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THE STABILITY OF HAUSDORFF DIMENSION FOR THE LEVEL SETS UNDER THE PERTURBATION OF CONFORMAL REPELLERS*

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Abstract Let M be a C^∞ compact Riemann manifold. $f : M \to M$ is a C^1 map and $\Lambda_f \subset M$ is a conformal repeller of f. Suppose $\varphi : M \to \mathbb{R}$ is a continuous function and let f_k be nonconformal perturbation of the map f. We consider the stability of Hausdorff dimension of level sets for Birkhorff average of potential function φ with respect to f_k and f.

Keywords Hausdorff dimension, conformal repellers, topological pressure, multifractal analysis.

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1. Introduction

The theory of multifractal analysis is a subfield of the dimension theory of dynamical systems. It studies the complexity of the level sets of invariant local quantities obtained from a dynamical system. For example, we can consider Birkhoff average, Lyapunov exponents, pointwise dimensions, or local entropies. These functions are usually only measurable and thus level sets are rarely manifold. Hence, in order to measure the complexity of these sets it is appropriate to use quantities such as the topological entropy or the Hausdorff dimension. The dimension spectrum has been extensively studied for Hölder continuous potentials for $C^{1+\alpha}$ conformal repellers Λ in [3,4,7,10]. Feng etc [6] consider the dimension spectrum for the Birkhoff average of continuous potentials on $C^{1+\alpha}$ conformal repellers Λ. Barrel etc [1] consider C^1 conformal repellers and potentials for which Φ is almost additive. Barreira etc [2] study the spectrum of $u-$dimension for the almost additive potential with a unique equilibrium measure. Cao [5] study the dimension spectrum of asymptotically additive potentials for C^1 average conformal repellers.

It is interesting to know whether the subtle structure of dimension survives after small perturbations of the original system. We will consider this problem in this paper. Let M be a C^∞ Riemann manifold, $\dim M = d$. Let U be an open subset of M and let $f : U \to M$ be a C^1 map. Suppose $\Lambda \subset U$ is a compact invariant set on which f is conformal expanding. Let $\varphi : M \to \mathbb{R}$ be a continuous potential

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function. For any $x \in \Lambda$, we define the ergodic limit, when it exists, as

$$\alpha(x) = \lim_{n\to\infty} S_{n,f}(x).$$

Given $\alpha \in \mathbb{R}$, we consider the level set:

$$L_\alpha = \{x \in \Lambda : \alpha(x) = \alpha\}.$$

The dimension spectrum $\mathcal{D} : \mathbb{R} \to \mathbb{R}$ is defined by

$$\mathcal{D}(\alpha) = dim_H L_\alpha.$$

If f_k is a nonconformal perturbation of f, then there exists a nonconformal repeller Λ_k such that $f_k|_{\Lambda_k}$ is topological conjugate to $f|_\Lambda$. We will consider the dimension spectrum for φ with respect to the map f_k on Λ_k and study the stability of Hausdorff dimension for level sets.

2. Preliminaries

In this section we briefly recall some notations about topological pressure and Hausdorff dimensions of sets.

2.1. Topological pressure

We first recall the notion of topological pressure (see Pesin [8] for more details). Let $f : X \to X$ be a continuous map. Given a finite cover $\mathcal{V}$, we denote $\mathcal{W}_n(\mathcal{V})$ the collection of vectors $V = (V_0, V_1, \ldots, V_n)$ with $V_0, V_1, \ldots, V_n \in \mathcal{V}$. For each $V \in \mathcal{W}_n(\mathcal{V})$, we write $m(V) = n$ and we consider the open set

$$X(V) = \bigcap_{k=0}^{n} f^{-k} V_k.$$

Now let φ be a continuous function. For each $V \in \mathcal{W}_n(\mathcal{V})$ we write

$$\varphi(V) = \begin{cases} \sup_{X(V)} S_n\varphi(x), \text{ if } X(V) \neq \emptyset, \\ -\infty, \qquad\qquad\quad \text{otherwise.} \end{cases}$$

Given a set $Z \subset X$ and $\alpha \in \mathbb{R}$, we define the function

$$M(Z, \alpha, \varphi, \mathcal{V}) = \lim_{n\to\infty} \inf_{\Gamma} \sum_{V\in\Gamma} \exp(-\alpha m(V) + \varphi(V)),$$

where the infimum is taken over all finite or countable collections $\Gamma \subset \bigcup_{k\geq n} \mathcal{W}_k(\mathcal{V})$ such that $\bigcup_{V\in\Gamma} X(V) \supset Z$.

We also define $P_Z(\varphi, \mathcal{V}) = \inf\{\alpha \in \mathbb{R} : M(Z, \alpha, \varphi, \mathcal{V}) = 0\}$. Then the limit

$$P_Z(\varphi) = \lim_{diam(\mathcal{V})\to 0} P_Z(\varphi, \mathcal{V})$$

is called the topogical pressure of φ in the set Z.

Next we give the definition of the topological pressure of subset using a seperated set. Let X be a compact metric space with a metric d and $f : X \to X$ is a continuous transformation. We define a new metric d_n on X by $d_n(x,y) = \max_{0\le i\le n-1} d(f^i x, f^i y)$ for $x, y \in X$. $B_n(x,\delta) = \{y \in X : d_n(x,y) < \delta\}$ is a ball centered at x with radius δ under the metric d_n. Now fix a potential function $\phi : X \to \mathbb{R}$. Given $Z \subset X$, $\delta > 0$ and $N \in \mathbb{N}$, let $\mathcal{P}(Z,N,\delta)$ be the collection of countable sets $\{(x_i,n_i) \subset Z \times \{N, N+1, \dots\}\}$ such that $Z \subset \bigcup_i B_{n_i}(x_i,\delta)$. For each $s \in \mathbb{R}$, consider the set functions

$$m_P(Z,s,\phi,N,\delta) = \inf_{\mathcal{P}(Z,N,\delta)} \sum_{(x_i,n_i)} exp(-n_i s + S_{n_i}\phi(x_i)),$$

$$m_P(Z,s,\phi,\delta) = \lim_{N\to\infty} m_P(Z,s,\phi,N,\delta).$$

This function is non-increasing in s, and takes values ∞ and 0 at all but at most one value of s. Denote the critical value of s by

$$\begin{aligned} P_Z(\phi,\delta) &= \inf\{s \in \mathbb{R} : m_P(Z,s,\phi,\delta) = 0\} \\ &= \sup\{s \in \mathbb{R} : m_P(Z,s,\phi,\delta) = \infty\}, \end{aligned}$$

we get $m_P(Z,s,\phi,\delta) = \infty$ when $s < P_Z(\phi,\delta)$ and 0 when $s > P_Z(\phi,\delta)$. The topological pressure of ϕ on Z is defined as

$$P_Z(\phi) = \lim_{\delta\to 0} P_Z(\phi,\delta).$$

The limit exists because given $\delta_1 < \delta_2$, we have $\mathcal{P}(Z,N,\delta_1) \subset \mathcal{P}(Z,N,\delta_2)$ and hence $m_P(Z,s,\phi,\delta_1) \ge m_P(Z,s,\phi,\delta_2)$, then $P_Z(\phi,\delta_1) \ge \mathcal{P}(Z,N,\delta_2)$.

2.2. Hausdorff dimensions of sets

Given a subset Z of X, for $s \ge 0$ and $\delta > 0$, define

$$\mathcal{H}^s_\delta(Z) = \inf\left\{\sum_i |U_i|^s : Z \subset \bigcup_i U_i, diam(U_i) \le \delta\right\}.$$

Note that $\mathcal{H}^s_\delta(Z)$ is decreasing in δ. Thus, the limit

$$\mathcal{H}^s(Z) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(Z)$$

exists(may be infinite). $\mathcal{H}^s(Z)$ is called $s-$dimensional Hausdorff measures of Z. And the hausdorff dimension of Z, denoted by $\dim_H Z$, is defined as follows:

$$\dim_H Z = \inf\{s : \mathcal{H}^s(Z) = 0\} = \sup\{s : \mathcal{H}^s(Z) = \infty\}.$$

3. Main result

Let M be a C^∞ compact Riemann manifold. $f : M \to M$ is a C^1 map and $\Lambda_f \subset M$ is a conformal repeller of f. Let $\mathcal{M}(\Lambda_f, f)$ the set of all f-invariant Borel probability measures supported on Λ_f. For each $\mu \in \mathcal{M}(\Lambda_f, f)$, denote by $h_\mu(f)$ the measure-theoretic entropy of f with respect to μ.

Suppose $\varphi : M \to \mathbb{R}$ is a continuous function. For $\alpha \in \mathbb{R}$, the level set L_α is defined as

$$L_\alpha \triangleq L_{f,\alpha} = \{x \in \Lambda_f | \lim_{n\to\infty} \frac{S_{n,f}\varphi(x)}{n} = \alpha\},$$

where $S_{n,f}\varphi(x) = \sum_{i=0}^{n-1} \varphi(f^i(x))$ is the Birkhoff sum of φ respect to f. It is easy to check that $L_\alpha \neq \emptyset$ if and only if $\alpha \in [\min_{\mu\in\mathcal{M}(\Lambda_f,f)} \int \varphi d\mu, \max_{\mu\in\mathcal{M}(\Lambda_f,f)} \int \varphi d\mu]$.

For every $k \in \mathbb{N}$, we now consider a C^1 map $f_k : M \to M$ which is C^1 close to f. Suppose the sequence $\{f_k\}$ converges to f as $k \to \infty$. By the structure stability of expanding maps, there exists $\Lambda_k \subset M$ which is a repeller(may not be conformal) of f_k such that $f_k|_{\Lambda_k}$ is topological conjugate to $f|_{\Lambda_f}$. More precisely, there exists a homeomorphism $\pi_k : \Lambda \to \Lambda_k$ satisfies $\pi_k \circ f = f_k \circ \pi_k$ and $\pi_k \to Id$ as $k \to \infty$.

For $\alpha_k \in [\min_{\mu\in\mathcal{M}(\Lambda_k,f_k)} \int \varphi d\mu, \max_{\mu\in\mathcal{M}(\Lambda_k,f_k)} \int \varphi d\mu]$, we denote the corresponding level set for f_k by

$$L_{\alpha_k} \triangleq L_{f_k,\alpha_k} = \{x \in \Lambda_k | \lim_{n\to\infty} \frac{S_{n,f_k}\varphi(x)}{n} = \alpha_k\}.$$

In the following we write

$$\underline{\alpha} = \min_{\mu\in\mathcal{M}(\Lambda_f,f)} \int \varphi d\mu, \ \ \overline{\alpha} = \max_{\mu\in\mathcal{M}(\Lambda_f,f)} \int \varphi d\mu$$

and

$$\underline{\alpha}_k = \min_{\mu\in\mathcal{M}(\Lambda_k,f_k)} \int \varphi d\mu, \ \ \overline{\alpha}_k = \max_{\mu\in\mathcal{M}(\Lambda_k,f_k)} \int \varphi d\mu$$

for convenience.

It is easy to check that $\lim_{k\to\infty} \underline{\alpha}_k = \underline{\alpha}$. In fact, suppose $\underline{m} \in \mathcal{M}(\Lambda, f)$ such that $\int \varphi d\underline{m} = \underline{\alpha}$, then $\underline{m}_k = \pi_k^* \underline{m} \in \mathcal{M}(\Lambda_k, f_k)$ and $\underline{\alpha}_k \leq \int \varphi d\underline{m}_k$. Hence

$$\limsup_{k\to\infty} \underline{\alpha}_k \leq \lim_{k\to\infty} \int \varphi d\underline{m}_k = \lim_{k\to\infty} \int \varphi \circ \pi_k d\underline{m} = \int \varphi d\underline{m} = \underline{\alpha}.$$

On the other hand, let $\lim_{n\to\infty} \underline{\alpha}_{k_n} = \liminf_{k\to\infty} \underline{\alpha}_k$. There exists $\underline{m}_{k_n} \in \mathcal{M}(\Lambda_{k_n}, f_{k_n})$ such that $\int \varphi d\underline{m}_{k_n} = \underline{\alpha}_{k_n}$. Suppose $\underline{m}$ is a limit point of $\{\underline{m}_{k_n}\}$, then $\underline{m} \in \mathcal{M}(\Lambda, f)$. Therefore

$$\underline{\alpha} \leq \int \varphi d\underline{m} = \lim_{n\to\infty} \int \varphi d\underline{m}_{k_n} = \liminf_{k\to\infty} \int \varphi d\underline{m}_k.$$

Similarly, we obtain that $\lim_{k\to\infty} \overline{\alpha}_k = \overline{\alpha}$. Thus for every $\alpha \in (\underline{\alpha}, \overline{\alpha})$ and every sequence $\{\alpha_k\}$ satisfies $\lim_{k\to\infty} \alpha_k = \alpha$, we conclude that $\alpha_k \in (\underline{\alpha}_k, \overline{\alpha}_k)$ for sufficiently large $k \in \mathbb{N}$.

The main result in this paper is the following theorem.

Theorem 3.1. *Suppose $\alpha \in (\underline{\alpha}, \overline{\alpha})$. If $\lim_{k\to\infty} \alpha_k = \alpha$, then*

$$\lim_{k\to\infty} \dim_H L_{\alpha_k} = \dim_H L_\alpha.$$

4. The proof of main result

In this section, we give the proof of main result Theorem 3.1. In order to prove theorem, we start with some lemmas.

Lemma 4.1. *For $\alpha \in (\underline{\alpha}, \overline{\alpha})$, we have*

$$\dim_H L_\alpha = \max_{\mu \in \mathcal{M}(\Lambda_f, f)} \left\{ \frac{h_\mu(f)}{\int \log \|Df\| d\mu} \Big| \int \varphi d\mu = \alpha \right\} = \min_{q \in \mathbb{R}} T(q, \alpha),$$

where $T(q, \alpha)$ is the unique root of equation $P(q(\varphi - \alpha) - t \log \|Df\|) = 0$.

This lemma is an immediate consequence of Theorem C in Cao [5], since Λ_f is a conformal repeller.

Lemma 4.2. *Suppose $\mu_k \in \mathcal{M}(\Lambda_k, f_k)$ for $k \in \mathbb{N}$, if $\mu_k \to \mu$ in the weak* topology in $\mathcal{M}(X)$, then $\mu \in \mathcal{M}(\Lambda_f, f)$ and $\limsup_{k\to\infty} h_{\mu_k}(f_k) \leq h_\mu(f)$.*

Proof. Since $\mu_k \in \mathcal{M}(\Lambda_k, f_k)$, for any continuous $g : M \to R$, we have

$$\int g \circ f_k d\mu_k = \int g d\mu_k.$$

Therefore

$$\begin{aligned}
\int g \circ f d\mu &= \lim_{k\to\infty} \int g \circ f d\mu_k \\
&= \lim_{k\to\infty} \Big(\int g \circ f d\mu_k - \int g \circ f_k d\mu_k + \int g \circ f_k d\mu_k \Big) \\
&= \lim_{k\to\infty} \int g \circ (f - f_k) d\mu_k + \lim_{k\to\infty} \int g \circ f_k d\mu_k \\
&= \lim_{k\to\infty} \int g d\mu_k \\
&= \int g d\mu.
\end{aligned}$$

This implies μ is f-invariant.

For every k, let $\overline{\mu}_k = (\pi_k^{-1})^* \mu_k$. Then $\overline{\mu}_k(B) = \mu_k(\pi_k(B))$ for any measurable $B \subset X$ and $\overline{\mu}_k \in \mathcal{M}(\Lambda_f, f)$. We claim that $\overline{\mu}_k \to \mu$ as $k \to \infty$. In fact, for any continuous $g : M \to \mathbb{R}$, we have

$$\int g d\overline{\mu}_k = \int g \circ \pi_k^{-1} d\mu_k = \int g \circ \pi_k^{-1} d\mu_k - \int g \circ \pi_k^{-1} d\mu + \int g \circ \pi_k^{-1} d\mu.$$

By using $\mu_k \to \mu$ and $\pi_k \to Id$, we have that $\lim_{k\to\infty} \int g d\overline{\mu}_k = \int g d\mu$, which implies $\overline{\mu}_k \to \mu$.

Since entropy is conjugacy invariant then we have $h_{\mu_k}(f_k) = h_{\overline{\mu}_k}(f)$. The upper semi-continuity of the map $\mu \mapsto h_\mu(f)$ implies that

$$\limsup_{k\to\infty} h_{\mu_k}(f_k) = \limsup_{k\to\infty} h_{\overline{\mu}_k}(f) \leq h_\mu(f).$$

□

Lemma 4.3. *If $\alpha_k \in (\underline{\alpha}_k, \overline{\alpha}_k)$, then*

$$t_*(k) \leq dim_H L_{\alpha_k} \leq t^*(k),$$

where $t_*(k), t^*(k)$ is the root of equation

$$P_{L_{\alpha_k}}(-t \log \|Df_k\|) = 0 \text{ and } P_{L_{\alpha_k}}(-t \log m(Df_k)) = 0$$

respectively.

Proof. First we proof that $dim_H L_{\alpha_k} \geq t_*(k)$. Without loss of generality, we assume $t_*(k) > 0$. Since the inequality holds for $t_*(k) = 0$. For every $0 < s < t_*(k)$, we have $P_{L_{\alpha_k}}(f_k, -s\log\|Df_k\|) > 0$ and denote it by A. Let ρ be small enough such that $s\rho < A$. Then there exists r_0 such that if $d(x,y) < r_0$ then

$$e^{-\rho} \leq \frac{Df_k(x)}{Df_k(y)} \leq e^{\rho}.$$

By the definition of the Haudorff measure, for every $\varepsilon > 0$, there is a cover $\mathcal{C}_\varepsilon = \{B(x_i,r_i), r_i < \varepsilon\}$ such that

$$\mathcal{H}^s_\varepsilon(L_{\alpha_k}) + 1 \geq \sum_{B(x_i,r_i)\in\mathcal{C}_\varepsilon} (2r_i)^s.$$

Fix $\delta > 0$, for every $B(x_i,r_i)$ there exists $n_i \in \mathbb{N}$ such that $B(x_i,r_i) \subset B_{n_i}(x_i,\delta)$ but $B(x_i,r_i) \nsubseteq B_{n_i+1}(x_i,\delta)$. Hence there exists $y \in B(x_i,r_i)$ such that $d(f_k^i(x_i), f_k^i(y)) < \delta$ for $i = 0,1,\cdots,n_i$ but $d(f_k^{n_i+1}(x_i), f_k^{n_i+1}(y)) \geq \delta$. Thus we have

$$\begin{aligned}\delta &\leq d(f_k^{n_i+1}(x_i), f_k^{n_i+1}(y)) \leq K\|Df_k^{n_i}(\xi)\| d(x_i,y)\\ &\leq K\Pi_{j=0}^{n_i-1}\|Df_k(f_k^j(\xi))\| r_i \leq Ke^{n_i\rho}\Pi_{j=0}^{n_i-1}\|Df_k(f_k^j(x_i))\| r_i.\end{aligned}$$

Therefore

$$r_i \geq \frac{\delta}{Ke^{n_i\rho}\Pi_{j=0}^{n_i-1}\|Df_k(f_k^j(x_i))\|}.$$

It implies that

$$\begin{aligned}\sum_{B(x_i,r_i)\in\mathcal{C}_\varepsilon}(2r_i)^s &\geq \frac{(2\delta)^s}{K^s}\sum_{B_{n_i}(x_i,\delta)}\frac{1}{e^{sn_i\rho}\Pi_{j=0}^{n_i-1}\|Df_k(f_k^j(x_i))\|^s}\\ &= \frac{(2\delta)^s}{K^s}\sum_{B_{n_i}(x_i,\delta)}\exp(-sS_{n_i,f_k}\log\|Df_k(x_i)\| - sn_i\rho)\\ &\geq \frac{(2\delta)^s}{K^s} m_P(L_{\alpha_k}, s\rho, -s\log\|Df_k\|, N, \delta),\end{aligned}$$

where $N = \min\{n_i\}$. Obviously, we can see that $N \to \infty$ as $\varepsilon \to 0$. From the definition of topological pressure and the fact that $s\rho < P_{L_{\alpha_k}}(f_k, -s\log\|Df_k\|)$, it follows that

$$\lim_{N\to\infty} m_P(L_{\alpha_k}, s\rho, -s\log\|Df_k\|, N, \delta) = \infty.$$

Thus we have $\lim_{\varepsilon\to\infty}\mathcal{H}^s_\varepsilon(L_{\alpha_k}) = \infty$. Hence $\dim_H L_{\alpha_k} \geq s$. The arbitrariness of $s < t_*(k)$ implies that $\dim_H L_{\alpha_k} \geq t_*(k)$.

Next we prove that $\dim_H L_{\alpha_k} \leq t^*(k)$.

For every $s > t^*(k)$, we have $P_{L_{\alpha_k}}(f_k, -s\log m(Df)) \triangleq B < 0$. Then there exists $\delta_0 > 0$ such that $P_{L_{\alpha_k}}(f_k, -s\log m(Df), \delta) < \frac{B}{2} < 0$ if $\delta < \delta_0$. Hence $m_P(L_{\alpha_k}, \frac{B}{2}, -s\log m(Df_k), \delta) = 0$. Let $\rho > 0$ be small enough and satisfy $s\rho < -\frac{B}{2}$. It has $e^{-\rho} \leq \frac{m(Df_k x)}{m(Df_k y)} \leq e^{\rho}$ if $d(x,y)$ is sufficiently small. Observe that

$$m_P(L_{\alpha_k}, \frac{B}{2}, -s\log m(Df_k), N, \delta)$$

$$
\begin{aligned}
&= \inf\left\{\sum_{B_{n_i}(x_i,\delta)} \exp(-\frac{B}{2}n_i - sS_{n_i}(\log m(Df_k(x_i))))\right\} \\
&= \inf\left\{\sum_{B_{n_i}(x_i,\delta)} \exp(-\frac{B}{2}n_i)\cdot \prod_{j=0}^{n_i-1}\Big(m(Df_k(f_k^j(x_i)))\Big)^{-s}\right\}
\end{aligned}
$$

where the infimum is taken over by the set of all covers $\mathcal{C}_\delta = \{B_{n_i}(x_i,\delta) : n_i \ge N\}$ of L_{α_k}. For any $y \in B_{n_i}(x,\delta)$, there exists $\xi \in B_{n_i}(x_i,\delta)$ such that

$$
\begin{aligned}
\delta &\ge d(f_k^{n_i}(x_i), f_k^{n_i}(y)) \ge m(Df_k^{n_i}(\xi))d(x_i,y) \\
&\ge \prod_{j=0}^{n_i-1} m(Df_k(f_k^j(\xi))d(x_i,y) \ge e^{-n_i\rho}\prod_{j=0}^{n_i-1} m(Df_k(f_k^j(x_i)))d(x_i,y).
\end{aligned}
$$

Therefore

$$
diam(B_{n_i}(x_i,\delta)) \le \frac{\delta e^{n_i\rho}}{\prod_{j=0}^{n_i-1} m(Df_k(f_k^j(x_i)))}.
$$

Thus

$$
\begin{aligned}
& m_P(L_{\alpha_k}, \frac{B}{2}, -s\log m(Df_k), N, \delta) \\
&\ge \inf\left\{\sum_{B_{n_i}(x_i,\delta)} \exp(-\frac{B}{2}n_i - sn_i\rho)\cdot \frac{1}{\delta^s}\cdot diam(B_{n_i}(x_i,\delta))^s\right\} \\
&\ge \inf\left\{\sum_{B_{n_i}(x_i,\delta)} \frac{1}{\delta^s}\cdot diam(B_{n_i}(x_i,\delta))^s\right\} \\
&\ge \frac{1}{\delta^s}\mathcal{H}^s_\delta(L_{\alpha_k}).
\end{aligned}
$$

Taking $N \to \infty$, we get

$$
\begin{aligned}
0 &= m_P(L_{\alpha_k}, \frac{B}{2}, -s\log m(Df_k), \delta) \\
&= \lim_{N\to\infty} m_P(L_{\alpha_k}, \frac{B}{2}, -s\log m(Df_k), N, \delta) \\
&\ge \frac{1}{\delta^s}\mathcal{H}^s_\delta(L_{\alpha_k}).
\end{aligned}
$$

This implies $\mathcal{H}^s_\delta(L_{\alpha_k}) = 0$ and hence $\mathcal{H}^s(L_{\alpha_k}) = 0$. This shows that $\dim_H(L_{\alpha_k}) \le s$. Since $s > t^*(k)$ is arbitrary, we obtain that $\dim_H(L_{\alpha_k}) \le t^*(k)$. □

Lemma 4.4. *$t_*(k), t^*(k), \alpha_k$ is defined as above, then*

$$
t_*(k) = \max_{\mu_k\in\mathcal{M}(\Lambda_k,f_k)}\left\{\frac{h_{\mu_k}(f_k)}{\int \log\|Df_k\|d\mu_k}\Big| \int\varphi d\mu_k = \alpha_k\right\}
$$

and

$$
t^*(k) = \max_{\mu_k\in\mathcal{M}(\Lambda_k,f_k)}\left\{\frac{h_{\mu_k}(f_k)}{\int \log m(Df_k)d\mu_k}\Big| \int\varphi d\mu_k = \alpha_k\right\}.
$$

Proof. By modifying the argument in the proof of Theorem 4.1 in Cao [5], we obtain that

$$P_{L_{\alpha_k}}(-t\log\|Df_k\|)=\max_{\mu_k\in\mathcal{M}(\Lambda_k,f_k)}\left\{h_{\mu_k}(f_k)-t\int\log\|Df_k\|d\mu_k\Big|\int\varphi d\mu_k=\alpha_k\right\}.$$

Then for every $\mu_k\in\mathcal{M}(\Lambda_k,f_k)$ with $\int\varphi d\mu_k=\alpha_k$, we have

$$h_{\mu_k}(f_k)-t_*(k)\int\log\|Df_k\|d\mu_k\le 0.$$

Thus

$$t_*(k)\ge\frac{h_{\mu_k}(f_k)}{\int\log\|Df_k\|d\mu_k}.$$

On the other hand, the upper semi-continuity of the map $\mu_k\to h_{\mu_k}(f_k)$ implies that there exists μ_k^* with $\int\varphi d\mu_k^*=\alpha_k$ such that

$$P_{L_{\alpha_k}}(-t_*(k)\log\|Df_k\|)=h_{\mu_k^*}(f_k)-t_*(k)\int\log\|Df_k\|d\mu_k^*=0.$$

Hence we have $t_*(k)=\frac{h_{\mu_k^*}(f_k)}{\int\log\|Df_k\|d\mu_k^*}$. Therefore

$$t_*(k)=\max_{\mu_k\in\mathcal{M}(\Lambda_k,f_k)}\left\{\frac{h_{\mu_k}(f_k)}{\int\log\|Df_k\|d\mu_k}\Big|\int\varphi d\mu_k=\alpha_k\right\}.$$

Similar arguments shows that

$$t^*(k)=\max_{\mu_k\in\mathcal{M}(\Lambda_k,f_k)}\left\{\frac{h_{\mu_k}(f_k)}{\int\log m(Df_k)d\mu_k}\Big|\int\varphi d\mu_k=\alpha_k\right\}.$$

□

For $\alpha_k\in(\underline{\alpha}_k,\overline{\alpha}_k)$, we now consider the equation

$$P(q(\varphi-\alpha_k)-t\log\|Df_k\|)=0 \tag{4.1}$$

where $q\in\mathbb{R}$. For fixed α_k,q, the function $p_k(t)=P(q(\varphi-\alpha_k)-t\log\|Df_k\|)$ is continuous and strictly decreasing. Moreover, $\lim_{t\to-\infty}p_k(t)=+\infty,\lim_{t\to\infty}p_k(t)=-\infty$. Then there exists a unique root $T_k(q,\alpha_k)$ of equation (4.1).

In the next we want to establish the equality $t_*(k)=\min_{q\in\mathbb{R}}T_k(q,\alpha_k)$. This process is a slightly modification of Proposition 4.3 in Cao [5]. Here we present it for completeness.

Lemma 4.5. *if $\alpha_k\in(\underline{\alpha}_k,\overline{\alpha}_k)$, then*

$$\inf_{q\in\mathbb{R}}P(q(\varphi-\alpha_k)-t_*(k)\log\|Df_k\|)\ge 0.$$

Proof. Given $\delta>0$ and $m\in\mathbb{N}$, we consider the set

$$L_{\delta,m}=\{x\in\Lambda_k:|S_{n,f_k}\varphi(x)-n\alpha_k|<\delta n\ for\ n\ge m\}.$$

It is easy to check that $L_{\alpha_k}\subset\bigcap_{\delta>0}\bigcup_{m\in\mathbb{N}}L_{\delta,m}$. Let $\mathcal{V}$ be a finite cover of Λ_k with sufficiently small diameter such that $|\varphi(x_1)-\varphi(x_2)|\le\delta$ for every $V\in\mathcal{V}$ and

$x_1, x_2 \in V$. Now take $\Gamma \subset \bigcup_{n \geq m} \mathcal{W}_n(\mathcal{V})$ such that $L_{\delta,m} \subset \bigcup_{V \subset \Gamma} X(V)$. Without loss of generality we assume that there is no V with $X(V) \bigcap L_{\delta,m} = \emptyset$. Then for any x in $X(V)$, there exists $y \in X(V) \bigcap L_{\delta,m} = \emptyset$ such that

$$\begin{aligned} |S_{m(V),f_k}\varphi(x) - m(V)\alpha_k| &\leq |S_{m(V),f_k}\varphi(x) - S_{m(V),f_k}\varphi(y)| \\ &\quad + |S_{m(V),f_k}\varphi(y) - m(V)\alpha_k| \\ &\leq m(V)\delta + m(V)\delta = 2m(V)\delta. \end{aligned}$$

It means that $-2m(V)\delta \leq S_{m(V),f_k}\varphi(V) - m(V)\alpha_k \leq 2m(V)\delta$ and hence $q(S_{m(V),f_k}\varphi(V) - m(V)\alpha_k) + 2|q|m(V)\delta \geq 0$. Then for every $\beta \in \mathbb{R}$, we have

$$\begin{aligned} &\sum_{V \in \Gamma} \exp(-\beta m(V) - t_*(k) S_{m(V),f_k} \log \|Df_k\|(V)) \\ &\leq \sum_{V \in \Gamma} \exp(-\beta m(V) + 2|q|m(V)\delta + q(S_{m(V),f_k}\varphi(V) - m(V)\alpha_k) \\ &\quad - t_*(k) S_{m(V),f_k} \log \|Df_k\|(V)) \\ &\leq \sum_{V \in \Gamma} \exp(-\beta m(V) + 2|q|m(V)\delta + [q(S_{m(V),f_k}\varphi - m(V)\alpha_k) \\ &\quad - t_*(k) S_{m(V),f_k} \log \|Df_k\|](V)). \end{aligned}$$

This implies that

$$\begin{aligned} &M(L_{\delta,m}, \beta, -t_*(k) S_{m(V),f_k} \log \|Df_k\|, \mathcal{V}) \\ &\leq M(L_{\delta,m}, \beta - 2|q|\delta, q(S_{m(V),f_k}\varphi - m(V)\alpha_k) - t_*(k) S_{m(V),f_k} \log \|Df_k\|, \mathcal{V}). \end{aligned}$$

Then

$$P_{L_{\delta,m}}(-t_*(k) \log \|Df_k\|, \mathcal{V}) - 2|q|\delta \leq P_{L_{\delta,m}}(q(\varphi - \alpha_k) - t_*(k) \log \|Df_k\|, \mathcal{V}).$$

Letting $diam(\mathcal{V}) \to 0$, we have

$$P_{L_{\delta,m}}(-t_*(k) \log \|Df_k\|) \leq P_{L_{\delta,m}}(q(\varphi - \alpha_k) - t_*(k) \log \|Df_k\|) + 2|q|\delta$$

for every $\delta > 0$ and $q \in \mathbb{R}$. On the other hand, by the definition of $t_*(k)$ we have

$$\begin{aligned} 0 &= P_{L_{\alpha_k}}(-t_*(k) \log \|Df_k\|) \leq P_{\bigcup_{m \in \mathbb{N}} L_{\delta,m}}(-t_*(k) \log \|Df_k\|) \\ &= \sup_{m \in \mathbb{N}} P_{L_{\delta,m}}(-t_*(k) \log \|Df_k\|) \leq P(q(\varphi - \alpha_k) - t_*(k) \log \|Df_k\|) + 2|q|\delta. \end{aligned}$$

Since $\delta > 0$ is arbitrary, we obtain $P(q(\varphi - \alpha_k) - t_*(k) \log \|Df_k\|) \geq 0$. This completes the proof of the lemma. □

Lemma 4.6. *if* $\alpha_k \in (\underline{\alpha}_k, \overline{\alpha}_k)$, *then*

$$\min_{q \in \mathbb{R}} P(q(\varphi - \alpha_k) - t_*(k) \log \|Df_k\|) = 0.$$

Proof. Let r_k be the distance of α_k to $\mathbb{R} \backslash (\underline{\alpha}_k, \overline{\alpha}_k)$. For $q \in \mathbb{R}$, define

$$F_k(q) = P(q(\varphi - \alpha_k) - t_*(k) \log \|Df_k\|).$$

Then $F_k(q)$ is continuous. Let $\beta = \alpha_k + \frac{r_k}{2}$ when $q > 0$, and $\beta = \alpha_k - \frac{r_k}{2}$ when $q < 0$. Then $\beta \in (\underline{\alpha}_k, \overline{\alpha}_k)$. It means that there exists $\mu_k \in \mathcal{M}(\Lambda_k, f_k)$ such that $\lim_{n\to\infty} \int \varphi d\mu_k = \beta$. Therefore

$$\begin{aligned} F_k(q) &= \max_{\mu}\{h_\mu(f_k) + \int (q(\varphi - \alpha_k) - t_*(k)\log\|Df_k\|)d\mu\} \\ &\geq h_{\mu_k}(f_k) + \int (q(\varphi - \alpha_k) - t_*(k)\log\|Df_k\|)d\mu_k \\ &= h_{\mu_k}(f_k) + \int (q(\varphi - \beta) + q(\beta - \alpha_k) - t_*(k)\log\|Df_k\|)d\mu_k \\ &= h_{\mu_k}(f_k) - t_*(k)\int \log\|Df_k\|d\mu_k + \frac{1}{2}|q|r_k. \end{aligned}$$

We note that the right-hand side of the inequality takes arbitrarily large values for $|q|$ sufficiently large. Thus there exists $M \in \mathbb{R}$ such that $F_k(q) \geq F_k(0)$ with $|q| > M$. The continuity of $F_k(q)$ implies that $F_k(q)$ attains a minimum at some points q_k with $|q_k| \leq M$. Thus for every $q \geq q_k, F_k(q) - F_k(q_k) \geq 0$. Let μ_q be the equilibrium of $q(\varphi - \alpha_k) - t_*(k)\log\|Df_k\|$. Then

$$\begin{aligned} F_k(q) - F_k(q_k) &\leq h_{\mu_q}(f_k) + \int (q(\varphi - \alpha_k) - t_*(k)\log\|Df_k\|)d\mu_q \\ &\quad -\Big(h_{\mu_q}(f_k) + \int (q_k(\varphi - \alpha_k) - t_*(k)\log\|Df_k\|)d\mu_q\Big) \\ &= (q - q_k)\int (\varphi - \alpha_k)d\mu_q. \end{aligned}$$

Hence $\int(\varphi - \alpha_k)d\mu_q \geq 0$. Now without loss of generality, suppose $\mu_q \to \nu_1$ as $q \to q_k$. The upper semi-continuity of entropy h_{μ_k} implies that

$$F_k(q_k) = \lim_{q\to q_k} F_k(q) \leq h_{\nu_1}(f_k) + \int (q_k(\varphi - \alpha_k) - t_*(k)\log\|Df_k\|)d\nu_1.$$

Thus ν_1 is an equilibrium of $q_k(\varphi - \alpha_k) - t_*(k)\log\|Df_k\|$ and $\int(\varphi - \alpha_k)d\nu_1 \geq 0$.

Similarly by considering the case $q \leq q_k$, then $F_k(q) - F_k(q_k) \geq 0$ and we can find an invariant measeure ν_2 such that ν_2 is an equilibrium of $q_k(\varphi - \alpha_k) - t_*(k)\log\|Df_k\|$ and $\int(\varphi - \alpha_k)d\nu_2 \leq 0$.

For $a \in [0,1]$, let $\mu_a = a\nu_1 + (1-a)\nu_2$. Then $p(a) = \int(\varphi - \alpha_k)d\mu_a$ is continuous on $[0,1]$ and $p(0) \leq 0, p(1) \geq 0$. Hence there exists a_0 such that $p(a_0) = \int(\varphi - \alpha_k)d\mu_{a_0} = 0$. Since ν_1, ν_2 are equilibriums of $q_k(\varphi - \alpha_k) - t_*(k)\log\|Df_k\|$, then μ_{a_0} is an equilibrium of $q_k(\varphi - \alpha_k) - t_*(k)\log\|Df_k\|$. If $F_k(q_k) > 0$, then $h_{\mu_{a_0}}(f_k) - t_*(k)\int \log\|Df_k\|d\mu_{a_0} > 0$. Therefore we have

$$t_*(k) < \frac{h_{\mu_{a_0}}}{\int \log\|Df_k\|d\mu_{a_0}}.$$

This contradicts the fact

$$t_*(k) = \max_{\mu_k \in \mathcal{M}(\Lambda_k, f_k)} \left\{ \frac{h_{\mu_k}(f_k)}{\int \log\|Df_k\|d\mu_k} \Big| \int \varphi d\mu_k = \alpha_k \right\}.$$

Hence $F_k(q_k) = 0$ and $t_*(k) = \frac{h_{\mu_{a_0}}}{\int \log\|Df_k\|d\mu_{a_0}}$. This completes the proof of the lemma. □

Remark 4.1. In the proof of lemma 4.6, we can prove that there exists $M \in \mathbb{R}$ such that for $k \in \mathbb{N}$ sufficient large, $F_k(q)$ attains a minimum at some points q_k with $|q_k| \leq M$. In fact, we only take $r_k \geq \tilde{r} > 0$ for some $\tilde{r}$ and sufficient large k.

Lemma 4.7. *If* $\alpha_k \in (\underline{\alpha}_k, \overline{\alpha}_k)$, *then* $t_*(k) = \min_{q\in\mathbb{R}} T_k(q, \alpha_k)$.

Proof. It follows from lemma 4.6 that if $\alpha_k \in (\underline{\alpha}_k, \overline{\alpha}_k)$, then there exists q_k such that

$$0 = F(q_k) = P(q_k(\varphi - \alpha_k) - t_*(k) \log \|Df_k\|).$$

Hence $t_*(k) = T_k(q_k, \alpha_k)$. On the other hand, for every $q \in \mathbb{R}, F(q) = P(q(\varphi - \alpha_k) - t_*(k) \log \|Df_k\|) \geq 0$ which implies $t_*(k) \leq T_k(q, \alpha_k)$ for every $q \in \mathbb{R}$. Hence $t_*(k) = \min_{q\in\mathbb{R}}\{T_k(q, \alpha_k)\}$. □

Now we move to the proof of **Theorem 3.1**.

Proof. It is sufficient to verify that

$$\limsup_{k\to\infty} \dim_H L_{\alpha_k} \leq \dim_H L_\alpha \leq \liminf_{k\to\infty} \dim_H L_{\alpha_k}.$$

By lemma 4.3 and lemma 4.4 we have

$$\dim_H L_{\alpha_k} \leq t^*(k) = \max_{\mu\in\mathcal{M}(\Lambda_k, f_k)} \left\{ \frac{h_{\mu_k}(f_k)}{\int \log m(Df_k) d\mu_k} \Big| \int \varphi d\mu_k = \alpha_k \right\}$$

for $\alpha_k \in (\underline{\alpha}_k, \overline{\alpha}_k)$. Supppose $\widetilde{\mu}_k$ is the f_k-invariant measure at which the maximum in the last equality can be attained. Then

$$\dim_H L_{\alpha_k} \leq \frac{h_{\widetilde{\mu}_k}(f_k)}{\int \log m(Df_k) d\widetilde{\mu}_k} \text{ and } \int \varphi d\widetilde{\mu}_k = \alpha_k.$$

Without loss of generality, we may assume $\widetilde{\mu}_k \to \mu^*$ as $k \to \infty$. It implies $\mu^* \in \mathcal{M}(\Lambda_f, f)$ and $\int \varphi d\mu^* = \lim_{k\to\infty} \int \varphi d\widetilde{\mu}_k = \lim_{k\to\infty} \alpha_k = \alpha$. Then by lemma 4.2 and lemma 4.1 we have

$$\begin{aligned}
\limsup_{k\to\infty} \dim_H L_{\alpha_k} &\leq \limsup_{k\to\infty} \frac{h_{\widetilde{\mu}_k}(f_k)}{\int \log m(Df_k) d\widetilde{\mu}_k} \\
&\leq \frac{h_{\mu^*}(f)}{\int \log m(Df) d\mu^*} \\
&\leq \max_{\mu\in\mathcal{M}(\Lambda, f)} \left\{ \frac{h_\mu(f)}{\int \log m(Df) d\mu} \Big| \int \varphi d\mu = \alpha \right\} \\
&= \dim_H L_\alpha.
\end{aligned}$$

On the other hand, by lemma 4.7

$$\dim_H L_{\alpha_k} \geq t_*(k) = \min_{q\in\mathbb{R}} T_k(q, \alpha_k).$$

Take q_k such that $\min_{q\in\mathbb{R}} T_k(q, \alpha_k) = T_k(q_k, \alpha_k)$. By Remark 4.1 we obtain that $|q_k| \leq M$ for every large k. This implies that the sequence $\{q_k\}$ has limit point. Without loss of generality, we may assume $\lim_{k\to\infty} q_k = q^*$. By Theorem 9.8 in Walters [9, p216] we have

$$P_{\Lambda_k}(f_k, q_k(\varphi - \alpha_k) - t \log \|Df_k\|) = P_\Lambda(f, (q_k(\varphi - \alpha_k) - t \log \|Df_k\|) \circ \pi_k).$$

Then

$$P_{\Lambda_k}(f_k, q_k(\varphi - \alpha_k) - t \log \|Df_k\|) \to P_\Lambda(f, q^*(\varphi - \alpha) - t \log \|Df\|)$$

as $k \to \infty$. Hence $\lim_{k\to\infty} T_k(q_k, \alpha_k) = T(q^*, \alpha)$.

Therefore for sufficiently small $\epsilon > 0$ we have

$$\liminf_{k\to\infty} \dim_H L_{\alpha_k} \geq T(q^*, \alpha) - \epsilon \geq \min_{q\in\mathbb{R}} T(q, \alpha) - \epsilon = \dim_H L_\alpha - \epsilon.$$

As ϵ is arbitrary, we obtain that

$$\liminf_{k\to\infty} \dim_H L_{\alpha_k} \geq \dim_H L_\alpha.$$

This implies the result. □

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