods in reversible cubic systems*, Comput. Math. Appl., 2000, 40, 771–782.

[22] L. Zou, X. Chen and W. Zhang, *Local bifurctions of critical periods for cubic Liénard equations with cubic damping*, J. Comput. Appl. Math., 2008, 222, 404–410.