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ON THE STRONG CONVERGENCE OF A PROJECTION-BASED ALGORITHM IN HILBERT SPACES

Liya Liu[1] and Xiaolong Qin[2,†]

Abstract In this paper, we introduce a new projection-based algorithm for solving variational inequality problems with a Lipschitz continuous pseudo-monotone mapping in Hilbert spaces. We prove a strong convergence of the generated sequences. The numerical behaviors of the proposed algorithm on test problems are illustrated and compared with previously known algorithms.

Keywords Strong convergence, variational inequality problem, inertial method, pseudo-monotone mapping.

MSC(2010) 49J40, 65K10, 90C33, 47J20.

1. Introduction

Throughout the paper, we always assume that H is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$ and C is a nonempty, convex and closed subset of H. Let $F : C \to H$ be a mapping. Recall that the classical variational inequality problem (VIP, for brevity) is to find $y \in C$ such that

$$\langle F(y), x - y\rangle \geq 0, \ \ \forall x \in C. \tag{1.1}$$

In this paper, we denote the solution set of the variational inequality problem by $VI(C, F)$. The variational inequality problem, serves as a powerful mathematical model, which unifies important concepts in applied mathematics, as special cases, complementarity problems, systems of nonlinear equations or equilibrium problems arising in several branches of applied sciences under a unified framework, see [9, 19, 32, 35] and references therein. Recently, much attention has been given to develop efficient and implementable numerical methods for solving variational inequality problems and related optimization problems, for instance, [3, 9, 10, 13, 14, 33] and references therein. In many cases, one considers the problem $VI(C, F)$ with some additional properties imposed on mapping F. Let us recall some related definitions here. A mapping $F : H \to H$ is said to be

(i) sequentially weakly continuous if for each sequence $\{x_n\}$, we have that $\{x_n\}$ converges weakly to x implies $F(x_n)$ converges weakly to Fx;

(ii) L-Lipschitz continuous iff there exists a positive constant $L > 0$ such that

$$\|Fx' - Fx\| \leq L\|x' - x\|, \ \ \forall x', x \in C;$$

[†]The corresponding author. Email address: qxlxajh@163.com (X. Qin)
[1]Institute of Fundamental and Frontier Sciences, University of Electronic Science and Technology of China, Chengdu, China
[2]Department of Mathematics, Zhejiang Normal University, Jinhua, China

(iii) monotone iff

$$\langle x^{'} - x, Fx^{'} - Fx\rangle \geq 0, \quad \forall x^{'}, x \in C;$$

(iv) pseudo-monotone iff

$$\langle Fx^{'}, x - x^{'}\rangle \geq 0 \Rightarrow \langle Fx, x - x^{'}\rangle \geq 0, \quad \forall x^{'}, x \in C.$$

Now we recall the definition of the projection operator. For any $y \in H$, there exists a unique point in C, denoted by $P_C(u)$, such that $\|y - P_C(u)\| \leq \|y - x\|, \ \forall x \in C$. The projection operator can be characterized by the following two properties

(i) $\langle P_C y - y, x - P_C y\rangle \geq 0, \ \forall x \in C$;

(ii) $\|P_C y - x\|^2 \leq \|y - x\|^2 - \|P_C y - y\|^2, \ \forall x \in C$.

It is known that x solves the $VI(C, F)$ if and only if x is an equilibrium point of the dynamical system, i.e.,

$$x = P_C(x - \beta Fx), \ \ \beta > 0.$$

A significant body of work on iteration methods for VIPs has accumulated in literature recently. Specifically, the so-called extragradient algorithm was proposed in 1976 by Korpelevich for solving saddle point problems and then extended to VIPs [22]. The algorithm takes the following form, for any $x_0 \in H$ and

$$\begin{cases} y_n = P_C(x_n - \alpha F(x_n)), \\ x_{n+1} = P_C(x_n - \alpha F(y_n)), n \geq 1, \end{cases} \tag{1.2}$$

where $F : C \to H$ is a monotone and L-Lipschitz continuous mapping and $\alpha \in \left(0, \frac{1}{L}\right)$. The iterative sequence $\{x_n\}$ converges to some point in $VI(C, F)$. Recently, this method was extensively analyzed and further studied; sees [11,15,24,26,36] and references therein. Later, Censor, Gibali, and Reich [16] proposed a subgradient extragradient algorithm in an Euclidean space. Starting with any point $x_0 \in H$, they defined a sequence $\{x_n\}_{n\geq 0}$ as

$$\begin{cases} y_n = P_C(x_n - \alpha F(x_n)), \\ T_n = \{w \in H : \langle x_n - \alpha F(x_n) - y_n, w - y_n\rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \alpha F(y_n)), n \geq 1, \end{cases} \tag{1.3}$$

where $F : C \to H$ is a monotone and L-Lipschitz continuous mapping and $\alpha \in \left(0, \frac{1}{L}\right)$ is a step-size parameter. In the context of algorithm (1.3), we can see that it is calculated only one projection onto a specific constructible half-space T_n, but not onto the general convex set C like in algorithm (1.2), on the second step. This is actually one of the subgradient half-spaces. Basically, the subgradient extragradient method is more applicable when a projection onto the general convex set C is a nontrivial problem. In general, this method is only known to be weakly convergent in the setting of infinite dimensional Hilbert spaces. Therefore, the natural question that arises is how to construct an algorithm which generates a strong convergent sequence in the framework of infinite dimensional Hilbert space. Aiming at this, an alternative modification to subgradient extragradient method (1.3) is the following algorithm, which was also proposed by Censor, Gibali and Reich in [17]. Given any point $x_0 \in H$,

$$\begin{cases} y_n = P_C(x_n - \alpha F(x_n)), \\ T_n = \{w \in H : \langle x_n - \alpha F(x_n) - y_n, w - y_n \rangle \leq 0\}, \\ z_n = (1 - \lambda_n) P_{T_n}(x_n - \alpha F(y_n)) + \lambda_n x_n, \\ C_n = \{w \in C : \|z_n - w\| \leq \|x_n - w\|\}, \\ Q_n = \{w \in C : \langle x_n - w, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{1.4}$$

where $F : C \to H$ is a monotone and L-Lipschitz continuous mapping, $\alpha \in \left(0, \frac{1}{L}\right)$ and $0 \leq \lambda_n \leq \alpha < 1$. They proved that the sequence $\{x_n\}$ generated by (1.4) converges strongly to $P_{VI(C,F)}(x_0)$.

In recent years, there have been increasing interests in studying inertial type algorithms, which were first proposed by Polyak [31], as an acceleration process to solve smooth convex minimization problems; see [1, 5, 12, 27]. Inertial type algorithms, which are two-step iterative and the second iterative step is defined by using previous two iterates. They are based on the heavy ball method of the two-order time dynamical system. Recently, some authors constructed fast iterative algorithms by using the inertial extrapolation, including inertial proximal algorithms, inertial forward-backward splitting algorithms, inertial Mann algorithms and inertial subgradient extragradient algorithms; see, e.g., [2, 4, 6, 7, 25, 28, 34] and the references therein.

In this paper, inspired and motivated by the mentioned works in literature and the ongoing research in these directions, we propose an inertial projection-based algorithm for solving pseudomonotone variational inequality problems. This algorithm is based on inertial ideas and hybrid gradient ideas. We also perform several numerical examples to support the convergence of the algorithm presented in this paper. This illustrates the numerical behaviors of our algorithm and compares them with the algorithms in [20, 29].

In order to prove our main result, we need the following two lemmas.

Lemma 1.1 (Minty lemma [18]). *Consider problem $VI(C, F)$ with C a nonempty, closed, convex subset of a real Hilbert space H and $F : C \to H$ pseudo-monotone and continuous. Then, $\hat{x}$ is a solution of $VI(C, F)$ if and only if $\langle x - \hat{x}, F(x) \rangle \geq 0, \forall x \in C$.*

Lemma 1.2 (Kadec-Klee property [8]). *Let $\{x_n\}$ be a sequence in H. If $\|x_n\| \to \|x\|$ and $x_n \rightharpoonup x$ as $n \to \infty$, then $x_n \to x$ as $n \to \infty$.*

2. The algorithm and its convergence

The following assumptions will be used through the rest of this paper.

(a) The feasible set C is a nonempty, closed, convex subset of the real Hilbert space H.

(b) The underline mapping $F : H \to H$ is pseudo-monotone, L-Lipschitz continuous and sequentially weakly continuous on bounded subsets of H.

(c) The solution set $VI(C, F)$ is nonempty.

Here we are in a position to design our algorithm, see Figure 1 for a further description.

Algorithm 2.1. (Inertial Hybrid Gradient Algorithm)

Initialization: Let $x_0, x_1 \in C$ be arbitrary initial points and let $\{\alpha_n\} \in (0, +\infty)$ and $\beta_n \in (0, a)$, where $a = \min\left\{1, \frac{1}{2L^2}\right\}$, be two real sequences. Assume that $\{\beta_n\}$ also satisfies $0 < \liminf_{n\to\infty} \beta_n$. Set $C_0 = H$, $Q_0 = H$.

Step 0: Set $n = 1$.

Step 1: Given the current iterates x_{n-1} and x_n, compute

$$\begin{aligned} y_n &= x_n + \alpha_n(x_n - x_{n-1}), \\ z_n &= P_C(x_n - \beta_n F y_n). \end{aligned} \tag{2.1}$$

Step 2: If $x_{n-1} = x_n = z_n$ or $F(y_n) = 0$, then stop. Otherwise, construct sets C_n and Q_n as

$$\begin{aligned} C_n &= \{w \in H : \langle x_n - z_n - \beta_n F y_n + \beta_n F z_n, z_n - w\rangle \geq 0\}, \\ Q_n &= \{w \in H : \langle x_n - w, x_0 - x_n\rangle \geq 0\},\ n \geq 1, \end{aligned} \tag{2.2}$$

and calculate

$$x_{n+1} = P_{C_n \cap Q_n} x_0. \tag{2.3}$$

Step 3: Replace n by $n + 1$; go to **step 1**.

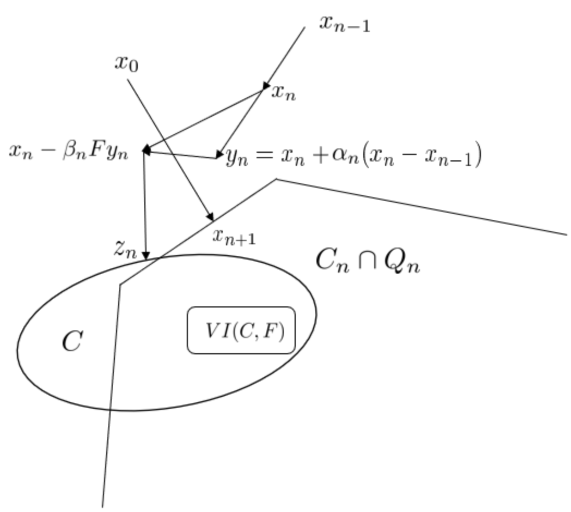

Figure 1. Iterative steps of Algorithm 2.1. The number of the projection onto the feasible set C is 1.

Sets C_n look slightly complicated in contrast to (1.4). However, it is only for a superficial examination, for a computation, it does not matter. In order to prove our main result, we give the following remark.

Remark 2.1. If $x_{n-1} = x_n = z_n$ in Algorithm 2.1, then $x_n \in VI(C, F)$.

Proof. Since $x_{n-1} = x_n = z_n$, it follows from (2.1) that

$$y_n = x_n + \alpha_n(x_n - x_{n-1}) = x_n.$$

According to projection characterization (i), we have

$$\langle \omega - z_n, z_n - (x_n - \beta_n F y_n)\rangle \geq 0, \ \ \forall \omega \in C,$$

which yields that $\beta_n \langle F x_n, \omega - x_n\rangle \geq 0, \ \forall \omega \in C$. Invoking $\beta_n > 0$, we find that $x_n \in VI(C, F)$. □

If $x_{n-1} = x_n = z_n$, we are at a solution of this variational inequality problem. In our convergence analysis, we will implicitly assume that this does not occur after finitely many iterations, so that Algorithm 2.1 generates three infinite sequences.

Theorem 2.1. *Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be three sequences generated by Algorithm 2.1. If assumptions (a), (b) and (c) hold, then the three sequences converge strongly to $z = P_{VI(C,F)}x_0$.*

Proof. Let $\omega \in VI(C, F)$. Invoking $z_n \in C$, we find that $\langle F\omega, z_n - \omega\rangle \geq 0$. In view of the pseudo-monotonicity of F and $\beta_n \geq 0$, we obtain that

$$\langle \beta_n F z_n, z_n - \omega\rangle \geq 0. \tag{2.4}$$

Due to (2.1), projection characterization (i) and $\omega \in C$, we have

$$\langle \omega - z_n, z_n - (x_n - \beta_n F y_n)\rangle \geq 0. \tag{2.5}$$

By combining (2.4) with (2.5), it further implies that

$$\langle z_n - \omega, x_n - z_n - \beta_n F y_n + \beta_n F z_n\rangle \geq 0. \tag{2.6}$$

It is evident that sets C_n and Q_n are closed and convex. Coming back to (2.6), we have that $VI(C, F) \subseteq C_n, \ \forall n \in \mathbb{N}$. Let us show by the mathematical induction that $VI(C, F) \subseteq Q_n$ for all $n \in N$. Recalling that $Q_0 = H$, it is obvious that $VI(C, F) \subseteq Q_0$ when $n = 0$. Suppose that $VI(C, F) \subseteq Q_n$. It is sufficient to prove that $VI(C, F) \subseteq Q_{n+1}$. Since $VI(C, F) \subseteq C_n \cap Q_n$ and $x_{n+1} = P_{C_n \cap Q_n} x_0$, we conclude that $\langle x_{n+1} - \omega, x_0 - x_{n+1}\rangle \geq 0, \forall \omega \in VI(C, F)$. Combining this with the definition of Q_n, we find that $\omega \in Q_{n+1}$. Since ω is chosen arbitrarily in $VI(C, F)$, we have that $VI(C, F) \subseteq Q_{n+1}$ and hence $VI(C, F) \subseteq C_n \cap Q_n$ for all $n \in \mathbb{N}$. So $\{x_n\}$ is well defined. Denote $z = P_{VI(C,F)}x_0$. Since $x_{n+1} = P_{C_n \cap Q_n} x_0$ and $z \in VI(C, F) \subseteq C_n \cap Q_n$, we have

$$\|z - x_0\| \geq \|x_{n+1} - x_0\|.$$

This implies that $\{x_n\}$ is a bounded sequence. Invoking (2.3) and substituting $w = x_{n+1}$ into C_n, one gets that

$$\langle x_n - z_n - \beta_n F y_n + \beta_n F z_n, z_n - x_{n+1}\rangle \geq 0,$$

which further implies that

$$\begin{aligned}
&\|z_n - x_{n+1}\|^2\\
&\le \langle z_n - x_{n+1}, x_n - x_{n+1} + \beta_n F z_n - \beta_n F y_n\rangle\\
&\le \langle z_n - x_{n+1}, x_n - x_{n+1}\rangle - \beta_n\langle z_n - x_{n+1}, F y_n - F z_n\rangle\\
&\le \frac{1}{2}(\|x_n - x_{n+1}\|^2 + \|x_{n+1} - z_n\|^2 - \|z_n - x_n\|^2) + \frac{\beta_n}{2}(L^2\|y_n - z_n\|^2 + \|z_n - x_{n+1}\|^2).
\end{aligned}$$

This is equivalent to

$$\begin{aligned}
&\|z_n - x_{n+1}\|^2\\
&\le \|x_n - x_{n+1}\|^2 - \|z_n - x_n\|^2 + \beta_n L^2\|y_n - z_n\|^2 + \beta_n\|z_n - x_{n+1}\|^2\\
&= \|x_n - x_{n+1}\|^2 - \|z_n - x_n\|^2 + \beta_n L^2\|x_n + \alpha_n(x_n - x_{n-1}) - z_n\|^2 + \beta_n\|z_n - x_{n+1}\|^2\\
&\le \|x_n - x_{n+1}\|^2 - \|z_n - x_n\|^2 + \beta_n\|z_n - x_{n+1}\|^2 + 2\beta_n L^2(\|z_n - x_n\|^2 + \|\alpha_n(x_{n-1} - x_n)\|^2)\\
&\le \|x_n - x_{n+1}\|^2 + (2\beta_n L^2 - 1)\|z_n - x_n\|^2 + 2\beta_n\alpha_n^2 L^2\|x_{n-1} - x_n\|^2 + \beta_n\|z_n - x_{n+1}\|^2.
\end{aligned}$$

By virtue of $\beta_n \in (0, a)$, where $a = \min\{\frac{1}{2L^2}, 1\}$, the above inequality implies that

$$\begin{aligned}
&(1 - 2\beta_n L^2)\|z_n - x_n\|^2\\
&\le \|x_{n+1} - x_n\|^2 + (\beta_n - 1)\|z_n - x_{n+1}\|^2 + 2\beta_n\alpha_n^2 L^2\|x_{n-1} - x_n\|^2\\
&\le \|x_{n+1} - x_n\|^2 + \alpha_n^2\|x_{n-1} - x_n\|^2.
\end{aligned} \tag{2.7}$$

It is immediately from (2.3) that $x_{n+1} \in C_n \cap Q_n \subseteq Q_n$, which, together with $x_n = P_{Q_n} x_0$, yields that

$$\|x_n - x_0\| \le \|x_{n+1} - x_0\|. \tag{2.8}$$

Combining this with (2.8), we obtain that $\lim_{n\to\infty} \|x_n - x_0\|$ exists. In addition, by putting together $x_n = P_{Q_n} x_0$ and $x_{n+1} \in Q_n$, it derives that

$$\|x_n - x_{n+1}\|^2 \le \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2.$$

Since $\lim_{n\to\infty} \|x_n - x_0\|$ exists, we arrive at

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{2.9}$$

In fact, by combining (2.7) with (2.9), we find that

$$(1 - 2\beta_n L^2)\|x_n - z_n\| \le \|x_n - x_{n+1}\|^2 + \alpha_n^2\|x_n - x_{n-1}\|^2 \to 0, \text{ as } n \to \infty. \tag{2.10}$$

Due to $\beta_n \in \left(0, \frac{1}{2L^2}\right)$, we have $1 - 2\beta_n L^2 > 0$. Invoking (2.10), we also have

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{2.11}$$

Indeed, by taking account of (2.1) and (2.9), we can see that

$$\lim_{n\to\infty} \alpha_n\|x_n - x_{n-1}\| = \lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{2.12}$$

It is immediately from (2.11) and (2.12) that

$$\lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{2.13}$$

According to the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to some $\hat{x} \in H$. From (2.11) and (2.12), we can see that both $\{y_{n_i}\}$ and $\{z_{n_i}\}$ also weakly converge to $\hat{x}$. We now show that $\hat{x} \in VI(C, F)$.

Indeed, from $z_n = P_C(x_n - \beta_n F y_n)$ and projection characterization (i), we conclude that

$$\langle z_{n_i} - (x_{n_i} - \beta_{n_i} F y_{n_i}), \omega - z_{n_i} \rangle \geq 0, \quad \forall \omega \in C,$$

which guarantees that

$$\langle F y_{n_i}, \omega - z_{n_i} \rangle \geq \frac{1}{\beta_{n_i}} \langle z_{n_i} - x_{n_i}, z_{n_i} - \omega \rangle, \quad \forall \omega \in C.$$

By rearranging the terms of the above inequality, we infer that

$$\langle F y_{n_i}, \omega - y_{n_i} \rangle \geq \frac{1}{\beta_{n_i}} \langle z_{n_i} - x_{n_i}, z_{n_i} - \omega \rangle - \langle F y_{n_i}, y_{n_i} - z_{n_i} \rangle, \forall \omega \in C.$$

Fixing $\omega \in C$ and taking the limit as $i \to \infty$ in the above inequality, invoking (2.11), (2.13) and $\liminf_{i\to\infty} \beta_{n_i} > 0$, we have

$$\liminf_{i\to\infty} \langle F y_{n_i}, \omega - y_{n_i} \rangle \geq 0. \tag{2.14}$$

Now we choose a positive real sequence $\{\epsilon_i\}$ decreasing and tending to 0. For each ϵ_i, we denote by m_i the smallest positive integer such that

$$\langle F(y_{n_j}), \omega - y_{n_j} \rangle + \epsilon_i \geq 0, \quad \forall j \geq m_i, \tag{2.15}$$

where the existence of m_i follows from (2.14). Since $\{\epsilon_i\}$ is decreasing, it is easy to see that sequence $\{m_i\}$ is increasing. For each i, $F(y_{n_{m_i}}) \neq 0$, set

$$t_{n_{m_i}} = \frac{F(y_{n_{m_i}})}{\|F(y_{n_{m_i}})\|^2}.$$

Note that $\langle F(t_{n_{m_i}}), t_{n_{m_i}} \rangle = 1$ for each i. It follows from (2.15) that

$$\langle F(y_{n_{m_i}}), \omega + \epsilon_i t_{n_{m_i}} - y_{n_{m_i}} \rangle \geq 0.$$

By the pseudo-monotonicity of F, we can conclude from the above inequality that

$$\langle F(\omega + \epsilon_i t_{n_{m_i}}), \omega + \epsilon_i t_{n_{m_i}} - y_{n_{m_i}} \rangle \geq 0. \tag{2.16}$$

On the other hand, we have that $\{y_{n_i}\}$ converges weakly to $\hat{x}$ as $i \to \infty$. Since F is sequentially weakly continuous on C , we have that $\{F(y_{n_i})\}$ converges weakly to $F(\hat{x})$. Assume $F(\hat{x}) \neq 0$ (otherwise, $\hat{x}$ is a solution). Due to the norm mapping is sequentially weakly lower semicontinuous, we obtain that

$$\liminf_{i\to\infty} \|F(y_{n_i})\| \geq \|F(\hat{x})\|.$$

From $\{y_{n_{m_i}}\} \subset \{y_{n_i}\}$ and $\epsilon_i \to 0$ as $i \to \infty$, we obtain

$$0 = \frac{0}{F(\hat{x})} \geq \lim_{i\to\infty} \frac{\epsilon_i}{\|F(y_{n_{m_i}})\|} = \lim_{i\to\infty} \|\epsilon_i t_{n_{m_i}}\| \geq 0.$$

By letting $i \to \infty$ in (2.16), we obtain

$$\langle F(\omega), \omega - \hat{x} \rangle \geq 0.$$

Combining this with Lemma 1.1, we get that $\hat{x} \in VI(C,F)$. In view of $x_{n_i} = P_{Q_{n_i}}(x_0)$ and $VI(C,F) \subseteq Q_{n_i}$, it follows from $z = P_{VI(C,F)}(x_0)$ and the lower semicontinuity of the norm that

$$\|x_0 - z\| \leq \|x_0 - \hat{x}\| \leq \liminf_{i\to\infty} \|x_0 - x_{n_i}\| \leq \limsup_{i\to\infty} \|x_0 - x_{n_i}\| \leq \|x_0 - z\|.$$

Hence, we have that

$$\lim_{i\to\infty} \|x_0 - x_{n_i}\| = \|x_0 - \hat{x}\|. \tag{2.17}$$

Due to (2.17) and $x_0 - x_{n_i} \rightharpoonup x_0 - \hat{x}$ as $i \to \infty$, which together with Lemma 1.2, amounts to $x_0 - x_{n_i} \to x_0 - \hat{x}$ as $i \to \infty$. This implies that $x_{n_i} \to \hat{x}$ as $i \to \infty$. For the reasons that $z \in VI(C,F) \subseteq Q_n$ and $x_n = P_{Q_n}x_0$, $\forall n \geq 1$, it follows from projection characterization (i) that $\langle x_0 - x_{n_i}, z - x_{n_i} \rangle \leq 0$. Accordingly, we conclude that

$$\|z - x_{n_i}\|^2 = \langle z - x_0, z - x_{n_i} \rangle + \langle x_0 - x_{n_i}, z - x_{n_i} \rangle \leq \langle z - x_0, z - x_{n_i} \rangle.$$

As $i \to \infty$, we conclude from $z = P_{VI(C,F)}x_0$ and $\hat{x} \in VI(C,F)$, that

$$\|z - \hat{x}\|^2 \leq \langle z - x_0, z - \hat{x} \rangle \leq 0.$$

Consequently, we derive that $z = \hat{x}$. Since the subsequence $\{x_{n_i}\}$ is arbitrarily chosen in $\{x_n\}$, it ensures that $x_n \to z$ as $n \to \infty$. From (2.11) and (2.12), we conclude that $y_n \to z$ and $z_n \to z$ as $n \to \infty$. This completes the proof. □

3. Numerical experiments

In this section, we consider several computational experiments in support of the convergence of the proposed algorithm. We also compare our method with some existing methods in literature.

Recall that the algorithm in [29] is as follows

Algorithm 3.1. (i) Choose $x_0 \in H, y_0 \in C$. The parameters λ, κ satisfy the following conditions $0 < \lambda < \frac{1}{2L}$ and $\kappa > \frac{1}{1-2\lambda L}$. Set $C_0 = Q_0 = H$.

(ii) Compute

$$\begin{aligned}
&y_{n+1} = P_C(x_n - \lambda F y_n),\\
&\varepsilon_n = \kappa\|x_n - x_{n-1}\|^2 + \lambda L\|y_n - y_{n-1}\|^2 - \left(1 - \frac{1}{\kappa} - \lambda L\right)\|y_{n+1} - y_n\|^2,\\
&C_n = \{w \in H : \|y_{n+1} - w\|^2 \leq \|x_n - w\|^2 + \varepsilon_n\},\\
&Q_n = \{w \in H : \langle w - x_n, x_0 - x_n \rangle \leq 0\},\\
&x_{n+1} = P_{C_n \cap Q_n}(x_0).
\end{aligned}$$

(iii) Set $n \leftarrow n + 1$, and go to (i).

We recall the hybrid extragradient method of Nadezhkina and Takahashi [30] is given as the following

Algorithm 3.2. (i) Choose $x_0 \in H$ and the parameter β such that $0 < \beta < \frac{1}{L}$. Set $C_0 = Q_0 = H$.

(ii) Compute

$$\begin{aligned} y_n &= P_C(x_n - \beta F x_n), \\ z_n &= P_C(x_n - \beta F y_n), \\ C_n &= \{w \in H : \|z_n - w\|^2 \le \|x_n - w\|^2\}, \\ Q_n &= \{w \in H : \langle w - x_n, x_0 - x_n \rangle \le 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n}(x_0). \end{aligned}$$

(iii) Set $n \leftarrow n + 1$, and go to (i).

All programs are written in Matlab (R2015b) and performed on a PC Desktop Intel(R) Core (TM) i5-8250U CPU @1.60GHz. We respectively apply different algorithms to solve the following convex feasibility problems.

Example 3.1. Consider the operator $F(x) = Ax + \iota$. This example is taken from [20], where

$$A = BB^T + C + D,$$

and B is a $k \times k$ matrix, C is a $k \times k$ skew-symmetric matrix, with their entries being generated randomly in $(-10, 10)$. D is a $k \times k$ diagonal matrix, whose diagonal entries are positive in $(0, 2)$ (hence A is positive symmetric definite), ι is a vector in $\mathbb{R}^k$. The feasible set $C \subset \mathbb{R}^k$ is a closed convex subset defined by $C = \{x \in \mathbb{R}^k : -3 \le x_i \le 6, i = 1, 2, \cdots, k\}$. It is clear that F is monotone and Lipschitz continuous with the constant $L = \|A\|$. We see that C above is a polyhedral convex set. The sets C_n and Q_n in Algorithms 2.1, 3.1, 3.2 are either a half-space or the whole space $\mathbb{R}^k$, thus $C_n \cap Q_n$ is also a polyhedral convex set. The projection onto $C_n \cap Q_n$ can be computed by Propositions 28.18 and 28.19 of [30].

We set the inertial parameter $\alpha_n = 1$, denote $a = \min\left\{1, \frac{1}{2L^2}\right\}$ and meanwhile choose the parameter β_n in $(0, a)$ for Algorithm 2.1. We randomly choose $\lambda \in \left(0, \frac{1}{2L}\right)$, $\kappa \in \left(\frac{1}{1-2\lambda L}, \infty\right)$ for Algorithm 3.1 and $\beta \in \left(0, \frac{1}{L}\right)$ for Algorithm 3.2. Recall that $\bar{x} \in VI(C, F)$ if and only if $\bar{x} = P_C(\bar{x} - \beta F(\bar{x}))$ for all $\beta > 0$, which means that we can used the sequence $D_n = x_n - P_C(x_n - \beta_n F x_n)$ $(n = 0, 1, 2, 3 \cdots)$ to study the convergence of Algorithms 2.1, 3.1 and 3.2. Note that, from the definition of the metric projection, if $\|D_n\| < \varepsilon$, then x_n can be considered as a ε-solution of the problem. We randomly choose the starting points in the range of $(0, 1)^k$ for Algorithms 2.1, 3.1 and 3.2. To illustrate that our proposed algorithm has a competitive performance compared with Algorithm 3.1 and 3.2, we describe the following numerical results shown in Figure 2 and Figure 3. We respectively take the number of iterations $n = 200, 500$ as the stopping criterion in the following experiments.

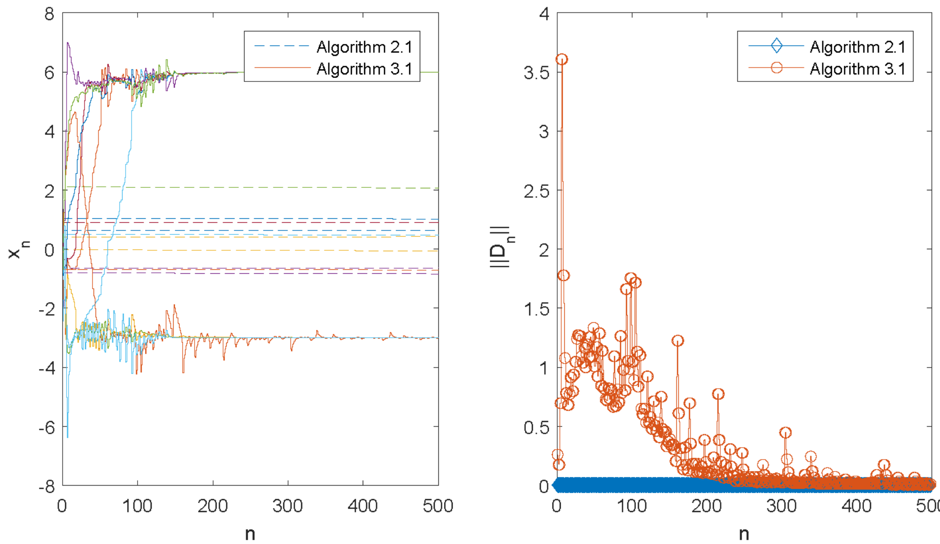

Figure 2. Behaviors of elements of $(x_n)_{10\times 1}$ and the value of $\|D_n\|$ with the number of iterations $n = 500$. Numerical results for Algorithms 2.1, 3.1.

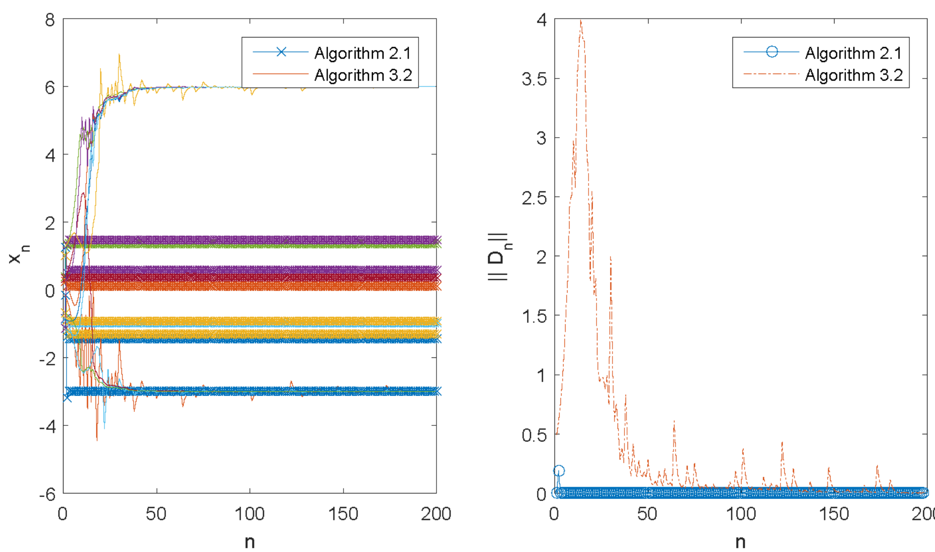

Figure 3. Behaviors of elements of $(x_n)_{10\times 1}$ and the value of $\|D_n\|$ with the number of iterations $n = 200$. Numerical results for Algorithms 2.1, 3.2.

The above two figures show that Algorithm 2.1 has a better behavior than Algorithms 3.1 and 3.2. It achieves a more stable and higher precision after a fewer steps. While the reduction of the sequence $\|D_n\|$ of Algorithms 3.1 and 3.2 have the oscillation with the increasing of the number of iterations. This illustrates that

Algorithms 2.1 significantly reduces the number of iterations. We can see that the effect of the acceleration of the inertial extrapolation to Algorithm 2.1 is obvious. The convergent point of $\|D_n\|$ is 0, means that the iterative sequences converge to the solution of this experiment. Furthermore, with different random initial points and parameters, the iterative sequences generated by each algorithm may converge to different solutions of the variational inequality problem.

Definition 3.1. A differential function $f : \mathbb{R}^n \to \mathbb{R}$ is pseudo-convex on C if for every pair of distinct points $x, y \in C$,

$$\langle \nabla f(x), y - x \rangle \geq 0 \Rightarrow f(y) \geq f(x).$$

In [23], Karamardian and Scchaible showed that a differentiable function is pseudo-convex if and only if its gradient is pseudo-monotone, which reveals the relationship between the variational inequality problem and the pseudo-convex optimization. Next, we indicate the significant class of applications to the variational inequality problem involving pseudo-monotone mappings instead of monotone mappings.

Example 3.2. Consider the following fractional programming problem [21]

$$\min\ f(x) = \tfrac{x^T Qx + a^T x + a_0}{b^T x + b_0},$$
$$\text{subject to } \ x \in C := \{x \in \mathbb{R}^k : b^T x + b_0 > 0\},$$

where

$$Q = \begin{pmatrix} 5 & -1 & 2 & 0 \\ -1 & 5 & -1 & 3 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 0 & 5 \end{pmatrix},\ a = \begin{pmatrix} 1 \\ -2 \\ -2 \\ 1 \end{pmatrix},\ b = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix},\ a_0 = -2,\ b_0 = 4.$$

We can see that Q is symmetric and positive definite in $\mathbb{R}^4$ and consequently f is pseudo-convex on $C = \{x \in \mathbb{R}^4 : b^T x + b_0 > 0\}$. We minimize f over C via Algorithm 2.1 with

$$F(x) := \nabla f(x) = \frac{(b^T x + b_0)(2Qx + a) - b(x^T Qx + a^T x + a_0)}{(b^T x + b_0)^2}.$$

We used the error sequences $D_n = x_n - P_C(x_n - \beta_n F x_n)$ and $E_n = x_n - y_n$ $(n = 0, 1, 2, 3 \cdots)$ to study the convergence of Algorithms 2.1.

We randomly choose the initial points in the range of $(0, 1)^4$ and take the number of iterations $n = 50$ as the stopping criterion. The values of error sequences $\{D_n\}$ and $\{E_n\}$ are represented by the y-axis, the number of iterations n is represented by the x-axis. We apply Algorithm 2.1 to solve this problem. The numerical results are shown in Figure 4. The convergence of $\{D_n\}$ and $\{E_n\}$ to $(0, 0, 0, 0)^T$ implies that the iterative sequences converge to the solution of this experiment. Furthermore, this problem has a unique solution $\hat{x} = (1, 1, 1, 1)^T \in C$.

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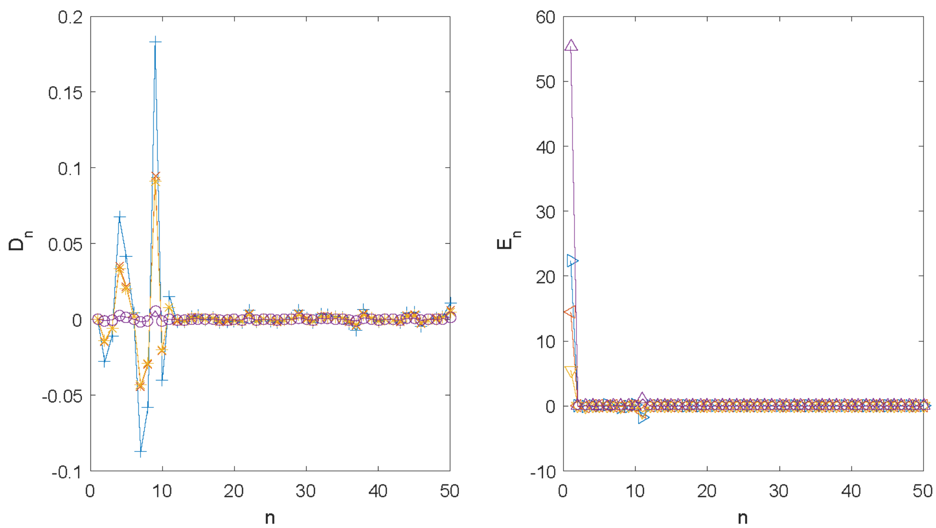

Figure 4. Behaviors of elements of $(D_n)_{4\times 1}$ and $(E_n)_{4\times 1}$ with the number of iterations $n = 50$. Numerical results for Algorithm 2.1.

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