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GLOBAL STABLE AND UNSTABLE MANIFOLDS FOR A CLASS OF SEMILINEAR EQUATIONS WITH SECTORIALLY DICHOTOMOUS OPERATOR*

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Abstract In this paper, the existence and smoothness of global stable and unstable manifolds at an equilibrium are established for a class of semilinear equations with sectorially dichotomous operator. As an application, an elliptic PDE in infinite cylindrical domain is discussed.

Keywords Semilinear equations, sectorially dichotomous operator, dichotomous solution, stable and unstable manifolds.

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1. Introduction

The invariant manifold theory is a powerful tool in analyzing the asymptotic behavior of a dynamical system or solution of differential equation in the vicinity of an equilibrium, and this theory is being driven to develop by some application problems arising in science and engineering(cf. Bates etc [3,4], Chow etc [6–9], Gal and Guo [14], Hirsch etc [17], Magal and Ruan [22], Sandstede etc [25], Schnaubelt [26], Wiggins [30], Zhang and Zhang [31], and references therein).

In this paper, we consider a class of abstract semilinear equations

$$\frac{dz(t)}{dt} = Sz(t) + H(z(t)), \quad t \in \mathbb{R}, z(t) \in \mathcal{Z}, \tag{1.1}$$

where $\mathcal{Z}$ is a Banach space, $S : \mathcal{D}(S) \subset \mathcal{Z} \to \mathcal{Z}$ is a sectorially dichotomous operator, and $H : \mathcal{O} \to \mathcal{Z}$ is a smooth map with $H(0) = 0$ and $D_zH(0) = 0$. Here, $\mathcal{O} \subset \mathcal{Z}_\alpha$ is an open set, $\mathcal{Z}_0 = \mathcal{Z}$, $\mathcal{Z}_\alpha \triangleq \mathcal{D}(S^\alpha)$ endows with the graph norm which satisfies the relation of continuous embedding $\mathcal{D}(S) \hookrightarrow \mathcal{Z}_\alpha \hookrightarrow \mathcal{Z}$ for $\alpha \in (0, 1)$, and S^α is the α-fractional power of S.

If the operator S of (1.1) is the infinitesimal generator of a strongly continuous semigroup or analytic semigroup on $\mathcal{Z}$, the invariant manifold theory of (1.1) has been established (cf. Henry [16], Chow and Lu [6,7], Bates and Jones [2], Vanderbauwhede and Iooss [29], and references therein). If S in (1.1) is the infinitesimal generator of a certain bisemigroup, then S has infinite spectrum in both sides of the imaginary axis on the complex plane, the Hille-Yosida theorem implies that (1.1)

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does not generate a semiflow or flow on $\mathcal{Z}$ for any given initial values. Hence, this Cauchy problem corresponding to (1.1) is often called *ill-posed* on $\mathcal{Z}$ (cf. [11, 12]). Even though the ill-posed Cauchy problem may not be solvable on $\mathcal{Z}$ for arbitrary initial values, some solutions can be defined on a subspaces of $\mathcal{Z}$ with suitable initial values. For example, ElBialy in [12] showed the existence of local Lipschitzian stable and unstable manifolds for the following system

$$\begin{cases} \frac{dx(t)}{dt} = Lx(t) + f(z(t)), \\ \frac{dy(t)}{dt} = Ry(t) + g(z(t)), \\ x(t_1) = x_1 \in \mathbf{X},\ y(t_2) = y_2 \in \mathbf{Y},\ z = (x, y) \in \mathbf{X} \times \mathbf{Y} = \mathbf{Z},\ t \in [t_1, t_2], \end{cases} \tag{1.2}$$

based on the existence of dichotomous mild solutions under the dichotomous initial conditions introduced by Latushkin and Layton [21]. Here $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$ are all Banach spaces, (L, R) is the infinitesimal generator of a hyperbolic and strongly continuous bisemigroup $(\{e^{Lt}\}_{t\geq 0}, \{e^{-Rt}\}_{t\geq 0})$, and the nonlinearity (f, g) are locally Lipschitz continuous in z.

Combined with the dichotomous initial conditions in [12, 21]:

$$\big\{(\widetilde{z}; t_1, t_2) | t_1 < t_2, \widetilde{z} = (x_1, y_2) \in \mathcal{Z}, x(t_1) = x_1 \in \mathcal{X}, y(t_2) = y_2 \in \mathcal{Y}\big\}, \tag{1.3}$$

the equation (1.1) can be transformed into the coupled system:

$$\begin{cases} \frac{dx(t)}{dt} = S_+ x(t) + F(z(t)), \\ \frac{dy(t)}{dt} = S_- y(t) + G(z(t)), \\ x(t_1) = x_1,\ y(t_2) = y_2,\ z = x + y \in \mathcal{X} \oplus \mathcal{Y} = \mathcal{Z},\ t \in [t_1, t_2], \end{cases} \tag{1.4}$$

where $S_+ := S|_{\mathcal{X}}$ and $-S_- := -S|_{\mathcal{Y}}$ are densely defined and sectorial operators on the Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ respectively, $F(z(t)) := P_+ H(z(t))$, $G(z(t)) := P_- H(z(t))$ and P_+ (P_-) is the projection of $\mathcal{Z}$ onto $\mathcal{X}$ $(\mathcal{Y})$ along $\mathcal{Y}$ $(\mathcal{X})$. We are interested in the asymptotic behavior of the dichotomous solution in a neighborhood of the equilibrium $z = 0$, and obtain the existence and smoothness of global stable and unstable manifolds for the system (1.4). The existence of global stable manifold is carried out by the Liapunov-Perron method, and the smoothness proof is built by the Lemma 2.1 in Chow and Lu [7] and the Henry's lemma [16, Lemma 6.1.6]. In addition, the existence of global unstable manifold can follow from the existence of global stable manifold by reversing time.

As an application of our results, we study the following elliptic equation in infinite cylindrical domain $\mathbb{R} \times \Omega$

$$\begin{aligned} u_{xx} + \Delta_y u + f(y, u, u_x, \nabla_y u) = 0, \quad (x, y, u) \in \mathbb{R} \times \Omega \times \mathbb{R}^m, \\ u(x, y) = 0, \quad x \in \mathbb{R},\ y \in \partial\Omega, \end{aligned} \tag{1.5}$$

where Ω is an open and bounded subset of $\mathbb{R}^n$ with smooth boundary, ∇_y is the gradient in the y-variable and Δ_y is the Laplace operator in the y-variable. We shows that the existence and asymptotic behavior of solutions for system (1.5) under some boundary value conditions.

It is worth pointing out that, the elliptic problem in infinite cylindrical domain formulated by the abstract form (1.1) could be transformed to a first order system consisting of a pair of semilinear coupled parabolic equations, and it is well known that the investigation of parabolic problem base on the theory of analytic semigroup is very importance, refer to the monographs [16,19]. So the assumption of sectorial dichotomy for the linear operator S in the equation (1.1) make it more realistic. In addition, one often encounter nonlinearities of PDEs depend not only on the unknown state variable but also on its derivatives, so it is reasonable to define the nonlinearities of the equation (1.1) in $\mathcal{Z}_\alpha$ which between $\mathcal{Z}$ and $\mathcal{D}(S)$.

This paper is organized as follows. In Section 2, some notations, definitions, hypothesises and lemmas are given. In Section 3, we devote to existence and smoothness of global stable and unstable manifolds. Last, we give an elliptic equation in infinite cylindrical domain as an example to illustrate our results.

2. Preliminaries

Let S be a linear operator with the domain $\mathcal{D}(S)$ and range $\mathcal{R}(S)$. We denote the resolvent set and the spectrum of S in the complex plane by $\rho(S)$ and $\sigma(S)$, respectively, denote $R(\lambda, S) \triangleq (\lambda I - S)^{-1}$ by its resolvent operator for $\lambda \in \rho(S)$, and denote $\Re\lambda$ by the real part of $\lambda \in \sigma(S)$. For any constant $\widetilde{\sigma} \in (0, \pi)$, we define the sector regions $\Sigma_{\widetilde{\sigma}} \triangleq \{\lambda \in \mathbb{C}\backslash\{0\} : |\arg\lambda| \leq \widetilde{\sigma}\}$ and $-\Sigma_{\widetilde{\sigma}} \triangleq \{\lambda \in \mathbb{C}\backslash\{0\} : |\arg(-\lambda)| \leq \widetilde{\sigma}\}$.

Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces and $\mathbb{U}$ be an open subset of $\mathbb{X}$, we denote $\mathcal{L}(\mathbb{X}, \mathbb{Y})$ by the bounded linear operator from $\mathbb{X}$ to $\mathbb{Y}$. Firstly we use the following notations to represent some function spaces:

(i) For any integer $k \geq 0$, let

$$C^k(\mathbb{U}, \mathbb{Y}) = \{f | f : \mathbb{U} \to \mathbb{Y} \text{ is } k\text{-times continuously differentiable}, \sup_{x\in\mathbb{U}} \|D_x^j f(x)\|_{\mathcal{L}(\mathbb{X}^j,\mathbb{Y})} < \infty \text{ for all } 0 \leq j \leq k\},$$

where D_x^j denotes the jth differentiation operator with respect to the variable x. $C^k(\mathbb{U}, \mathbb{Y})$ is a Banach space equipped with the norm

$$\|f\|_k = \max_{0\leq j\leq k} \sup_{x\in\mathbb{U}} \|D_x^j f(x)\|_{\mathcal{L}(\mathbb{X}^j,\mathbb{Y})},$$

where the space $\mathbb{X}^j$ denotes $\underbrace{\mathbb{X} \times \mathbb{X} \times \cdots \times \mathbb{X}}_{j}$.

(ii) Let $k \geq 0$ be an integer, $\gamma \in (0, 1]$, let

$$C^{k,\gamma}(\mathbb{U}, \mathbb{Y}) = \{f | f \in C^k(\mathbb{U}, \mathbb{Y}), \text{and } H_\gamma(D_x^k f) < \infty\},$$

$C^{k,\gamma}(\mathbb{U}, \mathbb{Y})$ is a Banach space equipped with the norm $\|f\|_{k,\gamma} = \max\big\{\|f\|_k, H_\gamma(D_x^k f)\big\}$, where $H_\gamma(D_x^k f) = \sup_{x,\widehat{x}\in\mathbb{U},x\neq\widehat{x}} \frac{\|D_x^k f(x) - D_{\widehat{x}}^k f(\widehat{x})\|_{\mathcal{L}(\mathbb{X}^k,\mathbb{Y})}}{\|x-\widehat{x}\|_{\mathbb{X}}^\gamma}$.

(iii) Let $J = (-\infty, t_0]$ or $[t_0, \infty)$ for $t_0 \in \mathbb{R}$. For a constant $\beta \in \mathbb{R}$, let

$$C_\beta(J, \mathbb{Y}) = \{y \in C^0(J, \mathbb{Y}) : \|y\|_{C_\beta} = \sup_{t\in J} e^{-\beta t}\|y(t)\|_{\mathbb{Y}} < \infty\},$$

which is a Banach space equipped with the norm $\|\cdot\|_{C_\beta}$.

Note that we also define the space $C^{k,\gamma}(\mathbb{U}, \mathbb{Y})$ for $\gamma = 0$, and specify $C^{k,0}(\mathbb{U}, \mathbb{Y}) := C^k(\mathbb{U}, \mathbb{Y})$ and $\|f\|_{k,0} := \|f\|_k$. Next, let me recall the definitions of sectorial and sectorially dichotomous operator, see [23,28].

Definition 2.1. Let $S : \mathcal{D}(S) \subset \mathcal{Z} \to \mathcal{Z}$ be a closed and linear operator.

(1) If there exists $\widetilde{\sigma} \in (0, \pi/2)$ and $M > 0$ such that $\sigma(S) \subset \mathbb{C} \setminus \Sigma_{\pi/2+\widetilde{\sigma}}$, and

$$\|R(\lambda, S)\|_{\mathcal{L}(\mathcal{Z})} \leq \frac{M}{|\lambda|} \text{ for } \lambda \in \Sigma_{\pi/2+\widetilde{\sigma}}, \tag{2.1}$$

then S is called **sectorial.**

(2) For two banach spaces $\mathcal{X}$ and $\mathcal{Y}$ satisfying $\mathcal{Z} = \mathcal{X} \oplus \mathcal{Y}$, set $S_+ := S|_{\mathcal{X}}$ and $S_- := S|_{\mathcal{Y}}$, if the following four conditions hold:
   (i) $i\mathbb{R} \subset \rho(S)$;
   (ii) $\mathcal{X}$ and $\mathcal{Y}$ are S-invariant, i.e.,$S(\mathcal{D}(S) \cap \mathcal{X}) \subset \mathcal{X}$ and $S(\mathcal{D}(S) \cap \mathcal{Y}) \subset \mathcal{Y}$;
   (iii) $\sigma(S_+) \subset \{\lambda \in \mathbb{C} : \Re\lambda < 0\}$, $\sigma(S_-) \subset \{\lambda \in \mathbb{C} : \Re\lambda > 0\}$;
   (iv) S_+ and $-S_-$ are densely defined and sectorial operators on $\mathcal{X}$ and $\mathcal{Y}$ respectively,

   then S is called **sectorially dichotomous** with respect to the decomposition $\mathcal{Z} = \mathcal{X} \oplus \mathcal{Y}$.

From Definition 2.1(2), we know that, there exists a bounded projection $P : \mathcal{Z} \to \mathcal{X}$ such that the sectorially dichotomous operator S can be reduced with respect to $\mathcal{Z} = \mathcal{X} \oplus \mathcal{Y}$, where $\mathcal{X} = \mathcal{R}(P)$ and $\mathcal{Y} = Ker(P)$. Namely, S admits the block matrix representation $S = \begin{pmatrix} S_+ & 0 \\ 0 & S_- \end{pmatrix}$, S_+ and S_- have the form of

$$S_+z = Sz, \forall z \in \mathcal{D}(S_+) := \{z \in \mathcal{D}(S) \cap \mathcal{X} : Sz \in \mathcal{X}\},$$

and

$$S_-z = Sz, \forall z \in \mathcal{D}(S_-) := \{z \in \mathcal{D}(S) \cap \mathcal{Y} : Sz \in \mathcal{Y}\}.$$

Besides, $\mathcal{D}(S) = \mathcal{D}(S_+) \oplus \mathcal{D}(S_-)$, S is also densely defined on $\mathcal{Z}$.

For the sectorially dichotomous operator S and a constant $\alpha \in (0, 1)$, α-fractional power of $-S_+, S_-$ and S can be defined and the relationship among them can also be obtained, see [10]. We denote $(-S_+)^\alpha$ and $(S_-)^\alpha$ by the α-fractional powers of $-S_+$ and S_-, respectively, and denote

$$\mathcal{X}_\alpha := \mathcal{D}((-S_+)^\alpha), \quad \mathcal{Y}_\alpha := \mathcal{D}((S_-)^\alpha), \quad \mathcal{Z}_\alpha := \mathcal{D}(S^\alpha)$$

by Banach spaces with norm $\|x\|_\alpha^{\mathcal{X}} = \|(-S_+)^\alpha x\|_{\mathcal{X}}$, $\|y\|_\alpha^{\mathcal{Y}} = \|(S_-)^\alpha y\|_{\mathcal{Y}}$ and $\|z\|_\alpha^{\mathcal{Z}} = \|S^\alpha z\|_{\mathcal{Z}}$ respectively. Moreover, $\mathcal{Z}_\alpha = \mathcal{X}_\alpha \oplus \mathcal{Y}_\alpha$ and $\mathcal{D}(S) \hookrightarrow \mathcal{Z}_\alpha \hookrightarrow \mathcal{Z}$. Note that $\mathcal{Z}_0 = \mathcal{Z}$, $\mathcal{X}_0 = \mathcal{X}$ and $\mathcal{Y}_0 = \mathcal{Y}$. Let $\|\cdot\|_{\mathcal{Z}}$ as $\|z\|_{\mathcal{Z}} = \|x + y\|_{\mathcal{Z}} = \max\{\|x\|_{\mathcal{X}}, \|y\|_{\mathcal{Y}}\}$ for $x \in \mathcal{X}$, $y \in \mathcal{Y}$ and $z \in \mathcal{Z}$.

Furthermore, from the sections 2.5 and 2.6 in [23], the following results hold.

Lemma 2.1. *S_+ and $-S_-$ generate uniformly exponentially stable, strongly continuous, analytic semigroups $\{T_+(t)\}_{t\geq 0}$ and $\{T_-(-t)\}_{t\geq 0}$ on $\mathcal{X}$ and $\mathcal{Y}$ respectively, where $T_+(t) := e^{tS_+}$ and $T_-(-t) := e^{-tS_-}$. Furthermore, there are real numbers $\beta_+ < 0 < \beta_-$ and some positive constants $M_0^+, M_0^-, M_\alpha^+, M_\alpha^-$ such that*

$$\|T_+(t)\|_{\mathcal{L}(\mathcal{X})} \leq M_0^+ e^{\beta_+ t}, \quad \|T_-(-t)\|_{\mathcal{L}(\mathcal{Y})} \leq M_0^- e^{-\beta_- t} \tag{2.2}$$

for $t \geq 0$, and

$$\|T_+(t)\|_{\mathcal{L}(\mathcal{X},\mathcal{X}_\alpha)} \leq M_\alpha^+ t^{-\alpha} e^{\beta_+ t}, \quad \|T_-(-t)\|_{\mathcal{L}(\mathcal{Y},\mathcal{Y}_\alpha)} \leq M_\alpha^- t^{-\alpha} e^{-\beta_- t} \tag{2.3}$$

for $t > 0$.

In particular, by the spectral mapping theorem [13, Corollary 4.3.12], $T_+(t)$ and $T_-(-t)$ are contraction semigroups, i.e., $\|T_+(t)\|_{\mathcal{L}(\mathcal{X})} \le 1$ and $\|T_-(-t)\|_{\mathcal{L}(\mathcal{Y})} \le 1$ for $t \ge 0$. Throughout this paper, we set

$$\widehat{M}_k^+ = \sup_{t>0} \|t^k S_+^k T_+(t)\|_{\mathcal{L}(\mathcal{X})}, \ \widehat{M}_k^- = \sup_{t>0} \|t^k S_-^k T_-(-t)\|_{\mathcal{L}(\mathcal{Y})},$$

and

$$\widehat{M}_\alpha^+ = \sup_{t>0} \|t^\alpha (-S_+)^\alpha T_+(t)\|_{\mathcal{L}(\mathcal{X})}, \ \widehat{M}_\alpha^- = \sup_{t>0} \|t^\alpha S_-^\alpha T_-(-t)\|_{\mathcal{L}(\mathcal{Y})},$$

From [19, Proposition 2.1.1(*iii*)], [27, Theorem 1.12], we have that $\widehat{M}_k^\pm < \infty, \widehat{M}_\alpha^\pm < \infty$ for every $k \in \mathbb{N} \cup \{0\}$ and $\alpha \in [0, 1)$.

In the following, we give the definitions of some dichotomous solutions and invariant manifolds for the system (1.4).

Definition 2.2. A function $z(t) : [t_1, t_2] \to \mathcal{Z}_\alpha$ is called **dichotomous solution** of system (1.4) for $-\infty < t_1 < t_2 < \infty$, if it satisfies the following three conditions:

- (i) $z(t) \in C^0([t_1, t_2], \mathcal{Z}_\alpha) \cap C^1((t_1, t_2), \mathcal{Z}_\alpha)$;
- (ii) $x(t) \in \mathcal{D}(S_+), y(t) \in \mathcal{D}(S_-)$ and $z(t) \in \mathcal{O}$ for $t \in (t_1, t_2)$;
- (iii) $x(t)$ and $y(t)$ satisfy the first and second equation in (1.4), respectively, for all $t \in (t_1, t_2)$,
  $moreover, x(t_1) = x_1, y(t_2) = y_2,$

where $z(t) = x(t) + y(t), x(t) \in \mathcal{X}_\alpha, y(t) \in \mathcal{Y}_\alpha$, and $\mathcal{O} \subset \mathcal{Z}_\alpha$ is an open set.

From Definition 2.2, the dichotomous solution $z(t)$ of system (1.4) satisfies the following dichotomous system of integral equations:

$$\begin{cases} x(t) = T_+(t - t_1)x_1 + \int_{t_1}^t T_+(t - s)F(z(s))\, ds, \\ y(t) = T_-(t - t_2)y_2 - \int_t^{t_2} T_-(t - s)G(z(s))\, ds. \end{cases} \tag{2.4}$$

If the limits of the integrals in (2.4) exist, we obtain a dichotomous solution on $I = \mathbb{R}$. In order to distinguishing some asymptotic behaviors of dichotomous solution clearly, two infinitely long dichotomous solutions deserve to be introduced similar to ElBily [12].

Definition 2.3. (1) We say that $z \in C^0([t_1, \infty), \mathcal{Z}_\alpha)$ is an **infinitely long forward dichotomous solution** if for all $t_1 \le \widehat{t}_1 < t_2 < \infty$, the restriction $z|_{[\widehat{t}_1, t_2]} \in C^0([\widehat{t}_1, t_2], \mathcal{Z}_\alpha)$ is a dichotomous solution. If $z(t_1) = \zeta = (x_1, y(t_1; x_1))$, we write the infinitely long forward dichotomous solution z as $z(\cdot; t_1, \zeta)$.

(2) We say that $z \in C^0((-\infty, t_2], \mathcal{Z}_\alpha)$ is an **infinitely long backward dichotomous solution** if for all $-\infty < t_1 < \widehat{t}_2 \le t_2$, the restriction $z|_{[t_1, \widehat{t}_2]} \in C^0([t_1, \widehat{t}_2], \mathcal{Z}_\alpha)$ is a dichotomous solution. If $z(t_2) = \zeta = (x(t_2; y_2), y_2)$, we write the infinitely long backward dichotomous solution z as $z(\cdot; t_2, \zeta)$.

The stable set $W^s(0)$ and the unstable set $W^u(0)$ of the equilibrium $z = 0$ are defined as

$$W^s(0) = \{\zeta \in \mathcal{Z}_\alpha : z(\cdot; t_1, \zeta) \text{ is defined for } t \in [t_1, \infty) \text{ and } \lim_{t \to \infty} z(t; t_1, \zeta) = 0\},$$

$$W^u(0) = \{\zeta \in \mathcal{Z}_\alpha : z(\cdot; t_2, \zeta) \text{ is defined for } t \in (-\infty, t_2] \text{ and } \lim_{t \to -\infty} z(t; t_2, \zeta) = 0\}.$$

It is trivial to check that $W^s(0)$ and $W^u(0)$ are both invariant to $\mathcal{Z}_\alpha$. The purpose of this paper is to show that $W^s(0)$ and $W^u(0)$ are indeed manifolds, before that, we give the definitions of global stable and unstable manifolds.

Definition 2.4. (1) Let $h^s : \mathcal{X}_\alpha \to \mathcal{Y}_\alpha$ be $C^{k,\gamma}$ with $h^s(0) = 0$, in addition, $D_x h^s(0) = 0$ for $k \geq 1$, such that $W^s(0) = \{(x, y) : y = h^s(x), x \in \mathcal{X}_\alpha\}$. Then $W^s(0)$ is called a $C^{k,\gamma}$ **global stable manifold** if $W^s(0)$ is invariant under the infinitely long forward dichotomous solution of (1.4), i.e., $(x_1, h^s(x_1)) \in W^s(0)$, then $z(t; t_1, \zeta) \in W^s(0)$ and $\zeta = x_1 + h^s(x_1)$, where $k \in \mathbb{N} \cup \{0\}$ and $\gamma \in [0, 1]$.

(2) Let $h^u : \mathcal{Y}_\alpha \to \mathcal{X}_\alpha$ be $C^{k,\gamma}$ with $h^u(0) = 0$, in addition, $D_y h^u(0) = 0$ for $k \geq 1$, such that $W^u(0) = \{(x, y) : x = h^u(y), y \in \mathcal{Y}_\alpha\}$. Then $W^u(0)$ is called a $C^{k,\gamma}$ **global unstable manifold** if $W^u(0)$ is invariant under the infinitely long backward dichotomous solution of (1.4), i.e., $(h^u(y_2), y_2) \in W^u(0)$, then $z(t; t_2, \zeta) \in W^u(0)$ and $\zeta = h^u(y_2) + y_2$, where $k \in \mathbb{N} \cup \{0\}$ and $\gamma \in [0, 1]$.

In addition, we introduce two lemmas by Chow and Lu [7, Lemma 2.1] and Henry [16, Lemma 6.1.6] respectively, which can be used to study $C^{k,\gamma}$ smoothness of invariant manifolds.

Lemma 2.2 (Lemma 2.1, [7])**.** *Let $\mathbb{X}$, $\mathbb{Y}$ be Banach spaces and $\mathbb{U}$ be an open subset of $\mathbb{X}$. Assume that $f : \mathbb{U} \to \mathbb{X}$ is locally Lipschitz continuous. Then f is continuously differentiable if and only if for every $x_0 \in \mathbb{U}$*

$$\|f(x + \Delta) - f(x) - f(x_0 + \Delta) + f(x_0)\|_{\mathbb{Y}} = o(\|\Delta\|_{\mathbb{X}})$$

as $(x, \Delta) \to (x_0, 0)$.

Lemma 2.3 (Lemma 6.1.6, [16])**.** *Let $\mathbb{X}$, $\mathbb{Y}$ be Banach spaces and $\mathbb{U}$ be an open subset of $\mathbb{X}$. Then a closed bounded ball in $C^{k,\gamma}(\mathbb{U}, \mathbb{Y})(0 < \gamma \leq 1, k \in \mathbb{N} \cup \{0\})$ is also a closed bounded subset in $C^0(\mathbb{U}, \mathbb{Y})$.*

Lemma 2.3 tells us that, a sequence $\{u_n\} \subset C^{k,\gamma}(\mathbb{U}, \mathbb{Y})$ and a map $u : \mathbb{U} \to \mathbb{Y}$ such that $\|u_n - u\|_{C^0(\mathbb{U},\mathbb{Y})} \to 0$ as $n \to \infty$, then $u \in C^{k,\gamma}(\mathbb{U}, \mathbb{Y})$.

To study the existence and $C^{k,\gamma}$ smoothness of invariant manifolds, we give further hypothesis for the nonlinear map H of the system (1.4).
(**H**) Let $F(z(t)) =: P_+ H(z(t))$, $G(z(t)) =: P_- H(z(t))$, and $\mathcal{O} \subset \mathcal{Z}_\alpha$ be an open subset. $F(0) = G(0) = 0$.

(1) $F(\cdot) \in C^{0,1}(\mathcal{O}, \mathcal{X})$ and $G(\cdot) \in C^{0,1}(\mathcal{O}, \mathcal{Y})$ if $k = 0$ and $\gamma = 1$.

(2) $F(\cdot) \in C^{k,\gamma}(\mathcal{O}, \mathcal{X})$ and $G(\cdot) \in C^{k,\gamma}(\mathcal{O}, \mathcal{Y})$ if $k \geq 1$ and $\gamma \in [0, 1]$. Moreover, $D_z F(0) = D_z G(0) = 0$.

3. Global stable and unstable manifolds

In this section, we give the global stable and unstable manifolds theorem as follows.

Theorem 3.1. *Assume that hypothesis (**H**) is satisfied for system* (1.4)*, and $\|F\|_{k,\gamma}$, $\|G\|_{k,\gamma}$ are sufficiently small. Then,*

(i) $W^s(0)$ is a unique $C^{k,\gamma}$ global stable manifold of system (1.4)*. And the infinitely*

long forward dichotomous solution on $W^s(0)$ take the form

$$
\begin{aligned}
x(t) &= T_+(t-t_1)x_1 + \int_{t_1}^{t} T_+(t-s)F(x(s), h^s(x(s)))\,ds,\\
y(t) &= h^s(x(t)) = -\int_t^{\infty} T_-(t-s)G(x(s), h^s(x(s)))\,ds.
\end{aligned}
\tag{3.1}
$$

for $-\infty < t_1 \le t < \infty$.

(i) $W^u(0)$ is a unique $C^{k,\gamma}$ global unstable manifold of system (1.4). *And the infinitely long backward dichotomous solution on $W^u(0)$ take the form*

$$
\begin{aligned}
x(t) &= h^u(y(t)) = \int_{-\infty}^{t} T_+(t-s)F(h^u(y(s)), y(s))\,ds,\\
y(t) &= T_-(t-t_2)y_2 - \int_t^{t_2} T_-(t-s)G(h^u(y(s)), y(s))\,ds.
\end{aligned}
\tag{3.2}
$$

for $-\infty < t \le t_2 < \infty$.

To prove the Theorem 3.1, we first give some lemmas in the following.

Lemma 3.1. *Assume that the nonlinear terms $F(z(t))$ and $G(z(t))$ of system* (1.4) *on arbitrary finite time interval $[t_1, t_2]$ satisfy the hypothesis (**H**-1). Then there is a unique dichotomous solution $z(t)$ for any $\widetilde{z} \in \overline{\mathcal{D}(S)}^{\alpha}$, where $\overline{\mathcal{D}(S)}^{\alpha}$ is the closure of $\mathcal{D}(S)$ in $\mathcal{Z}_\alpha$.*

Proof. Let

$$
\mathcal{E} \triangleq \Big\{ u \in C^0\big([t_1,t_2];\mathcal{Z}\big) : u_1(t_1) = (-S_+)^{\alpha}x_1, u_2(t_2) = (S_-)^{\alpha}y_2. \Big\},
$$

where $u(t) = u_1(t) + u_2(t)$, $u_1(t) \in \mathcal{X}$ and $u_2(t) \in \mathcal{Y}$. Obviously, $\mathcal{E}$ is a non-empty closed subset of $C^0\big([t_1,t_2];\mathcal{Z}\big)$ in the uniform C^0 norm, so it is a complete metric space with the induced metric $d_{\mathcal{E}}(u,v) = \max\limits_{t\in[t_1,t_2]} \|u(t)-v(t)\|_{\mathcal{Z}}$.

We define a mapping $\Psi : \mathcal{E} \to \mathcal{E}$ as follows:

$$
\begin{aligned}
(\Psi u)(t) =& T_+(t-t_1)S^{\alpha}x_1 + \int_{t_1}^{t} S^{\alpha}T_+(t-s)F(S^{-\alpha}u(s))ds\\
&+ T_-(t-t_2)S^{\alpha}y_2 - \int_t^{t_2} S^{\alpha}T_-(t-s)G(S^{-\alpha}u(s))ds,
\end{aligned}
\tag{3.3}
$$

that is,

$$
(\Psi u_1)(t) = T_+(t-t_1)(-S_+)^{\alpha}x_1 + \int_{t_1}^{t} (-S_+)^{\alpha}T_+(t-s)F(S^{-\alpha}u(s))ds, \tag{3.4}
$$

$$
(\Psi u_2)(t) = T_-(t-t_2)(S_-)^{\alpha}y_2 - \int_t^{t_2} (S_-)^{\alpha}T_-(t-s)G(S^{-\alpha}u(s))ds. \tag{3.5}
$$

Now we check the well-definedness of mapping Ψ firstly, so we consider the formulas (3.4) and (3.5).

From [1, Corollary 3.7.21], we know that

$$T_+(t-t_1)(-S_+)^\alpha x_1 \in C^0([t_1,t_2];\mathcal{X}) \cap C^\infty((t_1,t_2];\mathcal{X}),$$
$$T_-(t-t_2)(S_-)^\alpha y_2 \in C^0([t_1,t_2];\mathcal{Y}) \cap C^\infty([t_1,t_2);\mathcal{Y}).$$

On the other hand, we claim that there exists constants $\beta_1,\beta_2 \in (0,1-\alpha)$ such that

$$\mathcal{F}_\alpha(t) := \int_{t_1}^{t} (-S_+)^\alpha T_+(t-s)F(S^{-\alpha}u(s))ds \in C^{0,\beta_1}([t_1,t_2];\mathcal{X}), \tag{3.6}$$

and

$$\mathcal{G}_\alpha(t) := -\int_{t}^{t_2} (S_-)^\alpha T_-(t-s)G(S^{-\alpha}u(s))ds \in C^{0,\beta_2}([t_1,t_2];\mathcal{Y}), \tag{3.7}$$

which implies that

$$(\Psi u)(t) \in C^0([t_1,t_2];\mathcal{Z}) \cap C^{0,\beta_3}((t_1,t_2);\mathcal{Z}), \tag{3.8}$$

where $\beta_3 = \min\{\beta_1,\beta_2\}$.

Indeed, the continuity of $u(t)$ and the hypothesis (**H**-1) follow that $F(S^{-\alpha}u(t))$ and $G(S^{-\alpha}u(t))$ are bounded on $[t_1,t_2]$, there exist two positive constants $\overline{N}_1$ and $\overline{N}_2$ such that

$$\|F(S^{-\alpha}u(t))\|_{\mathcal{X}} \le \overline{N}_1,\ \|G(S^{-\alpha}u(t))\|_{\mathcal{Y}} \le \overline{N}_2. \tag{3.9}$$

for $t \in [t_1,t_2]$. Besides, assume that there exists β_1,β_2 and h_0 such that $\beta_1,\beta_2 \in (0,1-\alpha)$, $h_0 \in (0,1)$ and $t, t+h_0 \in [t_1,t_2]$. We discuss that whether (3.6) and (3.7) hold in two cases of α.

If $\alpha \in (0,1)$, by [23, Theorem 2.6.13(d)], there exists positive constants $C^+_{\beta_1}$, $C^-_{\beta_2}$, $\widehat{M}^+_{\alpha+\beta_1}$ and $\widehat{M}^-_{\alpha+\beta_2}$ such that

$$\begin{aligned}\|(T_+(h_0)-I)(-S_+)^\alpha T_+(t-s)\|_{\mathcal{L}(\mathcal{X})} &\le C^+_{\beta_1} h_0^{\beta_1} \|(-S_+)^{\alpha+\beta_1} T_+(t-s)\|_{\mathcal{L}(\mathcal{X})} \\ &\le C^+_{\beta_1} h_0^{\beta_1} \widehat{M}^+_{\alpha+\beta_1}(t-s)^{-\alpha-\beta_1},\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}\|(I-T_-(-h_0))(S_-)^\alpha T_-(t+h_0-s)\|_{\mathcal{L}(\mathcal{Y})} \le & C^-_{\beta_2} h_0^{\beta_2} \|(S_-)^{\alpha+\beta_2} T_-(t+h_0-s)\|_{\mathcal{L}(\mathcal{Y})} \\ \le & C^-_{\beta_2} h_0^{\beta_2} \widehat{M}^-_{\alpha+\beta_2}(s-t-h_0)^{-\alpha-\beta_2}.\end{aligned} \tag{3.11}$$

Then,

$$\begin{aligned}
&\|\mathcal{F}_\alpha(t+h_0)-\mathcal{F}_\alpha(t)\|_{\mathcal{X}} \\
\le& \|\int_{t_1}^{t} (T_+(h_0)-I)(-S_+)^\alpha T_+(t-s)F(S^{-\alpha}u(s))ds\|_{\mathcal{X}} \\
&+ \|\int_{t}^{t+h_0} (-S_+)^\alpha T_+(t+h_0-s)F(S^{-\alpha}u(s))ds\|_{\mathcal{X}} \\
\le& C^+_{\beta_1}\widehat{M}^+_{\alpha+\beta_1}\overline{N}_1 h_0^{\beta_1}\int_{t_1}^{t}(t-s)^{-\alpha-\beta_1}ds + \widehat{M}^+_{\alpha}\overline{N}_1\int_{t}^{t+h_0}(t+h_0-s)^{-\alpha}ds \\
=& \frac{C^+_{\beta_1}\widehat{M}^+_{\alpha+\beta_1}\overline{N}_1}{1-\alpha-\beta_1}(t-t_1)^{1-\alpha-\beta_1}h_0^{\beta_1} + \frac{\widehat{M}^+_{\alpha}\overline{N}_1}{1-\alpha}h_0^{1-\alpha} \\
\le& \left(\frac{C^+_{\beta_1}\widehat{M}^+_{\alpha+\beta_1}}{1-\alpha-\beta_1}(t_2-t_1)^{1-\alpha-\beta_1} + \frac{\widehat{M}^+_{\alpha}}{1-\alpha}\right)\overline{N}_1 h_0^{\beta_1}
\end{aligned} \tag{3.12}$$

and

$$
\begin{aligned}
&\|\mathcal{G}_\alpha(t+h_0)-\mathcal{G}_\alpha(t)\|_{\mathcal{Y}}\\
\le&\|\int_{t+h_0}^{t_2}(T_-(-h_0)-I)(S_-)^\alpha T_-(t+h_0-s)G(S^{-\alpha}u(s))ds\|_{\mathcal{Y}}\\
&+\|\int_t^{t+h_0}(S_-)^\alpha T_-(t-s)G(S^{-\alpha}u(s))ds\|_{\mathcal{Y}}\\
\le&C^-_{\beta_2}\widehat{M}^-_{\alpha+\beta_2}\overline{N}_2h_0^{\beta_2}\int_{t+h_0}^{t_2}(s-t-h_0)^{-\alpha-\beta_2}ds+\widehat{M}^-_\alpha\overline{N}_2\int_t^{t+h_0}(s-t)^{-\alpha}ds\\
=&\frac{C^-_{\beta_2}\widehat{M}^-_{\alpha+\beta_2}\overline{N}_2}{1-\alpha-\beta_2}(t_2-t-h_0)^{1-\alpha-\beta_2}h_0^{\beta_2}+\frac{\widehat{M}^-_\alpha\overline{N}_2}{1-\alpha}h_0^{1-\alpha}\\
\le&\left(\frac{C^-_{\beta_2}\widehat{M}^-_{\alpha+\beta_2}}{1-\alpha-\beta_2}(t_2-t_1)^{1-\alpha-\beta_2}+\frac{\widehat{M}^-_\alpha}{1-\alpha}\right)\overline{N}_2h_0^{\beta_2}.
\end{aligned}
\tag{3.13}
$$

If $\alpha=0$, we have

$$
\begin{aligned}
&\|\mathcal{F}_0(t+h_0)-\mathcal{F}_0(t)\|_{\mathcal{X}}\\
\le&\|\int_{t_1}^{t}ds\int_{t-s}^{t+h_0-s}S_+T_+(\tau)F(u(s))d\tau\|_{\mathcal{X}}+\|\int_t^{t+h_0}T_+(t+h_0-s)F(u(s))ds\|_{\mathcal{X}}\\
\le&\widehat{M}^+_1\overline{N}_1\int_{t_1}^{t}ds\int_{t-s}^{t+h_0-s}\frac{1}{\tau}d\tau+\widehat{M}^+_0\overline{N}_1h_0\\
\le&\widehat{M}^+_1\overline{N}_1\int_{t_1}^{t}\frac{ds}{(t-s)^{\beta_1}}\int_{t-s}^{t+h_0-s}\frac{1}{\tau^{1-\beta_1}}d\tau+\widehat{M}^+_0\overline{N}_1h_0\\
\le&\widehat{M}^+_1\overline{N}_1\int_{t_1}^{t}\frac{ds}{(t-s)^{\beta_1}}\int_0^{h_0}\frac{1}{\tau^{1-\beta_1}}d\tau+\widehat{M}^+_0\overline{N}_1h_0\\
\le&\left(\frac{\widehat{M}^+_1(t_2-t_1)^{\beta_1}}{\beta_1(1-\beta_1)}+\widehat{M}^+_0\right)\overline{N}_1h_0^{\beta_1}
\end{aligned}
\tag{3.14}
$$

and

$$
\begin{aligned}
&\|\mathcal{G}_0(t+h_0)-\mathcal{G}_0(t)\|_{\mathcal{Y}}\\
\le&\|\int_{t+h_0}^{t_2}ds\int_{s-t-h_0}^{s-t}-S_-T_-(-\tau)G(u(s))d\tau\|_{\mathcal{Y}}+\|\int_t^{t+h_0}T_-(t-s)G(u(s))ds\|_{\mathcal{Y}}\\
\le&\widehat{M}^-_1\overline{N}_2\int_{t+h_0}^{t_2}ds\int_{s-t-h_0}^{s-t}\frac{1}{\tau}d\tau+\widehat{M}^-_0\overline{N}_2h_0\\
\le&\widehat{M}^-_1\overline{N}_2\int_{t+h_0}^{t_2}\frac{ds}{(s-t-h_0)^{\beta_2}}\int_{s-t-h_0}^{s-t}\frac{1}{\tau^{1-\beta_2}}d\tau+\widehat{M}^-_0\overline{N}_2h_0\\
\le&\widehat{M}^-_1\overline{N}_2\int_{t+h_0}^{t_2}\frac{ds}{(s-t-h_0)^{\beta_2}}\int_0^{h_0}\frac{1}{\tau^{1-\beta_2}}d\tau+\widehat{M}^-_0\overline{N}_2h_0\\
\le&\left(\frac{\widehat{M}^-_1(t_2-t_1)^{\beta_2}}{\beta_2(1-\beta_2)}+\widehat{M}^-_0\right)\overline{N}_2h_0^{\beta_2}.
\end{aligned}
\tag{3.15}
$$

Therefore, (3.6) holds by the estimations (3.12) and (3.14), and (3.7) follows from the estimations (3.13) and (3.15).

Hence, Ψ maps $\mathcal{E}$ into itself. In the following, we show that Ψ is a contraction map on $\mathcal{E}$, which prove that Ψ has a unique fixed point on $\mathcal{E}$.

Indeed, for $u, v \in \mathcal{E}$,

$$\begin{aligned}
&\|(\Psi u)(t) - (\Psi v)(t)\|_{\mathcal{Z}} \\
=& \max\left\{ \| \int_{t_1}^{t} (-S_+)^{\alpha} T_+(t-s)[F(S^{-\alpha}u(s)) - F(S^{-\alpha}v(s))]ds\|_{\mathcal{X}}, \right. \\
& \left. \| \int_{t}^{t_2} (S_-)^{\alpha} T_-(t-s)[G(S^{-\alpha}u(s)) - G(S^{-\alpha}v(s))]ds\|_{\mathcal{Y}} \right\} \\
\le& \max\left\{ \frac{\widehat{M_\alpha^+}\|F\|_{0,1}}{1-\alpha}(t-t_1)^{1-\alpha}, \frac{\widehat{M_\alpha^-}\|G\|_{0,1}}{1-\alpha}(t_2-t)^{1-\alpha} \right\} d_{\mathcal{E}}(u,v).
\end{aligned}$$

By induction, we have

$$d_{\mathcal{E}}(\Psi^n u, \Psi^n v) \le R(n) d_{\mathcal{E}}(u,v),$$

where $R(n) = \frac{1}{n!}\left(\frac{\max\{\widehat{M_\alpha^+}\|F\|_{0,1}, \widehat{M_\alpha^-}\|G\|_{0,1}\}(t_2-t_1)^{1-\alpha}}{1-\alpha}\right)^n$. $R(n) < 1$ if n is large enough, one can apply the extension of the contraction mapping theorem to Ψ on $\mathcal{E}$ to obtain that there exists a unique fixed point $u \in \mathcal{E}$ of the mapping Ψ, that is, $\Psi u = u$.

From (3.8) and the fact that the composition of C^{0,β_3} functions is a C^{0,β_3} function, we know that $F(S^{-\alpha}u(t))$ and $G(S^{-\alpha}u(t))$ are uniformly β_3-Hölder continuous in t on (t_1, t_2). Let we consider the following linear non-homogeneous system for $t \in (t_1, t_2)$:

$$\begin{cases} \frac{dx(t)}{dt} = S_+ x(t) + f(t), \\ \frac{dy(t)}{dt} = S_- y(t) + g(t), \\ x(t_1) = x_1, y(t_2) = y_2, \end{cases} \tag{3.16}$$

where $f(t) = F(S^{-\alpha}u(t))$ and $g(t) = G(S^{-\alpha}u(t))$ are continuous in t. By [23, Corollary 4.3.3], the linear non-homogeneous system (3.16) has a unique solution pair (x, y), where $x \in C^0([t_1,t_2]; \mathcal{X}) \cap C^1((t_1,t_2); \mathcal{X})$ and $y \in C^0([t_1,t_2]; \mathcal{Y}) \cap C^1((t_1,t_2); \mathcal{Y})$ is given respectively by

$$x(t) = T_+(t-t_1)x_1 + \int_{t_1}^{t} T_+(t-s)F(S^{-\alpha}u(s))\, ds, \tag{3.17}$$

and

$$y(t) = T_-(t-t_2)y_2 - \int_{t}^{t_2} T_-(t-s)G(S^{-\alpha}u(s))\, ds. \tag{3.18}$$

It remains to show that

$$z(t) = x(t) + y(t) = S^{-\alpha}u(t), \quad t \in (t_1, t_2), \tag{3.19}$$

which proves that the function $z(t)$ is a dichotomous solution of the system (1.4), and $z \in C^0((t_1,t_2); \mathcal{Z}_\alpha) \cap C^1((t_1,t_2); \mathcal{Z}_\alpha)$. Especially, $z \in C^0([t_1,t_2]; \mathcal{Z}) \cap C^1((t_1,t_2); \mathcal{Z})$ is obvious when $\alpha = 0$.

Indeed, for $t \in (t_1, t_2)$, each term of (3.17) and (3.18) is in $\mathcal{D}(S)$ and is also in $\mathcal{D}(S^\alpha)$, then operating on both sides of (3.17) and (3.18) with S^α and adding them we obtain

$$\begin{aligned} S^\alpha(x(t)+y(t)) =& T_+(t-t_1)(-S)^\alpha x_1 + \int_{t_1}^t (-S)^\alpha T_+(t-s)F((-S)^{-\alpha}u(s))\,ds \\ &+ T_-(t-t_2)(-S)^\alpha y_2 - \int_t^{t_2} (-S)^\alpha T_-(t-s)G((-S)^{-\alpha}u(s))\,ds \\ =& u(t). \end{aligned} \tag{3.20}$$

This proves the formula (3.19).

Concerning the continuity of x with values in $\mathcal{X}_\alpha$ for $\alpha \in (0,1)$ up to $t = t_1$, from [19, Lemma 7.1.1], the function $t \mapsto x(t) - T_+(t-t_1)x_1$ belongs to $C^0([t_1,t_2],\mathcal{X}_\alpha)$, while $t \mapsto T_+(t-t_1)x_1$ belongs to $C^0([t_1,t_2],\mathcal{X}_\alpha)$ if and only if $x_1 \in \overline{\mathcal{D}(S_+)}^\alpha$. Therefore, $x \in C^0([t_1,t_2],\mathcal{X}_\alpha)$ if and only if $x_1 \in \overline{\mathcal{D}(S_+)}^\alpha$, where $\overline{\mathcal{D}(S_+)}^\alpha$ is the closure of $\mathcal{D}(S_+)$ in $\mathcal{X}_\alpha$. The similar argument follows the continuity of y with values in $\mathcal{Y}_\alpha$ for $\alpha \in (0,1)$ up to $t = t_2$. Thus, $z \in C^0([t_1,t_2],\mathcal{Z}_\alpha)$ if and only if $\widetilde{z} \in \overline{\mathcal{D}(S)}^\alpha$.

At last, the uniqueness of $z(t)$ follows from the uniqueness for solution of linear inhomogeneous system (3.16) and fixed point of the mapping Ψ. □

Lemma 3.2. *(i) $z(t) \in \mathcal{Z}_\alpha$, $t \geq t_1$ is an infinitely long forward dichotomous solution of system* (1.4) *if and only if $z(t)$ satisfies the integral equation*

$$z(t) = T_+(t-t_1)x_1 + \int_{t_1}^t T_+(t-s)F(z(s))ds - \int_t^\infty T_-(t-s)G(z(s))ds \tag{3.21}$$

and $x_1 \in \overline{\mathcal{D}(S_+)}^\alpha$.

(ii) $z(t) \in \mathcal{Z}_\alpha$, $t \leq t_2$ is an infinitely long backward dichotomous solution of system (1.4) *if and only if $z(t)$ satisfies the integral equation*

$$z(t) = T_-(t-t_2)y_2 - \int_{t_2}^t T_-(t-s)G(z(s))ds + \int_{-\infty}^t T_+(t-s)F(z(s))ds \tag{3.22}$$

and $y_2 \in \overline{\mathcal{D}(S_-)}^\alpha$.

Proof.

(i) Since $z(t)$, $t \geq t_1$ is an infinitely long forward dichotomous solution of (1.4), by the definition 2.3, for each $t_2 \in (t_1, \infty)$, we have

$$y(t) = T_-(t-t_2)y_2 - \int_t^{t_2} T_-(t-s)G(z(s))ds.$$

Let $t_2 \to \infty$, from the estimation (2.2), we obtain

$$y(t) = -\int_t^\infty T_-(t-s)G(z(s))ds.$$

Since

$$x(t) = T_+(t-t_1)x_1 + \int_{t_1}^t T_+(t-s)F(z(s))ds,$$

then from $z(t) = x(t) + y(t)$, we obtain (3.21). By Lemma 3.1, $x_1 \in \overline{\mathcal{D}(S_+)}^{\alpha}$ is obvious.

For converse part, Let $z(t) = x(t) + y(t)$, where

$$x(t) = T_+(t - t_1)x_1 + \int_{t_1}^{t} T_+(t - s)F(z(s))ds,$$

and

$$y(t) = -\int_{t}^{\infty} T_-(t - s)G(z(s))ds.$$

Since the right side of (3.21) is continuously differentiable, differentiate (3.21) with respect to t, we will have

$$\begin{cases} \frac{dx(t)}{dt} = S_+x(t) + F(z(t)), \\ \frac{dy(t)}{dt} = S_-y(t) + G(z(t)). \end{cases}$$

Moreover, for any $t_2 \in (t_1, \infty)$,

$$\begin{aligned} y(t) =& - T_-(t - t_2)\int_{t_2}^{\infty} T_-(t_2 - s)G(z(s))ds - \int_{t}^{t_2} T_-(t - s)G(z(s))ds \\ =& T_-(t - t_2)y(t_2) - \int_{t}^{t_2} T_-(t - s)G(z(s))ds. \end{aligned}$$

From above, $x(t_1) = x_1, y(t_2) = y_2$. In particular, for each $[\widehat{t_1}, t_2] \subset [t_1, \infty)$, $z(t)$ satisfies the Definition 2.2. The statement follows.

(ii) Evidenced by the same token for the case that $z(t)$, $t \leq t_2$, is bounded. □

In the following, set

$$\theta^+_{k,\gamma} := M^+_\alpha \|F\|_{k,\gamma} \int_0^\infty \mu^{-\alpha} e^{\zeta\mu} d\mu, \quad \theta^-_{k,\gamma} := M^-_\alpha \|G\|_{k,\gamma} \int_0^\infty \mu^{-\alpha} e^{\zeta\mu} d\mu,$$

for $\alpha \in (0, 1)$ and $\zeta < 0$. Note that $\theta^+_{k,\gamma}, \theta^-_{k,\gamma} < \infty$ since $\int_0^\infty \mu^{-\alpha} e^{\zeta\mu} d\mu$ converges. Moreover, take $\beta \in (-\eta, 0)$, where $\eta = \min\{-\beta_+, \beta_-\}$.

Lemma 3.3. *Assume that* $\max\{\theta^+_{0,1}, \theta^-_{0,1}\} < 1$ *holds. Then, for each* $x_1 \in \mathcal{X}_\alpha$, *the integral equation* (3.21) *has a unique solution* $z(t)$ *in* $C_\beta([t_1, \infty), \mathcal{Z}_\alpha)$ *such that* $\|z\|_{C_\beta} < \infty$.

Proof. Fix $x_1 \in \mathcal{X}_\alpha$, and define the operator T_{x_1} on the Banach space $C_\beta([t_1, \infty), \mathcal{Z}_\alpha)$ as

$$(T_{x_1}z)(t) = T_+(t-t_1)x_1 + \int_{t_1}^{t} T_+(t-s)F(z(s))ds - \int_{t}^{\infty} T_-(t-s)G(z(s))ds \quad (3.23)$$

for $t \geq t_1$. Obviously, $(T_{x_1}z)(t) \in \mathcal{Z}_\alpha$ is a continuous function in t for all $z \in C_\beta([t_1, \infty), \mathcal{Z}_\alpha)$. We shall show that T_{x_1} is a contraction in $C_\beta([t_1, \infty), \mathcal{Z}_\alpha)$, then the contraction mapping theorem yields the integral equation (3.21) has a unique solution $z(t)$ in $C_\beta([t_1, \infty), \mathcal{Z}_\alpha)$ such that $\|z\|_{C_\beta} < \infty$.

Since $F(0)=G(0)=0$, we have

$$\|(T_{x_1}z)(t)\|_\alpha^{\mathcal{Z}} \leq \max\big\{M_0^+e^{\beta_+(t-t_1)}\|x_1\|_\alpha^{\mathcal{X}}+\int_{t_1}^t M_\alpha^+\|F\|_{0,1}(t-s)^{-\alpha}e^{\beta_+(t-s)}\|z(s)\|_\alpha^{\mathcal{Z}}ds,$$
$$\int_t^\infty M_\alpha^-\|G\|_{0,1}(s-t)^{-\alpha}e^{-\beta_-(s-t)}\|z(s)\|_\alpha^{\mathcal{Z}}ds\big\}.$$

Then,

$$\begin{aligned}\|T_{x_1}z\|_{C_\beta} \leq & \sup_{t\geq t_1}\max\big\{M_0^+e^{(\beta_+-\beta)t}e^{-\beta_+t_1}\|x_1\|_\alpha^{\mathcal{X}} \\ & + M_\alpha^+\|F\|_{0,1}\int_{t_1}^t(t-s)^{-\alpha}e^{(\beta_+-\beta)(t-s)}\|z\|_{C_\beta}ds, \\ & M_\alpha^-\|G\|_{0,1}\int_t^\infty(s-t)^{-\alpha}e^{(\beta-\beta_-)(s-t)}\|z\|_{C_\beta}ds\big\} \\ \leq & M_0^+e^{-\beta t_1}\|x_1\|_\alpha^{\mathcal{X}}+\|z\|_{C_\beta}<\infty.\end{aligned}$$

Hence, T_{x_1} maps $C_\beta([t_1,\infty),\mathcal{Z}_\alpha)$ into itself.

Furthermore, for any $z,\widehat{z}\in C_\beta([t_1,\infty),\mathcal{Z}_\alpha)$, we have

$$\begin{aligned}\|T_{x_1}z-T_{x_1}\widehat{z}\|_{C_\beta} \leq & \sup_{t\geq 0}\max\big\{M_\alpha^+\|F\|_{0,1}\int_{t_1}^t(t-s)^{-\alpha}e^{(\beta_+-\beta)(t-s)}ds, \\ & M_\alpha^-\|G\|_{0,1}\int_t^\infty(s-t)^{-\alpha}e^{(\beta-\beta_-)(s-t)}ds\big\}\|z-\widehat{z}\|_{C_\beta} \\ < & \|z-\widehat{z}\|_{C_\beta}.\end{aligned}$$

Thus, $T_{x_1}:C_\beta([t_1,\infty),\mathcal{Z}_\alpha)\to C_\beta([t_1,\infty),\mathcal{Z}_\alpha)$ is a contraction. The proof is complete. $\square$

Hence, from the lemma 3.2, if $x_1\in\overline{\mathcal{D}(S_+)}^\alpha$, the fixed point $z(t)$ of T_{x_1} in the lemma 3.3 is the unique infinitely long forward dichotomous solution to system (1.4) in $C_\beta([t_1,\infty),\mathcal{Z}_\alpha)$ such that $P_+z(t_1)=x_1$. Obviously, $\|z\|_{C_\beta}<\infty$ implies $\|z(t)\|_\alpha^{\mathcal{Z}}\to 0$ as $t\to\infty$. In the following we will show that all these inifinitely long forward dichotomous solutions lie on the graph of a map $h^s:\mathcal{X}_\alpha\to\mathcal{Y}_\alpha$. Prior to this, a generalized Gronwall's inequality with singular kernel will be given, which generalizes the case $\alpha=1/2$ in [15, Lemma 6,p.33] to $\alpha=[0,1)$.

Lemma 3.4. *Let $\phi\in L^\infty(t_1,t_2)$ and $\phi\geq 0$. Assume that there exist $a>0$ and $\alpha\in[0,1)$ such that ϕ satisfies*

$$\phi(t)\leq\phi(t_1)+a\int_{t_1}^t(t-s)^{-\alpha}\phi(s)ds,\quad t\in[t_1,t_2].$$

Then there exists $K_0>1$ such that $\phi(t)\leq K_0\phi(t_1)$ on $[t_1,t_2]$.

Proof. Let $\phi^*(t):=ess\sup\limits_{s\in[t_1,t]}\phi(s)$. For $s\in[t_1,t]$, ε is small such that $1-\alpha-$

$a\varepsilon^{1-\alpha} > 0$, we have

$$\begin{aligned}\phi(s) \leq &\phi(t_1) + a\int_{t_1}^{s-\varepsilon}(s-\tau)^{-\alpha}\phi(\tau)d\tau + a\int_{s-\varepsilon}^{s}(s-\tau)^{-\alpha}\phi(\tau)d\tau \\ \leq &\phi(t_1) + \frac{a}{\varepsilon^\alpha}\int_{t_1}^{s}\phi(\tau)d\tau + \frac{a\varepsilon^{1-\alpha}}{1-\alpha}\phi^*(s) \\ \leq &\phi(t_1) + \frac{a}{\varepsilon^\alpha}\int_{t_1}^{t}\phi^*(\tau)d\tau + \frac{a\varepsilon^{1-\alpha}}{1-\alpha}\phi^*(t).\end{aligned}$$

Thus, $\left(1-\frac{a\varepsilon^{1-\alpha}}{1-\alpha}\right)\phi^*(t) \leq \phi(t_1) + \frac{a}{\varepsilon^\alpha}\int_{t_1}^{t}\phi^*(\tau)d\tau$. By the classical Gronwall's inequality, we get $\phi(t) \leq K_0\phi(t_1)$, where $K_0 = \frac{1-\alpha}{1-\alpha-a\varepsilon^{1-\alpha}}\exp\left(\frac{a(1-\alpha)}{\varepsilon^\alpha(1-\alpha)-a\varepsilon}(t_2-t_1)\right)$. □

The following lemma give the existence of global stable manifold of system (1.4).

Lemma 3.5. *Assume hypothesis (**H**-1) holds for system* (1.4) *and such that*

$$\theta_{0,1}^{-}\max\{1,\kappa\}KM_0^{+} \leq \kappa,\ \max\{\theta_{0,1}^{+}, \theta_{0,1}^{-}[1+(1+\kappa)K_1\theta_{0,1}^{+}]\} < 1$$

for some positive constants K, K_1, κ. Then there exists a unique $C^{0,1}$ global stable manifold.

Proof. Let $\kappa > 0$ be an arbitrary given number. Define a space Γ for the Lipschitz function h^s:

$$\Gamma = \{h^s | h^s \in C^{0,1}(\mathcal{X}_\alpha, \mathcal{Y}_\alpha), h^s(0) = 0, \|h^s\|_{0,1} \leq \kappa\}, \tag{3.24}$$

where $\|h^s\|_{0,1} \leq \kappa$ implies $\|h^s(x)\|_\alpha^{\mathcal{Y}} \leq \kappa\|x\|_\alpha^{\mathcal{X}}$ and $\|h^s(x) - h^s(\widehat{x})\|_\alpha^{\mathcal{Y}} \leq \kappa\|x-\widehat{x}\|_\alpha^{\mathcal{X}}$ for $x, \widehat{x} \in \mathcal{X}_\alpha$. Γ is complete metric space endowed with the induced metric $d_\Gamma(h^s, \widehat{h}^s) \triangleq \sup\limits_{x\in\mathcal{X}_\alpha}\{\|h^s(x) - \widehat{h}^s(x)\|_\alpha^{\mathcal{Y}}\}$.

Define the Lyapunov-Perron operator $\mathfrak{L}$ on the Lipschitz function h^s in Γ as follows:

$$\mathfrak{L}(h^s)(x_1) = -\int_{t_1}^{\infty} T_-(t_1-s)G(x(s), h^s(x(s)))ds, \tag{3.25}$$

where $x(t) = x(t; t_1, x_1, h^s)$ is the unique solution of the following system

$$\begin{cases}\frac{dx(t)}{dt} = S_+x(t) + F(x(t), h^s(x(t))), \quad t \geq t_1, \\ x(t_1) = x_1 \in \overline{\mathcal{D}(S_+)}^{\alpha}.\end{cases} \tag{3.26}$$

Since S_+ is the infinitesimal generator of strongly continuous and analytic semigroup $\{T_+(t)\}_{t\geq 0}$, from the hypothesis (**H**-1), $x(t)$ is well defined for all $t \geq t_1$, and $x(t)$ has the of form

$$x(t) = T_+(t-t_1)x_1 + \int_{t_1}^{t} T_+(t-s)F(x(s), h^s(x(s)))ds. \tag{3.27}$$

On the one hand, by (3.27) and Lemma 3.4, there exists a $K > 1$ such that

$$\|x(t)\|_\alpha^{\mathcal{X}} \leq KM_0^{+}e^{\beta(t-t_1)}\|x_1\|_\alpha^{\mathcal{X}}, \tag{3.28}$$

Then the integral on the right side of (3.25) converges. Indeed, it follows from

$$\begin{aligned}\|\mathfrak{L}(h^s)(x_1)\|_\alpha^{\mathcal{Y}} &\le \int_{t_1}^{\infty} M_\alpha^- \|G\|_{0,1} \max\{1,\kappa\}(s-t_1)^{-\alpha} e^{-\beta_-(s-t_1)} \|x(s)\|_\alpha^{\mathcal{X}} ds \\ &\le \theta_{0,1}^- \max\{1,\kappa\} K M_0^+ \|x_1\|_\alpha^{\mathcal{X}}.\end{aligned} \tag{3.29}$$

On the other hand, let $x_1, \widehat{x}_1 \in \overline{\mathcal{D}(S_+)}^\alpha$, and set $x(t) = x(t; t_1, x_1, h^s)$, $\widehat{x}(t) = x(t; t_1, \widehat{x}_1, h^s)$. From (2.2) and (2.3), we have

$$\begin{aligned}\|x(t) - \widehat{x}(t)\|_\alpha^{\mathcal{X}} \le & M_0^+ e^{\beta_+(t-t_1)} \|x_1 - \widehat{x}_1\|_\alpha^{\mathcal{X}} \\ & + \max\{1,\kappa\} M_\alpha^+ \|F\|_{0,1} \int_{t_1}^{t} (t-s)^{-\alpha} \|x(s) - \widehat{x}(s)\|_\alpha^{\mathcal{X}} ds.\end{aligned}$$

By the lemma 3.4, it yields that

$$\|x(t) - \widehat{x}(t)\|_\alpha^{\mathcal{X}} \le K M_0^+ e^{\beta(t-t_1)} \|x_1 - \widehat{x}_1\|_\alpha^{\mathcal{X}}. \tag{3.30}$$

In addition, we use the notation $x(t, h^s)$ to signifies the dependence of $x(t)$ on h^s. For $h^s, \widehat{h}^s \in \Gamma$,

$$\begin{aligned}&\|x(t,h^s) - x(t,\widehat{h}^s)\|_\alpha^{\mathcal{X}} \\ \le& \int_{t_1}^{t} \|T_+(t-s)\|_{\mathcal{L}(\mathcal{X},\mathcal{X}_\alpha)} \big[\|F(x(s,h^s), h^s(x(s,h^s))) - F(x(s,h^s), \widehat{h}^s(x(s,h^s)))\|_{\mathcal{X}} \\ &+ \|F(x(s,h^s), \widehat{h}^s(x(s,h^s))) - F(x(s,h^s), \widehat{h}^s(x(s,\widehat{h}^s)))\|_{\mathcal{X}} \\ &+ \|F(x(s,h^s), \widehat{h}^s(x(s,\widehat{h}^s))) - F(x(s,\widehat{h}^s), \widehat{h}^s(x(s,\widehat{h}^s)))\|_{\mathcal{X}}\big] ds \\ \le& \int_{t_1}^{t} M_\alpha^+ \|F\|_{0,1} (t-s)^{-\alpha} e^{\beta_+(t-s)} \big[d_\Gamma(h^s, \widehat{h}^s) + (1+\kappa)\|x(s,h^s) - x(s,\widehat{h}^s)\|_\alpha^{\mathcal{X}}\big] ds,\end{aligned}$$

then by Lemma 3.4, there exists a constant $K_1 > 1$ such that

$$\|x(t,h^s) - x(t,\widehat{h}^s)\|_\alpha^{\mathcal{X}} \le K_1 \theta_{0,1}^+ d_\Gamma(h^s, \widehat{h}^s). \tag{3.31}$$

If h^s is a fixed point of $\mathfrak{L}$ in Γ, then the graph of h^s is the global stable manifold. In the follows, we prove $\mathfrak{L}$ is a contraction map in Γ. First we show that $\mathfrak{L}(\Gamma) \subset \Gamma$.

Choose $\|G\|_{0,1}$ such that $\theta_{0,1}^- \max\{1,\kappa\} K M_0^+ \le \kappa$, by (3.29), we have $\|\mathfrak{L}(h^s)(x_1)\|_\alpha^{\mathcal{Y}} \le \kappa \|x_1\|_\alpha^{\mathcal{X}}$. Furthermore, we have

$$\begin{aligned}&\|\mathfrak{L}(h^s)(x_1) - \mathfrak{L}(h^s)(\widehat{x}_1)\|_\alpha^{\mathcal{Y}} \\ \le& \int_{t_1}^{\infty} M_\alpha^- \|G\|_{0,1} \max\{1,\kappa\} (s-t_1)^{-\alpha} e^{-\beta_-(s-t_1)} \|x(s) - \widehat{x}(s)\|_\alpha^{\mathcal{X}} ds \le \kappa \|x_1 - \widehat{x}_1\|_\alpha^{\mathcal{X}}.\end{aligned}$$

Besides, since $h^s(0) = 0$ and $G(0) = 0$, from the (3.25), $\mathfrak{L}(h^s)(x_1) \in \mathcal{Y}_\alpha$ and $\mathfrak{L}(h^s)(0) = 0$ are obvious. Thus, $\mathfrak{L}(\Gamma) \subset \Gamma$.

Furthermore, for $h^s, \widehat{h}^s \in \Gamma$, by (3.31) and assume $\theta_{0,1}^-[1 + (1+\kappa)K_1\theta_{0,1}^+] < 1$, we obtain

$$
\begin{aligned}
&\|\mathfrak{L}(h^s)(x_1)-\mathfrak{L}(\widehat{h}^s)(x_1)\|_\alpha^{\mathcal{Y}}\\
\le&\int_{t_1}^{t}\|T_-(t_1-s)\|_{\mathcal{L}(\mathcal{Y},\mathcal{Y}_\alpha)}\big[\|G(x(s,h^s),h^s(x(s,h^s)))-G(x(s,h^s),\widehat{h}^s(x(s,h^s)))\|_{\mathcal{Y}}\\
&+\|G(x(s,h^s),\widehat{h}^s(x(s,h^s)))-G(x(s,h^s),\widehat{h}^s(x(s,\widehat{h}^s)))\|_{\mathcal{Y}}\\
&+\|G(x(s,h^s),\widehat{h}^s(x(s,\widehat{h}^s)))-G(x(s,\widehat{h}^s),\widehat{h}^s(x(s,\widehat{h}^s)))\|_{\mathcal{Y}}\big]ds\\
\le&\int_{t_1}^{t}M_\alpha^+\|G\|_{0,1}(s-t_1)^{-\alpha}e^{-\beta_-(s-t_1)}\big[d_\Gamma(h^s,\widehat{h}^s)+(1+\kappa)\|x(s,h^s)-x(s,\widehat{h}^s)\|_\alpha^{\mathcal{X}}\big]ds\\
<&d_\Gamma(h^s,\widehat{h}^s).
\end{aligned}
\tag{3.32}
$$

Hence, $\mathfrak{L}$ is a contraction map in Γ, then Banach fixed point theorem follows that there exists a unique fixed point h^s of $\mathfrak{L}(h^s)=h^s$ in Γ. From the lemma 3.3, (3.27) and (3.25), all infinitely long forward dichotomous solutions of the system (1.4) with $\|z\|_{C_\beta}<\infty$ are contained in the graph of h^s defined by

$$
h^s(x_1)=-\int_{t_1}^{\infty}T_-(t_1-s)G(x(s),h^s(x(s)))ds. \tag{3.33}
$$

This means that $W^s(0)=\{(x(t),y(t)):y(t)=h^s(x(t)),x(t)\in\mathcal{X}_\alpha\}$.

To prove that $W^s(0)=\{(x(t),y(t)):y(t)=h^s(x(t)),x(t)\in\mathcal{X}_\alpha\}$ is $C^{0,1}$ global stable manifold, it remains to prove the invariance of $W^s(0)$. Let $x(t),t\ge t_1$, be a solution for (3.26), $(x_1,h^s(x_1))\in W^s(0)$ then denote $y(t):=h^s(x(t))$ for $t\ge t_1$. This defines a curve $(x(t),y(t))\in W^s(0)$, $t\ge t_1$, through the point $(x_1,h^s(x_1))\in W^s(0)$, it suffices to prove $y(t)$ satisfies

$$
\frac{dy(t)}{dt}=S_-y(t)+G(x(t),h^s(x(t))),\ t\in[t_1,t_2]. \tag{3.34}
$$

for all $t_2\in(t_1,\infty)$. And the equation (3.34) indeed has a unique solution $y(t)$ which remains bounded as $t_2\to\infty$, namely

$$
y(t)=-\int_t^{\infty}T_-(t-s)G(x(s),h^s(x(s)))ds.
$$

Thus, $z(t)=x(t)+h^s(x(t))$ is the unique infinitely long forward dichotomous solution of the system (1.4) with $z(t_1)=x_1+h^s(x_1)$.

The proof is complete. □

In the following, we shall focus on the smoothness of h^s obtained in Lemma 3.5.

Lemma 3.6. *For the ill-posed system* (1.4)*, assume in addition to the hypothesises of Lemma 3.5 that Hypothesis (**H**-2) holds for $k=1$ and $\gamma=0$, and* $(1+\kappa)K_2\theta_{1,0}^-\theta_{1,0}^+<1$ *for some positive constants K_2. Then there exists a unique C^1 global stable manifold.*

Proof. By Lemma 3.5, we obtain a unique $C^{0,1}$ global stable manifold characterized by the graph of $h^s\in\Gamma$, where h^s and Γ refers to (3.33) and (3.24) respectively. we shall proceed to prove that h^s is C^1 provided F and G are C^1 and $(1+\kappa)K_2\theta_{1,0}^-\theta_{1,0}^+<1$ for some positive constants K_2.

Defined

$$\lambda(h^s,x_0)=\limsup_{(x_1,\Delta)\to(x_0,0)}\frac{\|h^s(x_1+\Delta)-h^s(x_1)-h^s(x_0+\Delta)+h^s(x_0)\|_\alpha^{\mathcal{Y}}}{\|\Delta\|_\alpha^{\mathcal{X}}}. \tag{3.35}$$

By Lemma 2.2, we need to prove $\lambda(h^s,x_0)=0$ for every $x_0\in\overline{\mathcal{D}(S_+)}^\alpha$.

Here we use the notation $x(t,x_1,h^s)$ to represent the solution of (3.26). By (3.27) and the Taylor expression, we have

$$\begin{aligned}
&x(t,x_1+\Delta,h^s)-x(t,x_1,h^s)-x(t,x_0+\Delta,h^s)+x(t,x_0,h^s)\\
=&\int_{t_1}^t T_+(t-s)\big\{D_xF(x(s,x_0,h^s),h^s(x(s,x_0,h^s)))[x(s,x_1+\Delta,h^s)-x(s,x_1,h^s)\\
&-x(s,x_0+\Delta,h^s)+x(s,x_0,h^s)]+D_{h^s}F(x(s,x_0,h^s),h^s(x(s,x_0,h^s)))\\
&[h(x(s,x_1+\Delta,h^s))-h(x(s,x_1,h^s))-h(x(s,x_0+\Delta,h^s))+h(x(s,x_0,h^s))]\\
&+\mathfrak{R}_2(x)\big\}ds
\end{aligned} \tag{3.36}$$

where $\mathfrak{R}_2(x)$ represents the sum of higher order Taylor expressions of F in (3.36) at the point $(x(s,x_0,h^s),h^s(x(s,x_0,h^s)))$.

By Lemma 3.4, there exists a $K_2>1$ such that

$$\begin{aligned}
&\|x(t,x_1+\Delta,h^s)-x(t,x_1,h^s)-x(t,x_0+\Delta,h^s)+x(t,x_0,h^s)\|_\alpha^{\mathcal{X}}\\
&\qquad\le K_2[\theta_{1,0}^+\|\Delta\|_\alpha^{\mathcal{X}}\sup_{t\ge t_1}[\lambda(h^s,x(t,x_0,h^s))]+o(\|\Delta\|_\alpha^{\mathcal{X}})]
\end{aligned} \tag{3.37}$$

as $(x_1,\Delta)\to(x_0,0)$.

Then, similarly as (3.36), by (3.30), (3.33), (3.37) and the Taylor expression,

$$\begin{aligned}
&\|h^s(x_1+\Delta)-h^s(x_1)-h^s(x_0+\Delta)+h^s(x_0)\|_\alpha^{\mathcal{Y}}\\
\le&\int_{t_1}^\infty M_\alpha^-\|G\|_1(s-t_1)^{-\alpha}e^{-\beta_-(s-t_1)}(1+\kappa)\\
&\times\big\{\|x(s,x_1+\Delta,h^s)-x(s,x_1,h^s)-x(s,x_0+\Delta,h^s)+x(s,x_0,h^s)\|_\alpha^{\mathcal{X}}\\
&+2\kappa\|x(s,x_0+\Delta,h^s)-x(s,x_0,h^s)\|_\alpha^{\mathcal{X}}\big\}ds+o(\|\Delta\|_\alpha^{\mathcal{X}})\\
\le&(1+\kappa)K_2\theta_{1,0}^+\theta_{1,0}^-\|\Delta\|_\alpha^{\mathcal{X}}\sup_{t\ge t_1}[\lambda(h^s,x(t,x_0,h^s))]+o(\|\Delta\|_\alpha^{\mathcal{X}})
\end{aligned} \tag{3.38}$$

as $(x_1,\Delta)\to(x_0,0)$, and it yields

$$\lambda(h^s,x_0)\le(1+\kappa)K_2\theta_{1,0}^+\theta_{1,0}^-\sup_{t\ge t_1}\lambda(h^s,x(t,x_0,h^s))<\infty. \tag{3.39}$$

Because $x(t,x(s,x_0,h^s),h^s)=x(t+s,x_0,h^s)$ for $t+s\ge t_1$, we have

$$\sup_{t\ge t_1}\lambda(h^s,x(t,x_0,h^s))\le(1+\kappa)K_2\theta_{1,0}^+\theta_{1,0}^-\sup_{t\ge t_1}\lambda(h^s,x(t,x_0,h^s)).$$

Since $(1+\kappa)K_2\theta_{1,0}^+\theta_{1,0}^-<1$, then $\sup_{t\ge t_1}\lambda(h^s,x(t,x_0,h^s))=0$. By (3.39), $\lambda(h^s,x_0)=0$. Thus, h^s is C^1. Moreover, by (3.33), $Dh^s(0)=0$.

The proof is complete. □

Remark 3.1. With the same arguments as Lemma 3.6 and Lemma 2.2, the conclusion of Lemma 3.6 can be improved to the $C^k(k \geq 2)$ case provided in addition to the existence of $C^{k-1,1}$ global stable manifold that F and G are C^k along with sufficiently small $\|F\|_k$ and $\|G\|_k$. While we omit the details and shift attention to the $C^{k,\gamma}(k \geq 1, \gamma \in (0,1])$ smoothness later.

Lemma 3.7. *For the ill-posed system* (1.4)*, assume in addition to the hypothesises of Lemma 3.5 that $\beta_+ < (1+\gamma)\beta$ and hypothesis (**H**-2) holds for $\gamma \in (0,1]$ with sufficiently small $\|F\|_{k,\gamma}$ and $\|G\|_{k,\gamma}$. Then there exists a unique $C^{k,\gamma}$ global stable manifold.*

Proof. Set $\Gamma_k = \{h^s \in \Gamma \cap C^{k,\gamma}(\mathcal{X}_\alpha, \mathcal{Y}_\alpha) : Dh^s(0) = 0, \|h^s\|_{k,\gamma} \leq \kappa\}$ for $k \geq 1$, where Γ refers to (3.24). By Lemma 3.5, we obtain a unique $C^{0,1}$ global stable manifold characterized by the graph of $h^s \in \Gamma$, where h^s refers to (3.33). In the following, we shall continue to prove h^s is $C^{k,\gamma}$ for $\gamma \in (0,1]$. From (3.25) and Lemma 2.3, it suffices to show $\mathfrak{L}(\Gamma_k) \subset \Gamma_k$ in the C^0 norm.

Step I: Prior to this, we need to prove that $x(t)$ defined by (3.27) is $C^{k,\gamma}$ in x_1 and satisfies the estimates on the derivatives up to order k and Hölder derivatives of $D^k_{x_1}x(t)$.

Now we define the space

$$L_\beta = \Big\{\phi : \mathbb{R} \times \overline{\mathcal{D}(S_+)}^\alpha \to \mathcal{X}_\alpha | \phi(t_1, \xi) = \xi, \forall \xi \in \overline{\mathcal{D}(S_+)}^\alpha, \phi(\cdot, \xi) \in C^1(\mathbb{R}),$$
$$\phi(t,\cdot) \in C^{k,\gamma}\big(\overline{\mathcal{D}(S_+)}^\alpha\big), \|\phi(t,\cdot)\|_{C^0(\mathcal{X}_\alpha, \mathcal{X}_\alpha)} \leq KM_0^+ e^{\beta(t-t_1)},$$
$$\|D^i_\xi \phi(t,\cdot)\|_{C^0(\mathcal{X}_\alpha, \mathcal{X}_\alpha)} \leq KM_0^+ e^{\beta(t-t_1)}, \ i = 1, \cdots, k,$$
$$H_\gamma(D^k_\xi \phi(t,\cdot)) \leq KM_0^+ e^{\beta(t-t_1)}\Big\}, \tag{3.40}$$

where K, M_0^+ refer to (3.28). Obviously, L_β is a complete metric space with respect to the induced metric $d_L(\phi, \widehat{\phi}) \triangleq \|\phi - \widehat{\phi}\|_{C_\beta}$.

For any $x \in L_\beta$, define

$$[\mathfrak{T}x](t, x_1) = T_+(t-t_1)x_1 + \int_{t_1}^t T_+(t-s)F(x(s,x_1), h^s(x(s,x_1)))ds. \tag{3.41}$$

To prove that $x(t)$ which defined by (3.27) is $C^{k,\gamma}$ in x_1 provided $h^s \in \Gamma_k$, it suffices to prove $\mathfrak{T}$ is a contraction map in L_β. We first prove that $\mathfrak{T}(L_\beta) \subset L_\beta$.

Obviously, $[\mathfrak{T}x](t_1, x_1) = x_1$. For any $x \in L_\beta$, $[\mathfrak{T}x]$ being C^1 in t and $C^{k,\gamma}$ in x_1 follow from the fact that, for any $l \in \mathbb{N}$ and $\gamma \in (0,1]$, the composition of $C^{l,\gamma}$ functions is a $C^{l,\gamma}$ function. Moreover, by (3.28), we know that $\|[\mathfrak{T}x]\|_{C^0(\mathcal{X}_\alpha, \mathcal{X}_\alpha)} \leq KM_0^+ e^{\beta(t-t_1)}$.

Differentiating (3.41) in x_1, it yields

$$\begin{aligned} D_{x_1}[\mathfrak{T}x](t,x_1) =& T_+(t-t_1) + \int_{t_1}^t T_+(t-s)\big[D_x F(s,x_1) D_{x_1} x(s,x_1) \\ &+ D_{h^s} F(s,x_1) D_x h^s(x(s,x_1)) D_{x_1} x(s,x_1)\big]ds, \end{aligned} \tag{3.42}$$

where $F(s,x_1) := F(x(s,x_1), h^s(x(s,x_1)))$. Note that the integral in (3.42) converges. Indeed, since $x \in L_\beta$ and $h^s \in \Gamma_k$, choose $\|F\|_{1,\gamma}$ so small that $\theta^+_{1,0} \leq$

$\frac{K-1}{K(1+\kappa)}$, we have

$$\begin{aligned}
&\|D_{x_1}[\mathfrak{T}x]\|_{C^0(\mathcal{X}_\alpha,\mathcal{X}_\alpha)}\\
&\leq M_0^+e^{\beta_+(t-t_1)}+\int_{t_1}^t M_\alpha^+(t-s)^{-\alpha}e^{\beta_+(t-t_1)}\|F\|_1(1+\kappa)KM_0^+e^{\beta(s-t_1)}ds\\
&\leq [1+M_\alpha^+\|F\|_1K(1+\kappa)\int_{t_1}^t(t-s)^{-\alpha}e^{(\beta_+-\beta)(t-s)}ds]M_0^+e^{\beta(t-t_1)}\\
&\leq KM_0^+e^{\beta(t-t_1)}.
\end{aligned}$$

Furthermore, for $2\leq i\leq k$,

$$\begin{aligned}
D_{x_1}^i[\mathfrak{T}x](t,x_1)=\int_{t_1}^t &T_+(t-s)\big[D_xF(s,x_1)D_{x_1}^ix(s,x_1)\\
&+D_{h^s}F(s,x_1)D_xh^s(x(s,x_1))D_{x_1}^ix(s,x_1)+\mathfrak{R}_i(s,x_1)\big]ds,
\end{aligned}\tag{3.43}$$

where $\mathfrak{R}_i(s,x_1)$ is a sum of monomials whose factors are derivatives of F and of h^s up to order i and of x up to order $i-1$. Note that (3.43) is well-defined because $[\mathfrak{T}x]$ is $C^{k,\gamma}$ in x_1 and the integral in (3.43) converges. Moreover, by choosing $\|F\|_i$ sufficiently small, we can obtain

$$\|D_{x_1}^i[\mathfrak{T}x]\|_{C^0(\mathcal{X}_\alpha,\mathcal{X}_\alpha)}\leq KM_0^+e^{\beta(t-t_1)}.$$

The only thing that remains to prove to ensure that $\mathfrak{T}(L_\beta)\subset L_\beta$ is the estimate on $H_\gamma(D_{x_1}^k[\mathfrak{T}x])$. For all $x_1,\widehat{x}_1\in\overline{\mathcal{D}(S_+)}^\alpha$, $x\in L^\beta$ and $h^s\in\Gamma_k$, we have

$$\begin{aligned}
&D_{x_1}^k[Tx](t,x_1)-D_{\widehat{x}_1}^k[Tx](t,\widehat{x}_1)\\
\leq&\int_{t_1}^t T_+(t-s)\big[D_xF(s,x_1)D_{x_1}^kx(s,x_1)-D_xF(s,\widehat{x}_1)D_{\widehat{x}_1}^kx(s,\widehat{x}_1)\\
&+D_{h^s}F(s,x_1)D_xh^s(x(s,x_1))D_{x_1}^kx(s,x_1)-D_{h^s}F(s,\widehat{x}_1)D_xh^s(x(s,\widehat{x}_1))D_{\widehat{x}_1}^kx(s,\widehat{x}_1)\\
&+\mathfrak{R}_i(s,x_1)-\mathfrak{R}_i(s,\widehat{x}_1)\big]ds.
\end{aligned}\tag{3.44}$$

Each difference terms in the right side of (3.44) contain the factors $D_{x_1}x(s,x_1)$ and $D_{\widehat{x}_1}x(s,\widehat{x}_1)$, we use the triangle inequality to estimate (3.44) in the C^0 norm and assume $\beta_+<(1+\gamma)\beta$ and $\|F\|_{k,\gamma}$ being sufficiently small, then we can obtain $H_\gamma(D^k[Tx])\leq KM_0^+e^{\beta(t-t_1)}$.

This finishes the verification of $\mathfrak{T}(L_\beta)\subset L_\beta$.

In addition, for any $x,\widehat{x}\in L_\beta$, we have

$$\begin{aligned}
&\|[\mathfrak{T}x](t,x_1)-[\mathfrak{T}\widehat{x}](t,x_1)\|_\alpha^{\mathcal{X}}\\
\leq&\|\int_{t_1}^t T_+(t-s)[F(x(s,x_1),h^s(x(s,x_1)))-F(\widehat{x}(s,x_1),h^s(\widehat{x}(s,x_1)))]\|_\alpha^{\mathcal{X}}\\
\leq&\int_{t_1}^t M_\alpha^+(t-s)^{-\alpha}e^{\beta_+(t-s)}\|F\|_1\max\{1,\kappa\}\|x(s,x_1)-\widehat{x}(s,x_1)\|_\alpha^{\mathcal{X}}ds\\
\leq&\max\{1,\kappa\}\theta_{1,0}^+e^{\beta t_1}d_L(x,\widehat{x}).
\end{aligned}\tag{3.45}$$

Since $\theta_{1,0}^+\leq\frac{K-1}{K(1+\kappa)}$, we have $\theta_{1,0}^+\leq\frac{1}{\max\{1,\kappa\}}$ then it follows that $d_L([\mathfrak{T}x],[\mathfrak{T}\widehat{x}])<d_L(x,\widehat{x})$. Thus, the contraction mapping theorem yields that $\mathfrak{T}$ has a fixed point x on L_β.

Step II: Now we prove that $\mathfrak{L}(\Gamma_k) \subset \Gamma_k$.

Since $x \in L_\beta$ and $h^s \in \Gamma_k$, $\mathfrak{L}(h^s)$ is $C^{k,\gamma}$ in x_1. This is again a result that the composition of $C^{k,\gamma}$ functions is a $C^{k,\gamma}$ functions. Differentiable $\mathfrak{L}(h^s)(x_1)$ with respect to x_1, we have

$$\begin{aligned} D_{x_1}\mathfrak{L}(h^s)(x_1) = -\int_{t_1}^{\infty} T_-(t_1 - s)\big[&D_x G(s, x_1) D_{x_1} x(s, x_1) \\ &+ D_{h^s} G(s, x_1) D_x h^s(x(s, x_1)) D_{x_1} x(s, x_1)\big] ds, \end{aligned} \tag{3.46}$$

where we use the notation $G(s, x_1)$ to signifies $G(x(s, x_1), h^s(x(s, x_1)))$. Choose $\|G\|_1$ small such that $\theta_{1,0}^- \leq \frac{\kappa}{(1+\kappa)KM_0^+}$, it yields

$$\|D_{x_1}\mathfrak{L}(h^s)\| \leq \int_{t_1}^{\infty} M_\alpha^-(s - t_1)^{-\alpha} e^{-\beta_-(s-t_1)} \|G\|_{1,\gamma}(1+\kappa)KM_0^+ e^{\beta(s-t_1)} ds \leq \kappa.$$

Besides, $D_{x_1}\mathfrak{L}(h^s)(0) = 0$.

Furthermore, for $2 \leq i \leq k$,

$$\begin{aligned} D_{x_1}^i\mathfrak{L}(h^s)(x_1) = -\int_{t_1}^{\infty} T_-(t_1 - s)\big[&D_x G(s, x_1) D_{x_1}^i x(s, x_1) \\ &+ D_{h^s} G(s, x_1) D_x h^s(x(s, x_1)) D_{x_1}^i x(s, x_1) + \widehat{\mathfrak{R}}_i(x_1)\big] ds, \end{aligned} \tag{3.47}$$

$\widehat{\mathfrak{R}}_i(x_1)$ is a sum of monomials whose factors are derivatives of G and x up to order $i-1$ and h^s up to i. Note that all the terms in $\widehat{\mathfrak{R}}_i(x_1)$ contain at least one factor which is a derivative of G. Hence, assuming $\|G\|_i$ is sufficiently small, we can obtain $\|D_{x_1}^i\mathfrak{L}(h^s)\| \leq \kappa$.

In addition, for $x_1, x_1 \in \overline{D(S_+)}^\alpha$,

$$\begin{aligned} &D_{x_1}^k[\mathfrak{L}(h^s)](x_1) - D_{\widehat{x}_1}^k[\mathfrak{L}(h^s)](\widehat{x}_1) \\ &\leq \int_{t_1}^{t} T_+(t-s)\big[D_x G(s, x_1) D_{x_1}^k x(s, x_1) - D_x G(s, \widehat{x}_1) D_{\widehat{x}_1}^k x(s, \widehat{x}_1) \\ &\quad + D_{h^s} G(s, x_1) D_x h^s(x(s, x_1)) D_{x_1}^k x(s, x_1) - D_{h^s} G(s, \widehat{x}_1) D_x h^s(x(s, \widehat{x}_1)) D_{\widehat{x}_1}^k x(s, \widehat{x}_1) \\ &\quad + \widehat{\mathfrak{R}}_i(x_1) - \widehat{\mathfrak{R}}_i(\widehat{x}_1)\big] ds. \end{aligned} \tag{3.48}$$

Each difference terms in the right side of (3.48) contain the factors $D_{x_1}x(s, x_1)$ and $D_{\widehat{x}_1}x(s, \widehat{x}_1)$, we use the triangle inequality to estimate (3.48) in the C^0 norm and assume $\beta_+ < (1+\gamma)\beta$ and $\|G\|_{k,\gamma}$ being sufficiently small, then we can obtain $H_\gamma(D_{x_1}^k[\mathfrak{L}(h^s)]) \leq \kappa$. Thus, $\mathfrak{L}(\Gamma_k) \subset \Gamma_k$.

By Lemma 2.3, Γ_k is a non-empty closed subset of $\Gamma(\subset C^0(\mathcal{X}_\alpha, \mathcal{Y}_\alpha))$ in the C^0 norm, and since $\mathfrak{L}$ has a fixed point h^s in Γ, $\mathfrak{L}(\Gamma_k) \subset \Gamma_k$ implies that the fixed point h^s of $\mathfrak{L}$ also lies in Γ_k and is therefore of class $C^{k,\gamma}$. Thus, $W^s(0)$ is the unique global stable manifold.

The proof is complete. □

We are now in the position to prove Theorem 3.1.

Proof of Theorem 3.1.

(i) From Lemma 3.1, Lemma 3.2, Lemma 3.3 and lemma 3.5, for each $x_1 \in \overline{\mathcal{D}(S_+)}^\alpha$, there is a unique point $\zeta = (x_1, h^s(x_1))$ such that system (1.4) has a unique infinitely long forward dichotomous solution $z(t;t_1,\zeta) \in C_\beta([t_1,\infty),\mathcal{Z}_\alpha)$ which defined by

$$\begin{aligned} z(t) =& T_+(t-t_1)x_1 + \int_{t_1}^{t} T_+(t-s)F(x(s),h^s(x(s)))ds \\ &- \int_t^\infty T_-(t-s)G(x(s),h^s(x(s)))ds \end{aligned} \tag{3.49}$$

for $t \geq t_1$, and $\lim_{t\to\infty} z(t;t_1,\zeta) = 0$. Then, Lemma 3.5, Lemma 3.6 and Lemma 3.7 follow that $W^s(0) = \{(x(t),y(t)) : y(t) = h^s(x(t)), x(t) \in \mathcal{X}_\alpha\}$ contains $z(t;t_1,\zeta)$, and is the unique $C^{k,\gamma}$ global stable manifold of system (1.4), where h^s is defined by (3.33). Moreover, $x(t)$ and $y(t)$ on $W^s(0)$ have the form (3.1).

(ii) In order to obtain a unique global unstable manifold for system (1.4), we begin by considering (2.4) which written in the following form:

$$\begin{cases} x(t) = e^{S_+(t-t_1)}x_1 + \int_{t_1}^{t} e^{S_+(t-s)}F(z(s))\,ds, \\ y(t) = e^{S_-(t-t_2)}y_2 - \int_t^{t_2} e^{S_-(t-s)}G(z(s))\,ds \end{cases} \tag{3.50}$$

for $-\infty < t_1 \leq t \leq t_2 < \infty$.

Let $\tau = -t, \tau_1 = -t_2$ and $\tau_2 = -t_1$. Set $\widetilde{x}(\tau) := x(-\tau), \widetilde{y}(\tau) := y(-\tau)$ and $\widetilde{z}(\tau) := z(-\tau)$. Then $z(t)(= x(t)+y(t)) : [t_1,t_2] \to \mathcal{Z}_\alpha$ is the dichotomous solution of (3.50) if and only if $\widetilde{z}(\tau)(= \widetilde{x}(\tau)+\widetilde{y}(\tau)) : [\tau_1,\tau_2] \to \mathcal{Z}_\alpha$ is the dichotomous solution of the following system

$$\begin{cases} \widetilde{x}(\tau) = e^{(-S_+)(\tau-\tau_2)}\widetilde{x}(\tau_2) - \int_\tau^{\tau_2} e^{(-S_+)(\tau-s)}(-F)(\widetilde{z}(s))\,ds, \\ \widetilde{y}(\tau) = e^{(-S_-)(\tau-\tau_1)}\widetilde{y}(\tau_1) + \int_{\tau_1}^{\tau} e^{(-S_-)(\tau-s)}(-G)(\widetilde{z}(s))\,ds \end{cases} \tag{3.51}$$

for $-\infty < \tau_1 \leq \tau \leq \tau_2 < \infty$, where $(-F)(\widetilde{z}) := -F(\widetilde{z})$ and $(-G)(\widetilde{z}) := -G(\widetilde{z})$. By Theorem 3.1(1), system (3.51) exists a $C^{k,\gamma}$ unique global stable manifold $W^s(0) = \{(\widetilde{y}(\tau),\widetilde{x}(\tau)) : \widetilde{x}(\tau) = h^s(\widetilde{y}(\tau)), \widetilde{y}(\tau) \in \mathcal{Y}_\alpha\}$, where

$$h^s(\widetilde{y}(\tau_1)) = -\int_{\tau_1}^{\infty} e^{(-S_+)(\tau_1-s)}(-F)(h^s(\widetilde{y}(s)),\widetilde{y}(s))\,ds.$$

This implies that system (1.4) has a $C^{k,\gamma}$ unique global unstable manifold $W^u(0) = \{(x(t),y(t)) : x(t) = h^u(y(t)), y(t) \in \mathcal{Y}_\alpha\}$, where

$$h^u(y_2) = \int_{-\infty}^{t_2} T_+(t_2-s)F(h^u(y(s)),y(s))\,ds.$$

Moreover, for each $y_2 \in \overline{\mathcal{D}(S_-)}^\alpha$, there is a unique point $\zeta = (h^u(y_2),y_2)$ such that system (1.4) has a unique infinitely long backward dichotomous solution $z(t;t_2,\zeta) \in C_{-\beta}((-\infty,t_2],\mathcal{Z}_\alpha)$ which defined by

$$\begin{aligned} z(t) =& T_-(t-t_2)y_2 - \int_t^{t_2} T_-(t-s)G(h^u(y(s)),y(s))\,ds \\ &+ \int_{-\infty}^{t} T_+(t-s)F(h^u(y(s)),y(s))\,ds \end{aligned}$$

for $t \leq t_2$, and $\lim_{t\to-\infty} z(t;t_2,\zeta) = 0$. In addition, $x(t)$ and $y(t)$ on $W^u(0)$ has the form (3.2).

The proof is complete. $\square$

4. Elliptic equations in infinite cylindrical domain

Consider the elliptic equation with Dirichlet boundary condition on $\partial\Omega$

$$\begin{aligned} u_{xx} + \Delta_y u + f(y, u, u_x, \nabla_y u) = 0, \quad (x, y, u) \in \mathbb{R} \times \Omega \times \mathbb{R}^m, \\ u(x, y) = 0, \quad x \in \mathbb{R}, \ y \in \partial\Omega \end{aligned} \tag{4.1}$$

in infinite cylindrical domain $\mathbb{R} \times \Omega$, where Ω is an open and bounded subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, ∇_y is the gradient in the y-variable and Δ_y is the Laplace operator in the y-variable. The function $(y, u, v, w) \longmapsto f(y, u, v, w)$ is defined in $\Omega \times \mathcal{V}$, and has value in $\mathbb{R}^m$, where $\mathcal{V} \subset \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{mn}$ is an open subset and it is also the higher order term of (u, v, w).

Assume that f is globally Lipschitz continuous with respect to (u, v, w). Precisely, we assume that there exists a $\mathcal{K}_0 > 0$ such that

$$|f(y, u_1, v_1, w_1) - f(y, u_2, v_2, w_2)|_{\mathbb{R}^m} \leq \mathcal{K}_0(|u_1 - u_2|_{\mathbb{R}^m} + |v_1 - v_2|_{\mathbb{R}^m} + |w_1 - w_2|_{\mathbb{R}^{mn}}) \tag{4.2}$$

for $(u_1, v_1, w_1), (u_2, v_2, w_2) \in \mathcal{V}$.

Based on the idea from Kirchgässner [18], we consider (4.1) as an evolution equation by treating the unbounded spatial variable x as time variable. We first transfer the problem of elliptic equation (4.1) to a abstract semilinear problem.

Let $A := -\Delta_y$. Then A is a closed operator on $X := L^2(\Omega)$ with dense domain $X^1 := \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$, and it is positive, symmetric with compact inverse. Moreover, $\sigma(A) = \{\lambda_n : n \in \mathbb{Z}_+\}$ is a discrete set that $\lambda_n \geq \lambda_{n-1} > 0$ and $\lambda_n \to \infty$ as $n \to \infty$, the corresponding eigenfunctions $\{e_n : n \in \mathbb{Z}_+\}$ of A can be chosen to form an orthonormal basis for X, and in terms of this basis the operator A can be represented by $Au = \sum_{n=1}^{\infty} \lambda(u, e_n)e_n$, where $(\cdot, \cdot)$ be a inner product on X. In particular, $-A$ is sectorial on X.

Based on the choice of X and the properties of A, by [24, Exercise 3.10, p87], we know that the fractional power $A^{-\alpha/2}(0 < \alpha \leq 1)$ can be defined as $A^{-\alpha/2}u = \sum_{n=1}^{\infty} \lambda^{-\alpha/2}(u, e_n)e_n$. Then $A^{\alpha/2}$ can be defined as $A^{\alpha/2} := (A^{-\alpha/2})^{-1}$,

$$A^{\alpha/2}u = \sum_{n=1}^{\infty} \lambda^{\alpha/2}(u, e_n)e_n, \tag{4.3}$$

and

$$X^{\alpha/2} := \mathcal{D}(A^{\alpha/2}) = \{u : \|A^{\alpha/2}u\|_X < +\infty\}$$

is a Hilbert space when endowed with the inner product

$$(u_1, u_2)_{\alpha/2} := (A^{\alpha/2}u_1, A^{\alpha/2}u_2),$$

which gives rise to a corresponding norm $\|u\|_{\mathcal{D}(A^{\alpha/2})} = \|A^{\alpha/2}u\|_X$. Note that $X^{\alpha_2/2} \subset X^{\alpha_1/2}$ if $0 < \alpha_1 < \alpha_2 \leq 1$. In particular, $\mathcal{D}(A^{1/2}) = H_0^1(\Omega)$, $\sigma(A^{1/2}) = \{\sqrt{\lambda_n} : n \in \mathbb{Z}_+\}$ and $-A^{1/2}$ is the infinitesimal generator of a strongly continuous and analytic semigroup $\{e^{-A^{1/2}t}\}_{t \geq 0}$ on X.

Set $\mathcal{Z} := X \times X$ endowed with the norm $\|z\|_{\mathcal{Z}} = \max\{\|z_1\|_X, \|z_2\|_X\}$ for $z_1 \in X$, $z_2 \in X$ and $z = (z_1, z_2)^{\mathbf{T}} \in \mathcal{Z}$. By utilizing the factorized method in [5, section 2], we can write the linear part of equation (4.1) on $\mathcal{Z}$ into the following abstract linear equation

$$\frac{dz(x)}{dx} = Sz(x), \quad x \in \mathbb{R}, \ z(x) \in \mathcal{Z}, \tag{4.4}$$

where $z = (p, q)^{\mathbf{T}}$, $p(x), q(x) \in X$ and

$$p := -v + A^{1/2}u, \qquad q := v + A^{1/2}u. \tag{4.5}$$

Besides,

$$S = \begin{pmatrix} -A^{1/2} & 0 \\ 0 & A^{1/2} \end{pmatrix} \tag{4.6}$$

with $\mathcal{D}(S) = X^{1/2} \times X^{1/2}$. Obviously, $\overline{\mathcal{D}(S)} = \mathcal{Z}$ and $\sigma(S) = \sigma(-A^{1/2}) \cup \sigma(A^{1/2})$. Thus, S is a densely defined and hyperbolic bisectorial operator on $\mathcal{Z}$. Furthermore, we have the following lemma.

Lemma 4.1. *The operator S of equation* (4.4) *is sectorially dichotomous on $\mathcal{Z}$.*

Proof. For $z \in \mathcal{Z}$, there exist two bounded and complementary projections $P_+ : (p, q)^{\mathbf{T}} \mapsto (p, 0)^{\mathbf{T}}$ and $P_- : (p, q)^{\mathbf{T}} \mapsto (0, q)^{\mathbf{T}}$, which refer to [11, Appendix A.7], such that

$$\mathcal{Z} = \mathcal{Z}_+^p \oplus \mathcal{Z}_-^q,$$

where $\mathcal{Z}_+^p = \{z \in \mathcal{Z} : q = 0\} = X \times \{0\}$ and $\mathcal{Z}_-^q = \{z \in \mathcal{Z} : p = 0\} = \{0\} \times X$.

Since $\mathcal{D}(S) = [\mathcal{D}(S) \cap \mathcal{Z}_+^p] \oplus [\mathcal{D}(S) \cap \mathcal{Z}_-^q]$, S map $\mathcal{D}(S) \cap \mathcal{Z}_+^p$ and $\mathcal{D}(S) \cap \mathcal{Z}_-^q$ into $\mathcal{Z}_+^p$ and $\mathcal{Z}_-^q$ respectively, then $\mathcal{Z}_+^p$ and $\mathcal{Z}_-^q$ are S-invariant.

By decomposition of $\mathcal{Z}$, S can be reduced to the block matrix representation $S = \left(\begin{smallmatrix} S_+ & 0 \\ 0 & S_- \end{smallmatrix}\right)$, where

$$S_+ := S|_{\mathcal{Z}_+} = \begin{pmatrix} -A^{1/2} & 0 \\ 0 & 0 \end{pmatrix}, \qquad S_- := S|_{\mathcal{Z}_-} = \begin{pmatrix} 0 & 0 \\ 0 & A^{1/2} \end{pmatrix}.$$

Obviously, $\mathcal{D}(S_+) = \mathcal{D}(S) \cap \mathcal{Z}_+^p$, $\mathcal{D}(S_-) = \mathcal{D}(S) \cap \mathcal{Z}_-^q$. Specifically,

$$\mathcal{D}(S_+) = X^{1/2} \times \{0\}, \quad \mathcal{D}(S_-) = \{0\} \times X^{1/2}.$$

Moreover, $\sigma(S) = \sigma(S_+) \cup \sigma(S_-)$, $\sigma(S_+) = \sigma(-A^{1/2})$ and $\sigma(S_-) = \sigma(A^{1/2})$.

Besides, since S_+ and $-S_-$ are sectorial operators on $\mathcal{Z}_+^p$ and $\mathcal{Z}_-^q$, respectively,

$$\overline{\mathcal{D}(S_+)} = \mathcal{Z}_+^p, \quad \overline{\mathcal{D}(S_-)} = \mathcal{Z}_-^q.$$

S_+ generates strongly continuous and analytic semigroup

$$\left\{ \begin{pmatrix} e^{-A^{1/2}t} & 0 \\ 0 & 0 \end{pmatrix} \right\}_{t \geq 0}$$

on $\mathcal{Z}_+^p$ and $-S_-$ generates strongly continuous and analytic semigroup

$$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & e^{-A^{1/2}t} \end{pmatrix} \right\}_{t \geq 0}$$

on $\mathcal{Z}_-^q$, both are uniformly exponentially stable. Hence, S is sectorially dichotomous on $\mathcal{Z}$. $\square$

Based on the Lemma 4.1, we can constrcut the fractional power S^α and obtain its domain. The propery of A follows that sectorial operator $-S_+$ is also positive, symmetric with compact inverse, and the corresponding eigenfunctions $\{\widehat{e}_n : n \in \mathbb{Z}_+\}$ of $-S_+$ can be chosen to form an orthonormal basis for $X^{1/2} \times \{0\}$. Thus, by [24, Exercise 3.10, p87], we can define fractional power $(-S_+)^\alpha$ as

$$(-S_+)^\alpha \widehat{u} = \sum_{n=1}^{\infty} \lambda^{\alpha/2} (\widehat{u}, \widehat{e}_n) \widehat{e}_n.$$

By (4.3), we can rewrite $(-S_+)^\alpha$ as $\begin{pmatrix} A^{\alpha/2} & 0 \\ 0 & 0 \end{pmatrix}$. $(S_-)^\alpha = \begin{pmatrix} 0 & 0 \\ 0 & A^{\alpha/2} \end{pmatrix}$ can be defined in the same way above. From above, we can define the fractional power

$$S^\alpha := \begin{pmatrix} (-S_+)^\alpha & 0 \\ 0 & (S_-)^\alpha \end{pmatrix} \tag{4.7}$$

for $\alpha \in (0,1)$, along with $\mathcal{D}(S^\alpha) = \mathcal{D}((-S_+)^\alpha) \oplus \mathcal{D}((S_-)^\alpha) = X^{\alpha/2} \times X^{\alpha/2}$, $\mathcal{D}((-S_+)^\alpha) = X^{\alpha/2} \times \{0\}$ and $\mathcal{D}((S_-)^\alpha) = \{0\} \times X^{\alpha/2}$. Moreover, $\mathcal{D}(S_+) \subset \mathcal{D}((-S_+)^\alpha) \subset \mathcal{Z}_+^p$ and $\mathcal{D}(S_-) \subset \mathcal{D}((S_-)^\alpha) \subset \mathcal{Z}_-^q$. Therefore, we can define

$$\mathcal{Z}_\alpha := \mathcal{D}(S^\alpha) \text{ with the norm } \|z\|_\alpha = \|S^\alpha z\|_{\mathcal{Z}}$$

for $\alpha \in (0,1)$.

Let $\widehat{\mathcal{O}}$ be the open subset in $\mathcal{Z}_\alpha$ consisting of all the variables such that the range of (u, u_x, ∇_y) is contained in $\mathcal{V}$. From (4.2), set $v = u_x$ and the function

$$\widetilde{f} : \widehat{\mathcal{O}} \to X, \quad \widetilde{f}(u,v)(y) = f(y, u(y), v(y), \nabla_y u(y)).$$

So $\widetilde{f}$ is continuous, and (4.2) follows that $\widetilde{f}$ satisfies

$$\|\widetilde{f}(\widehat{u},\widehat{v}) - \widetilde{f}(\overline{u},\overline{v})\|_X \le \mathcal{K}_0 \cdot \max\{\|\widehat{u} - \overline{u}\|_{X^{\alpha/2}}, \|\widehat{v} - \overline{v}\|_{X^{\alpha/2}}\} \tag{4.8}$$

for $(\widehat{u},\widehat{v}), (\overline{u},\overline{v}) \in \widehat{\mathcal{O}}$.

Combining with (4.4), the whole equation (4.1) on $\mathcal{Z}$ can be written as the following abstract semilinear equation

$$\frac{dz}{dx} = Sz + H(x,z), \quad x \in \mathbb{R}, \quad z(x) \in \mathcal{Z}, \tag{4.9}$$

where $H(z) = \begin{pmatrix} \widetilde{f}(u,v) \\ -\widetilde{f}(u,v) \end{pmatrix}$, $\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2} \begin{pmatrix} A^{-1/2} & A^{-1/2} \\ -I & I \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$, $z = (p,q)^{\mathbf{T}}$. Thus, $H(z)$ is well defined in $\mathcal{Z}_\alpha$, with values in $\mathcal{Z}$. Moreover, $\|H(\widehat{z}) - H(\overline{z})\|_{\mathcal{Z}} \le \mathcal{K}_0 \|\widehat{z} - \overline{z}\|_\alpha$ for $\widehat{z}, \overline{z} \in \widehat{\mathcal{O}}$, and $H(0) = 0$.

We now state the following results for (4.1) as follows.

Theorem 4.1. *Assume that system* (4.1) *satisfies the condition* (4.2) *with a sufficiently small* $\mathcal{K}_0$. *Then,*

(i) for each $[x_1, x_2]$ *and* $(u(x_1,\cdot), u_x(x_1,\cdot))$, $(u(x_2,\cdot), u_x(x_2,\cdot)) \in H_0^1(\Omega) \times L^2(\Omega)$ *such that* $p(x_1), q(x_2) \in X^{\alpha/2}$, *then system* (4.1) *has a unique solution* $u(x,y) : [x_1,x_2] \times \Omega \to \mathbb{R}^m$, *such that* $u, u_x \in C([x_1,x_2] \times \Omega, \mathbb{R}^m)$ *and* $u_{xx} \in C((x_1,x_2) \times \Omega, \mathbb{R}^m)$, *where* p *and* q *refer to* (4.5).

(ii) *There is a unique infinite dimensional $C^{0,1}$ global stable manifold $\mathcal{W}^s$ in $H_0^1(\Omega)\times L^2(\Omega)$ such that any solution of* (4.1) *with initial condition $(u,u_x)\in \mathcal{W}^s$ satisfies $\|u(x,\cdot)\|_{H_0^1(\Omega)}\to 0$ and $\|u_x(x,\cdot)\|_{L^2(\Omega)}\to 0$ as $x\to\infty$.*

(iii) *There is a unique infinite dimensional $C^{0,1}$ global unstable manifold $\mathcal{W}^u$ in $H_0^1(\Omega)\times L^2(\Omega)$ such that any solution of* (4.1) *with initial condition $(u,u_x)\in \mathcal{W}^u$ satisfies $\|u(x,\cdot)\|_{H_0^1(\Omega)}\to 0$ and $\|u_x(x,\cdot)\|_{L^2(\Omega)}\to 0$ as $x\to-\infty$.*

Proof. Let $\overline{p}=(p,0)^{\mathbf{T}}$ and $\overline{q}=(0,q)^{\mathbf{T}}$. Based on the discussion above, equation (5.3) can be transformed into the following semilinear system:

$$\begin{cases}\frac{d}{dx}\overline{p}=S_+\overline{p}+P_+H(z),\\ \frac{d}{dx}\overline{q}=S_-\overline{q}+P_-H(z),\end{cases}\tag{4.10}$$

where $z=\overline{p}+\overline{q}\in\mathcal{Z}_+^p\oplus\mathcal{Z}_-^q=\mathcal{Z}$, $\overline{p}\in\mathcal{Z}_+^p$ and $\overline{q}\in\mathcal{Z}_-^q$. Under the hypothesises of f in (4.2), $P_\pm H(z)$ in (4.10) satisfy the hypothesis (**H**-1).

(i) For each $[x_1,x_2]$, by (4.5), take $(u(x_1,y),u_x(x_1,y))$, $(u(x_2,y),u_x(x_2,y))\in H_0^1(\Omega)\times L^2(\Omega)$ in (4.1) such that $p(x_1),q(x_2)\in X^{\alpha/2}$, which is equivalent to that we choose the dichotomous initial condition

$$(\overline{p}(x_1),\overline{q}(x_2)):=\{(p(x_1),0)^{\mathbf{T}},(0,q(x_2))^{\mathbf{T}}\}\in\mathcal{Z}_\alpha\tag{4.11}$$

on $[x_1,x_2]$ for (4.10). Note that the closure of $\mathcal{D}(S)$ in $\mathcal{Z}_\alpha$ is $\mathcal{Z}_\alpha$.

Hence, Lemma 3.1 yields that, system (4.10) with the above dichotomous initial condition (4.11) has a unique dichotomous solution $z(x)$ on $\mathcal{Z}_\alpha$, $x\in[x_1,x_2]$ such that $p,q\in C([t_1,t_2],X^{\alpha/2})$ and $p,q\in C^1((t_1,t_2),X^{\alpha/2})$. It follows that elliptic equation (4.1) has a local solution $u(x,y):[x_1,x_2]\times\Omega\to\mathbb{R}^m$ such that $u,u_x\in C([x_1,x_2]\times\Omega,\mathbb{R}^m)$ and $u_{xx}\in C((x_1,x_2)\times\Omega,\mathbb{R}^m)$.

(ii) From Lemma 3.5, under the condition (4.11), it follows that system (4.10) has a unique infinite dimensional $C^{0,1}$ global stable manifold $W^s(0)$ given by the graph of $C^{0,1}$ map

$$h^s:\mathcal{X}_\alpha\to\mathcal{Y}_\alpha,\quad \overline{q}=h^s(\overline{p}),\tag{4.12}$$

where $\mathcal{X}_\alpha=X^{\alpha/2}\times\{0\}$ and $\mathcal{Y}_\alpha=\{0\}\times X^{\alpha/2}$. Besides, we take $\mathcal{K}_0$ so small such that $\|h^s\|_{0,1}<1$. In fact, the map h^s can be viewed as a map from $X^{\alpha/2}$ to itself, and $q=h^s(p)$.

Corresponding to (u,u_x)-coordinates, $\mathcal{X}_\alpha$ and $\mathcal{Y}_\alpha$ can be written as

$$\mathcal{X}_\alpha=\{(2A^{1/2}u,0),u\in X^{1/2}\},\quad \mathcal{Y}_\alpha=\{(0,2A^{1/2}u),u\in X^{1/2}\}.\tag{4.13}$$

Note that the above statement (i) follows that $A^{1/2}u\in X^{\alpha/2}\subset X$ in (4.13). Thus, h^s can be represented by a $C^{0,1}$ map

$$\widetilde{h}^s:X^{1/2}\to X^{1/2},\quad \widetilde{h}^s(u)=(2A^{1/2})^{-1}h^s(2A^{1/2}u).\tag{4.14}$$

Moreover, by (4.14) and direct calculation, $\|\widetilde{h}^s(u_1)-\widetilde{h}^s(u_2)\|_{X^{1/2}}\le\|h^s\|_{0,1}\|u_1-u_2\|_{X^{1/2}}$, it implies that $\|\widetilde{h}^s\|_{0,1}\le\|h^s\|_{0,1}<1$.

Let $z_0=(p_0,h^s(p_0))$, $p_0=2A^{1/2}u_0$, be a point on the stable manifold of system (4.10) as given by (4.12). By $\begin{pmatrix}u\\ v\end{pmatrix}=\frac{1}{2}\begin{pmatrix}A^{-1/2} & A^{-1/2}\\ -I & I\end{pmatrix}\begin{pmatrix}p\\ q\end{pmatrix}$, and (4.14), z_0 in the (u,u_x)-coordinates is given by

$$u=u_0+\widetilde{h}^s(u_0),\qquad u_x=A^{1/2}(-u_0+\widetilde{h}^s(u_0)).\tag{4.15}$$

Then global Lipschitz inverse function theorem follows that there exists a $C^{0,1}$ map $\widehat{h}^s \triangleq (I+\widetilde{h}^s)^{-1} : X^{1/2} \to X^{1/2}$ such that $u_0 = \widehat{h}^s(u)$.

Since the composition of $C^{0,1}$ functions is a $C^{0,1}$ function, there exists a $C^{0,1}$ map $\overline{h}^s$ such that

$$\overline{h}^s : X^{1/2} \to X, \quad u_x = \overline{h}^s(u) = A^{1/2}(-\widehat{h}^s(u) + \widetilde{h}^s(\widehat{h}^s(u))). \tag{4.16}$$

Hence, system (4.1) has a unique infinite dimensional $C^{0,1}$ global stable manifold $\mathcal{W}^s = \{(u, \overline{h}^s(u)), u \in H_0^1(\Omega)\}$ in $H_0^1(\Omega) \times L^2(\Omega)$, where $\overline{h}^s$ refers to (4.16). Moreover, any solution of (4.1) with initial condition $(u, u_x) \in \mathcal{W}^s$ satisfies $\|u(x,\cdot)\|_{H_0^1(\Omega)} \to 0$ and $\|u_x(x,\cdot)\|_{L^2(\Omega)} \to 0$ as $x \to \infty$.

(iii) The assertions about the $C^{0,1}$ global unstable manifold follows from those about the $C^{0,1}$ global stable manifolds by reversing the direction of "time" variable x. □

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