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# HEREDITARY EFFECTS OF EXPONENTIALLY DAMPED OSCILLATORS WITH PAST HISTORIES*

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**Abstract** This paper presents hereditary effects of exponentially damped oscillators with past histories. Unlike the classical viscously damped oscillators, the nonviscously damped ones involve damping forces which depend on time-histories of vibrating motions via convolution integrals. As a result, equations of motion of such systems are a set of coupled second-order Volterra integro-differential equations. In this work, initial value problems for the integro-differential equations are revisited. The initial conditions should contain time-histories of vibrating motions. Then, initialization response of exponentially damped oscillators is obtained. It is used to characterize the hereditary effects on the dynamic response. At last, stability of initialization response is proved from the theoretical viewpoint and verified by numerical simulations. This reveals that the hereditary effects gradually recede with increasing of time.

**Keywords** Nonviscously damping, dynamical systems, integro-differential equation, initialization problems, hereditary effects.

**MSC(2010)** 45D05, 45J05.

## 1. Introduction

Viscoelastic materials have seen broad applications in vibration control engineering due to their high damping capacity [8]. The modeling of constitutive relations of viscoelastic materials is of great importance for analysis and design of viscoelastically damped structures. However, it is difficult to be modeled because the mechanical behavior of viscoelastic materials is highly dependent on various factors such as time, temperature, the vibrating frequency and so on [22].

Integral constitutive models of viscoelastic materials are derived based on mechanical properties of stress relaxation and creep. The stress relaxation functions and creep functions are memory and hereditary kernels in the integral constitutive equations. They can be expressed by a series of exponential functions [13, 24],

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power-law functions [14], Mittage-Leffler functions [14], or other types of functions. Integral constitutive models are superior to the differential ones in many aspects: (i) the fading memory property can be characterized and time-history of loading acting on materials can be recorded; (ii) the stress relaxation functions or creep functions are easily and directly obtained via experiment data fitting; (iii) other factors such as temperature and the ageing affects can be conveniently included in the model. When integral constitutive relations are used as damping models in viscoelastically damped structures, equations of motion of such systems are a set of coupled second-order integro-differential equations. The presence of the "integral" term makes the vibration analysis and control design more complicated than the classical ones. The integral type damping models are also called nonviscously damping models. The corresponding oscillators are called nonviscously damped oscillators.

Researches on nonviscously damped oscillators are mainly concentrated on two types: one is the exponentially damped oscillators, where the damping forces are expressed by exponentially fading memory kernels; the other is the fractional-order oscillators, where the viscoelastic relaxation functions are characterized by power-law functions or Mittage-Leffler functions. Adhikari and his colleagues have systematically investigated the structural dynamics with exponentially damped models in [1–3]. They have discovered that, unlike the classical viscously damped oscillators, an exponentially damped single-degree-of-freedom oscillator has three eigenvalues. The complex conjugate pair of roots corresponds to the vibration motion and the third one corresponds to a purely dissipative motion. Dynamics of exponentially damped oscillators is governed by both of the viscous damping factor and the nonviscous damping factor. For the dynamic analysis of multi-degree-of-freedom systems, they have developed a state-space approach using additional dissipation coordinates and the configuration space method. It was shown that the characteristic equation for an $N$ degree-of-freedom system is more than $N$ and the modes are divided into elastic modes and nonviscous modes. In [4], an analytical solution using modal superposition are developed for the analysis of an exponentially damped solid rod. In [10], a method has been proposed to calculate eigensolution derivatives for the nonviscously damped systems. In [19], a state-space method has been proposed to identify modal and physical parameters for the nonviscously damped systems. A closed-form approximation expression of the eigenvalues for non-viscous, non-proportional systems has been derived in [11].

Fractional-order oscillators are another type of nonviscously damped oscillators. They are under extensive investigations for the last three decades. The constitutive models involving fractional-order derivatives have been viewed more accurate and concise than other ones [15]. As a result, researches on fractional oscillators have been expected to be a promising work for structural dynamics analysis and control design. However, the fractional differential equations of motions are difficult to deal with due to the presence of the weakly singular kernels. Studies on dynamic responses of fractional-order oscillators have been reviewed in [20]. Asymptotically steady state behavior of fractional oscillators has been presented in [16, 17, 21]. The criteria for the existence and the behavior of solutions have been obtained in [5, 9, 23]. In [26] , expression of mechanical energy in fractional oscillators has been determined and energy regeneration and dissipation have been obtained. In [27, 28], vibration controls have been designed using sliding mode control technique for several oscillators.

The damping forces in nonviscously damped oscillators depend on the past history of motion via convolution integrals. As a result, dynamics of the nonviscously damped oscillators is said to have memory. However, the memory and hereditary properties of fractional-order systems have been neglected and ambiguous for a long time [18]. Considering the memory effect and prehistory of fractional oscillators, the historical effects and initialization problems have been proposed in [7,12]. Stability of initialization response of fractional oscillators has been proved based on the unit impulse response function in [25]. Initial value problems for general fractional order systems have been studied in [6, 29, 30]. In [29], the complexity and the importance of the initial value problem have been presented by simulation examples. The named aberration phenomenon has been revealed by generalizing the infinite dimensional property and the long memory property. In [30], the term pre-initial process has been used to describe the long memory property of fractional order systems. A method of segmented linearization has been proposed to fit the time-varying initialization function. In [6], a practical method has been proposed to estimate the exact initial states of fractional order systems.

In short, nonviscously damped oscillators are capable of characterizing dynamics of viscoelastically damped structures in vibration engineering. They are mainly classified into two types: exponentially damped ones and fractional-order ones. This paper focuses on the hereditary effects of the exponentially damped oscillators with past histories. The main objective is to prove stability of initialization response and to show that the hereditary effects gradually recede with time. For this end, we revisit initial value problems for nonvicously damped oscillators in Section 2. We declare that knowledge of the equations of motion, along with the initial displacement and velocity is insufficient to determine the dynamics behaviors. The initial conditions should also contain past history of response velocity. In Section 3, we obtain initialization response of exponentially damped oscillators. It can be used to characterize the hereditary effects of history on the dynamic response. In Section 4, we prove that hereditary effects on the initialization response recede to zero with increasing of time. Numerical simulations are carried out to verify this phenomenon.

## 2. Initialization for nonviscously damped oscillators

The integral constitutive relations of viscoelastic materials are represented by the following integro-differential equation of Volterra type:

$$\sigma(t) = \int_{-\infty}^{t} G(t-\tau)\,\dot{\varepsilon}(\tau)\,d\tau, \tag{2.1}$$

where $\sigma(t)$ is the stress, $\varepsilon(t)$ is the stain, $G(t)$ is the stress relaxation function. The lower terminal in the integral is $-\infty$ because the stress of viscoelastic materials is dependent on all the time histories of the strain [22].

Fig.1 shows an single-degree-of-freedom oscillator with an viscoelastic damper. When Eq.(2.1) is used to characterize the damping force, i.e.,

$$f_d(t) = c\int_{-\infty}^{t} G(t-\tau)\,\dot{x}(\tau)\,d\tau,$$

the equation of motion of the oscilaator can be expressed as

$$m\ddot{x}(t) + c\int_{-\infty}^{t} G\left(t-\tau\right)\dot{x}\left(\tau\right)d\tau + kx(t) = f\left(t\right), \tag{2.2}$$

where $m$ is the mass, $k$ is the stiffness, $c$ is the damping coefficient, $x\left(t\right)$ is the displacement, $f\left(t\right)$ is the external force acting on the system.

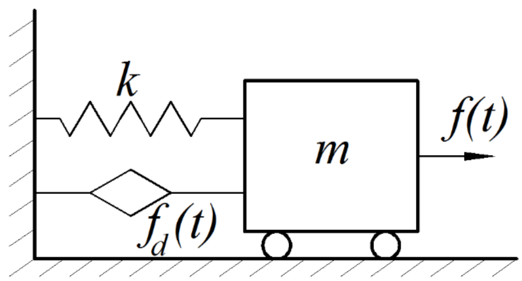



**Figure 1.** Single-degree-of-freedom exponentially damped oscillator, damping force $f_d\left(t\right) = c\int_{-\infty}^{t} G\left(t-\tau\right)\dot{x}\left(\tau\right)d\tau$.

The integral term in Eq.(2.2) makes the dynamic models different from the classical ones. It contains not only the information of vibrating displacement $x\left(t\right)$ and velocity $\dot{x}\left(t\right)$, but also the time-histories of velocity $\dot{x}\left(t\right)$. This implies that, unlike the viscously damped systems, the equation of motion, the instantaneous displacement and velocity are insufficient to predict the dynamic behaviors. Time-histories of motion should be added to initial conditions to fully determine the dynamics of nonvicously damped oscillators. As a result, the dynamic equation with past history is described as

$$\begin{cases} m\ddot{x}(t) + c\int_{-\infty}^{t} G\left(t-\tau\right)\dot{x}\left(\tau\right)d\tau + kx(t) = f\left(t\right),\ t>0, \\ x(0)=x_0,\ \dot{x}(0)=v_0, \\ \dot{x}(t)=v(t),\ \ -\infty\ <t<0, \end{cases} \tag{2.3}$$

where $t=0$ is the initial time and the lower terminal in the integral $t=-\infty$ is the starting time of vibration. In reality it is more reasonable to set the starting time as $t=-a$, which means that the system is at quiescent before $t=-a$ and begins vibrating at $t=-a$.

To be more familiar with the dynamic systems with memory, the integral term in Eq.(2.3) can be separated into two part:

$$\int_{-a}^{t} G\left(t-\tau\right)\dot{x}\left(\tau\right)d\tau = \int_{-a}^{0} G\left(t-\tau\right)\dot{x}\left(\tau\right)d\tau + \int_{0}^{t} G\left(t-\tau\right)\dot{x}\left(\tau\right)d\tau.$$

The first part characterizes the hereditary effects of the histories of motion on the system dynamics, which is denoted as $\psi\left(t\right)$:

$$\psi\left(t\right) = \int_{-a}^{0} G\left(t-\tau\right)\dot{x}\left(\tau\right)d\tau = \int_{-a}^{0} G\left(t-\tau\right)v\left(\tau\right)d\tau. \tag{2.4}$$

$\psi(t)$ contains the time histories of motion, acts as an internal force and influence the behavior of dynamic systems after the initial time $t = 0$. Eq.(2.3) can be rewritten as

$$\begin{cases} m\ddot{x}(t) + c\int_0^t G(t-\tau)\dot{x}(\tau)\,d\tau + kx(t) = f(t) - c\psi(t),\ t > 0, \\ x(0) = x_0,\ \dot{x}(0) = v_0. \end{cases} \tag{2.5}$$

**Remark 2.1.** If the nonvicously damped oscillator is at rest before $t = 0$, there is no vibrating motion from past histories, i.e.,

$$x(t) = 0, -\infty < t < 0.$$

In this special case, the equation of motion is

$$m\ddot{x}(t) + c\int_0^t G(t-\tau)\dot{x}(\tau)\,d\tau + kx(t) = f(t).$$

The initial values contain only the initial displacement $x(0)$ and the initial velocity $\dot{x}(0)$.

It is a special case of Eq.(2.3) and has been studied by Adhikari and his colleagues [6]. Because the system is quiescent before $t = 0$, there's no hereditary effect on the system dynamics. However, in the present work, the more general case with past histories is considered. Hereditary effects on such systems are needed to be studied.

**Remark 2.2.** In Eqs.(2.1)-(2.5), $G(t)$ is a relaxation function of viscoelastic materials and characterizes the mechanical properties. In practical applications, $G(t)$ can be effectively expressed by several types of decaying functions and further determined by fitting experimental data. In the following sections, the exponentially decaying function $e^{-\mu t}$ is chosen, i.e.,

$$G(t) = \mu e^{-\mu t}, \mu > 0.$$

The corresponding oscillator is named as exponentially damped oscillator.

The other two functions to specify $G(t)$ are power-law function $t^{-\alpha}$, $0 < \alpha < 1$ and Mittage-Leffler function $E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k+\beta)}$, $\alpha > 0, \beta > 0$ . In both of the cases, the integral constitutive equation becomes fractional-order derivative. The corresponding oscillator is named as fractional-order oscillator.

## 3. Hereditary effects on the dynamic response

Now we are ready to study the initialization response of exponentially damped oscillators with memories, which characterizes the hereditary effects of the vibrating motion from the past history. For this purpose, the external acting force $f(t)$ is not considered and is set to be zero. In this case, the equation of motion with initial condition is

$$\begin{cases} m\ddot{x}(t) + c\int_0^t G(t-\tau)\dot{x}(\tau)\,d\tau + kx(t) = -c\psi(t),\ t > 0, \\ x(0) = x_0,\ \dot{x}(0) = v_0. \end{cases} \tag{3.1}$$

where $G(t) = \mu e^{-\mu t}, \mu > 0$.

Applying the Laplace transform to both sides of Eq.(3.1), we obtain

$$m\left(s^2\bar{x}(s) - sx_0 - v_0\right) + c\left(\frac{\mu s}{\mu + s}\bar{x}(s) - \frac{\mu}{\mu + s}x_0\right) + k\bar{x}(s) = -c\bar{\psi}(s), \quad (3.2)$$

where $\bar{x}(s)$ is the Laplace transform of $x(t)$, $\bar{\psi}(s)$ is the Laplace transform of $\psi(t)$. After rearranging Eq.(3.2), one has

$$\left(ms^2 + \frac{c\mu s}{\mu + s} + k\right)\bar{x}(s) = -c\bar{\psi}(s) + msx_0 + \frac{c\mu}{\mu + s}x_0 + mv_0. \quad (3.3)$$

We denote that $\bar{d}(s) = ms^2 + \frac{c\mu s}{\mu+s} + k$ and $\bar{h}(s) = \frac{1}{\bar{d}(s)}$.

From Eq.(3.3), the solution of $\bar{x}(s)$ can be derived as

$$\bar{x}(s) = -c\bar{h}(s)\bar{\psi}(s) + mx_0 s\bar{h}(s) + c\mu x_0\frac{\bar{h}(s)}{s + \mu} + mv_0\bar{h}(s). \quad (3.4)$$

Taking the inverse Laplace transform of Eq.(3.4), one derives

$$\begin{aligned} x(t) = &-c\int_0^t h(t-\tau)\psi(\tau)\,d\tau + mx_0\dot{h}(t) \\ &+ c\mu x_0\int_0^t h(t-\tau)e^{-\mu\tau}d\tau + mv_0 h(t), \end{aligned} \quad (3.5)$$

where $h(t)$ is inverse Laplace transform of $\bar{h}(s)$.

Next we determine the expression of $h(t)$. It has been shown in [1] that $\bar{d}(s)$ has zeros at $s = s_j, j = 1, 2, 3$:

$$s_1 = -\alpha + \beta i, s_2 = -\alpha - \beta i, s_1 = -\gamma, \quad (3.6)$$

where $\alpha, \beta, \gamma > 0$.

Furthermore, $\bar{h}(s)$can be expressed in the pole-residue form as

$$\bar{h}(s) = \sum_{j=1}^{3}\frac{R_j}{s - s_j},$$

where $R_j$ are the residues and calculated as

$$R_i = \operatorname*{Res}_{s=s_j}\bar{h}(s) = \lim_{s\to s_j}(s - s_j)\bar{h}(s) = \frac{1}{\lim\limits_{s\to s_j}\frac{ms^2 + \frac{c\mu s}{\mu+s} + k}{s - s_j}} = \frac{1}{\frac{\partial\bar{d}(s)}{\partial s}|_{s=s_j}}.$$

Taking the inverse Laplace transform of Eq.(3.7), one derives

$$h(t) = L^{-1}\left\{\bar{h}(s)\right\} = \sum_{j=1}^{3}R_j e^{-s_j t}. \quad (3.7)$$

Eq.(3.6)and Eq.(3.5) determine the initialization response of the oscillators. It represents the hereditary effects of past histories of motions from the starting time at $t = -a$.

# 4. Stability of initialization response

Section 2 and Section 3 have shown the effects of past histories of motions on initial conditions and dynamic response. In this section, we proceed to show the hereditary effects on stability of initialization response. We will prove theoretically that this influence will gradually recede with increasing of time, and then carry out numerical simulations to verify this statement.

## 4.1. Theoretical stability analysis

From Eq.(3.5), it is clear that the initialization response involves four parts. The last three terms decreased as time increases. Next we will prove that the first term also decays with increasing of time.

Inserting Eq.(3.6) into the first term in Eq.(3.5), we have

$$\begin{aligned}-c\int_0^t h\left(t-\tau\right)\psi\left(\tau\right)d\tau = -c\int_0^t R_1 e^{s_1(t-\tau)}\psi\left(\tau\right)d\tau - c\int_0^t R_1 e^{s_2(t-\tau)}\psi\left(\tau\right)d\tau \\ - c\int_0^t R_3 e^{s_3(t-\tau)}\psi\left(\tau\right)d\tau.\end{aligned} \tag{4.1}$$

Substituting Eq.(3.6) into Eq.(4.1), we have

$$\begin{aligned}&-c\int_0^t h\left(t-\tau\right)\psi\left(\tau\right)d\tau \\ &= -c\int_0^t R_1 e^{-(\alpha-\beta i)(t-\tau)}\psi\left(\tau\right)d\tau - c\int_0^t R_1 e^{-(\alpha+\beta i)(t-\tau)}\psi\left(\tau\right)d\tau \\ &\quad - c\int_0^t R_3 e^{-\gamma(t-\tau)}\psi\left(\tau\right)d\tau \\ &= -2cR_1\int_0^t e^{-\alpha(t-\tau)}\cos\beta\left(t-\tau\right)\psi\left(\tau\right)d\tau - cR_3\int_0^t e^{-\gamma(t-\tau)}\psi\left(\tau\right)d\tau.\end{aligned} \tag{4.2}$$

We denote that

$$I_1 = -2cR_1\int_0^t e^{-\alpha(t-\tau)}\cos\beta\left(t-\tau\right)\psi\left(\tau\right)d\tau, \tag{4.3}$$

$$I_2 = -cR_3\int_0^t e^{-\gamma(t-\tau)}\psi\left(\tau\right)d\tau. \tag{4.4}$$

Then Eq.(4.2) is simplified as

$$-c\int_0^t h\left(t-\tau\right)\psi\left(\tau\right)d\tau = I_1 + I_2.$$

Substituting Eq.(2.4) into Eq.(4.2), one has

$$\begin{aligned}|I_1| &= 2cR_1\left|\int_0^t e^{-\alpha(t-\tau)}\cos\beta\left(t-\tau\right)\psi\left(\tau\right)d\tau\right| \\ &= 2cR_1\left|\int_0^t e^{-\alpha(t-\tau)}\cos\beta\left(t-\tau\right)d\tau\int_{-a}^0 G\left(\tau-\tau_1\right)v\left(\tau_1\right)d\tau_1\right|.\end{aligned} \tag{4.5}$$

It is reasonable to suppose that the response velocity before initial time is bounded, i.e.,

$$|v(t)| \le M, t \in [-a, 0].$$

Noting that $|\cos(\beta t)| \le 1$, then Eq.(4.4) yields

$$|I_1| \le 2cR_1 M \left| \int_0^t e^{-\alpha(t-\tau)} d\tau \int_{-a}^0 G(\tau - \tau_1) d\tau_1 \right|. \tag{4.6}$$

Substituting $G(t) = \mu e^{-\mu t}$ into Eq.(16) yields

$$\begin{aligned} |I_1| &\le 2cR_1 M \left(1 - e^{-\mu a}\right) \left| \int_0^t e^{-\alpha(t-\tau)-\mu\tau} d\tau \right| \\ &= 2cR_1 M \left(1 - e^{-\mu a}\right) e^{-\alpha t} \left| \int_0^t e^{(\alpha-\mu)\tau} d\tau \right| \\ &= \frac{2cR_1 M \left(1 - e^{-\mu a}\right)}{|\alpha - \mu|} \left| e^{-\mu t} - e^{-\alpha t} \right|. \end{aligned} \tag{4.7}$$

Due to the fact that $\mu, \alpha > 0$, it is clear to see that

$$\lim_{t \to \infty} I_1 = 0. \tag{4.8}$$

Substituting Eq.(4) into Eq.(14), one has

$$\begin{aligned} |I_2| &= cR_3 \left| \int_0^t e^{-\gamma(t-\tau)} \psi(\tau) d\tau \right| \\ &= cR_3 \left| \int_0^t e^{-\gamma(t-\tau)} d\tau \int_{-a}^0 G(\tau - \tau_1) v(\tau_1) d\tau_1 \right| \\ &\le McR_3 \left(1 - e^{-\mu a}\right) \left| \int_0^t e^{-\gamma(t-\tau)-\mu\tau} d\tau \right| \\ &= McR_3 \left(1 - e^{-\mu a}\right) e^{-\gamma t} \left| \int_0^t e^{(\gamma-\mu)\tau} d\tau \right| \\ &= \frac{McR_3 \left(1 - e^{-\mu a}\right)}{|\gamma - \mu|} \left| e^{-\mu t} - e^{-\gamma t} \right|. \end{aligned} \tag{4.9}$$

Because $\mu, \gamma > 0$, it is also clear to see that

$$\lim_{t \to \infty} I_2 = 0. \tag{4.10}$$

By now, we have proved that $\lim_{t \to \infty} I_1 = 0$ and $\lim_{t \to \infty} I_2 = 0$. This implies that the first term of Eq.(3.5) decreases to zero, i.e.,

$$-c \int_0^t h(t - \tau) \psi(\tau) d\tau \to 0 \text{ as } t \to \infty.$$

It is apparent that the last three terms in Eq.(3.5) also decrease to zero. Consequently, the solution of equation of motion, which has been determined in Eq.(10) converges to zero, i.e.,

$$x(t) \to 0 \text{ as } t \to \infty. \tag{4.11}$$

In other words, the initialization response of the oscillator will gradually decay in amplitude with increasing of time. In practical applications, it reveals a fact that memories from past histories of motion have no influence on the stability of such systems.

### 4.2. Experimental stability analysis

In this section, numerical simulations are carried out to verify stability of initialization response of such systems. The mass of the oscillator is taken to be $m = 5kg$, the stiffness $k = 500N/s$, the damping coefficient $c = 40N \cdot s/m$. Other parameters are chosen as $\mu = 5$ and $a = 4s$.

By introducing the following new variables:

$$x_1 = x, x_2 = \dot{x}, x_3 = \int_0^t e^{-\mu(t-\tau)}\dot{x}(\tau)\,d\tau,$$

Eq.(3.1) can be transformed into

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -\frac{k}{m}x_1 - \frac{c\mu}{m}x_3 - \frac{c}{m}\psi(t), \\ \dot{x}_3 = x_2 - \mu x_3. \end{cases} \quad (4.12)$$

Vibrating motions in time period $[-a, 0]$ are specified by the following four history functions,

$$v_1(t) = \cos 5t,\ v_2(t) = 0.5\cos 5t,\ v_3(t) = \sin 5t,\ v_4(t) = 1.$$

In terms of Eq.(2.4), the initialization functions are calculated respectively as

$$\psi_1(t) = 0.5e^{-\mu t}, \psi_2(t) = 0.25e^{-\mu t}, \psi_3(t) = -0.5e^{-\mu t},\ \text{and}\ \psi_4(t) = 1.03e^{-5t}.$$

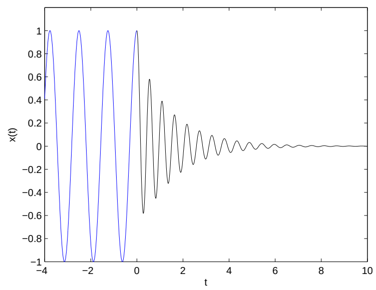
Initial conditions for Eq.(4.12) are respectively as

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0.5\\0\\0\end{pmatrix}, \begin{pmatrix}0\\-5\\0\end{pmatrix},\ \text{and}\ \begin{pmatrix}1\\0\\0\end{pmatrix}.$$

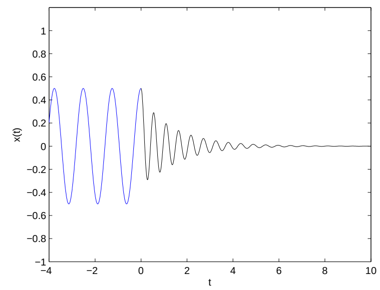
Fig.2 show the initialization responses of the oscillators with different initialization functions. From the above figures, it is clear that although past histories of motion are different, vibrating motions always gradually decay with increasing of time. Consequently, the stability of initialization response has been verified by these numerical simulations.

## 5. Conclusions

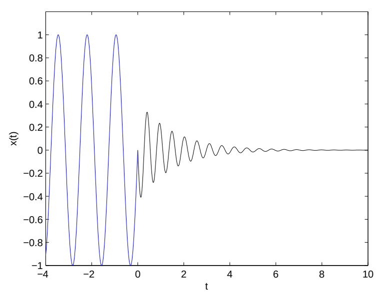
The main contribution of this paper is to prove stability of initialization response of exponentially damped oscillators and to show that the hereditary effects gradually recede with time. From the theoretical analysis and numerical simulations, it has been shown that initialization response of such systems gradually recedes with increasing of time, in spite of different histories of vibrating motions. This implies that although motions from past histories affect the dynamic behavior of such systems, they have no influence in the stability. This phenomenon brings a fact to practical applications: for linear single-degree-of-freedom exponentially damped oscillators without external forces, vibrations always cease due to internal damping, whatever past histories of vibrating motions are. Further research is needed to investigate



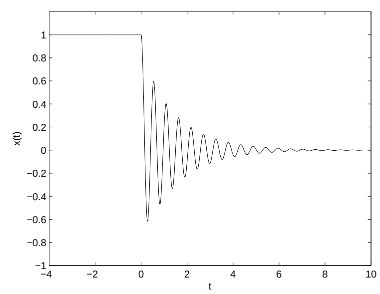

(a) Initialization response with past history $v_1(t) = \cos 5t$.




(b) Initialization response with past history $v_2(t) = 0.5\cos 5t$.




(c) Initialization response with past history $v_3(t) = \sin 5t$.




(d) Initialization response with past history $v_4(t) = 1$.

**Figure 2.** Initialization responses with different past histories of vibrating motion.

hereditary effects on the vibration control design in order to propose more accurate and effective control laws.

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# References

[1] S. Adhikari, *Structural Dynamic Analysis with Generalized Damping Models*, John Wiley & Sons, Hoboken, 2014.

[2] S. Adhikari, *Dynamic response characteristics of a nonviscously damped oscillator*, ASME J. Appl. Mech., 2008, 75(1), 148–155.

[3] S. Adhikari and J. Woodhouse, *Quantification of non-viscous damping in discrete linear systems*, J. Sound Vib., 2003, 260(3), 499–518.

[4] García-Barruetabeña, J., et al., *Dynamics of an exponentially damped solid rod: Analytic solution and finite element formulations*, Int. J. Solids. Struct., 2012, 49(34), 590–598.

[5] H. Beyer and S. Kempfle, *Definition of physically consistent damping laws with fractional derivatives*, ZAMM J. Appl. Math. Mech., 1995. 75(8), 623–635.

[6] B. Du, Y. H. Wei, S. Liang, et al, *Estimation of exact initial states of fractional order systems*, Nonlinear Dynam., 2016, 86(3), 2061–2070.

[7] M. Fukunaga, *On initial value problems of fractional differential equations*, I. J. Appl. Math., 2002, 9(2), 219–236.

[8] R. A. Ibrahim, *Recent advances in nonlinear passive vibration isolators*, J. Sound Vib., 2008, 314(3–5), 371–452.

[9] S. Kempfle, I. Schäfer and H. Beyer, *Fractional calculus via functional calculus: theory and applications*, Nonlinear Dynam., 2002, 29(1–4), 99–127.

[10] L. Li, Y. J. Hu, X. L. Wang, et al, *Computation of Eigensolution Derivatives for Nonviscously Damped Systems Using the Algebraic Method*, AIAA J., 2012, 50(10), 2282–2284.

[11] M. Lázaro, *Closed-form eigensolutions of nonviscously, nonproportionally damped systems based on continuous damping sensitivity*, J. Sound Vib., 2018, 413, 368–382.

[12] C. F. Lorenzo, and T. T. Hartley, *Initialization of Fractional-Order Operators and Fractional Differential Equations*, ASME J. Comput. Nonlinear Dyn., 2008, 3(2), 021101.

[13] A. Muravyov, *Forced vibration responses of a viscoelastic structure*, J. Sound Vib., 1998, 218(5), 892–907.

[14] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity*, Imperial College Press, London, 2010.

[15] M. D. Paola, A. Pirrotta, and A. J. M.o. M. Valenza, *Visco-elastic behavior through fractional calculus: An easier method for best fitting experimental results*, Mech. Mater., 2011. 43(12), 799–806.

[16] J. Padovan, S. Chung, and Y. H. Guo, *Asymptotic steady state behavior of fractionally damped systems*, J. Franklin I., 1987, 324(3), 491–511.

[17] J. Padovan and Y. Guo, *General response of viscoelastic systems modelled by fractional operators*, J. Franklin I., 1988. 325(2), 247–275.

[18] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution And Some of Their Applications*, Academic Press, San Diego, CA, 1999.

[19] A. Reggio, A. M. De, R. Betti, *A state-space methodology to identify modal and physical parameters of non-viscously damped systems*, Mech. Syst. Signal Pr., 2013, 41(1–2), 380–395.

[20] Y. A. Rossikhin and M. V. Shitikova, *Application of fractional calculus for dynamic problems of solid mechanics: novel trends and recent results*, ASME Appl. Mech. Rev., 2010, 63(1), 010801.

[21] Y. A. Rossikhin and M. V. Shitikova, *Analysis of rheological equations involving more than one fractional parameters by the use of the simplest mechanical systems based on these equations*, Mech. Time-Depend. Mat., 2001. 5(2), 131–175.

[22] M. T. Shaw and W. J. Macknight, *Introduction to Polymer Viscoelasticity*, John Wiley & Sons, New York, 2005.

[23] I. Schäfer and S. Kempfle, *Impulse responses of fractional damped systems.* Nonlinear Dynam., 2004, 38(1–4), 61–68.

[24] J. Woodhouse, *Linear damping models for structural vibration*, J. Sound Vib., 1998, 215(3), 547–569.

[25] C. X. Wu, J. Yuan, B. Shi, *Stability of initialization response of fractional oscillators*, J. Vibroeng., 2016, 139(1), 4148–4154.

[26] J. Yuan, Y. A. Zhang, J. M. Liu, et al, *Mechanical energy and equivalent differential equations of motion for single-degree-of-freedom fractional oscillators*, J. Sound Vib., 2017. 397, 192–203.

[27] J. Yuan, Y. A. Zhang, J. M. Liu, et al, *Sliding mode control of vibration in single-degree-of-freedom fractional oscillators*, ASME J. Dyn. Syst., 2017, 139(11), 114503.

[28] Y. A. Zhang, J. Yuan, J. M. Liu, et al, *Lyapunov functions and sliding mode control for two degrees-of-freedom and multidegrees-of-freedom fractional oscillators*, ASME J. Vib. Acoust., 2017. 139(1), 011014.

[29] Y. Zhao, Y. H. Wei, Y. Q. Chen, et al, *A new look at the fractional initial value problem: the aberration phenomenon*, ASME J. Comput. Nonlinear Dyn., 2018, 13(12), 121004.

[30] Y. Zhao, Y. H. Wei, J. Shuai, et al, *Fitting of the initialization function of fractional order systems*, Nonlinear Dynam., 2018, 93(3), 1589–1598.