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NEW CANONICAL FORMS OF SELF-ADJOINT BOUNDARY CONDITIONS FOR REGULAR DIFFERENTIAL OPERATORS OF ORDER FOUR*

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Abstract In this paper, we find new canonical forms of self-adjoint boundary conditions for regular differential operators of order two and four. In the second order case the new canonical form unifies the coupled and separated canonical forms which were known before. Our fourth order forms are similar to the new second order ones and also unify the coupled and separated forms. Canonical forms of self-adjoint boundary conditions are instrumental in the study of the dependence of eigenvalues on the boundary conditions and for their numerical computation. In the second order case this dependence is now well understood due to some surprisingly recent results given the long history and voluminous literature of Sturm-Liouville problems. And there is a robust code for their computation: SLEIGN2.

Keywords Differential operators, boundary conditions, self-adjoint, canonical forms.

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1. Introduction

Consider the regular Sturm-Liouville equation

$$-(py')' + qy = \lambda wy,\ J = (a, b),\ -\infty \le a < b \le +\infty, \tag{1.1}$$

with coefficients functions $p,\ q,\ w$ satisfying

$$\frac{1}{p},\ q,\ w \in L^1(J, \mathbb{R}),\ p > 0,\ w > 0,\ \text{a.e. on } J, \tag{1.2}$$

and boundary conditions

$$AY(a) + BY(b) = 0,\ A,\ B \in M_2(\mathbb{C}),\ Y = \begin{pmatrix} y \\ py' \end{pmatrix}. \tag{1.3}$$

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Eq. (1.1) generates a minimal operator S_{min} and a maximal operator S_{max} with domains D_{min} and D_{max}, respectively. The self-adjoint operators S in the Hilbert space $L^2(J,w)$ generated by (1.1), (1.2), and (1.3) satisfy

$$S_{min} \subset S = S^* \subset S_{max}. \tag{1.4}$$

It is clear that these operators S differ from each other only by their domains. These domains $D(S)$ are characterized by matrices $A,\ B$ which satisfy

$$\mathrm{rank(A:B)} = 2,\ \mathrm{AE_2A^*} = \mathrm{BE_2B^*},\ \mathrm{E_2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{1.5}$$

That is, the linear manifold $D(S)$ defined by

$$D(S) = \{y \in D_{max} : (1.3) \text{ and } (1.5) \text{ hold}\} \tag{1.6}$$

is a self-adjoint domain and every operator S satisfying (1.4) is determined this way.

Here and below, $M_n(\mathbb{X})$ denotes the n by n matrices with entries from $\mathbb{X} = \mathbb{C}$ the complex numbers, or $\mathbb{X} = \mathbb{R}$, the real numbers, and $(A : B)$ denotes the n by $2n$ matrix whose first n columns are those of A in the same order and the last n are those of B in the same order.

It is well known that the self-adjoint condition (1.5) can be classified into two mutually exclusive classes: coupled and separated, and these classes have the following canonical form representations:

$$(A : B) = \left(e^{i\gamma}K : I_2\right),\quad K \in M_2(\mathbb{R})\,,\ \det(K) = 1,\ -\pi < \gamma \leq \pi; \tag{1.7}$$

$$\begin{aligned} &\cos(\alpha)\, y(a) - \sin(\alpha)\, (py')(a) = 0,\ \alpha \in [0,\pi), \\ &\cos(\beta)\, y(b) - \sin(\beta)\, (py')(b) = 0,\ \beta \in (0,\pi]; \end{aligned} \tag{1.8}$$

respectively. Here I_2 denotes the 2 by 2 identity matrix.

These representations (1.7), (1.8) make it possible to define the eigenvalues $\lambda_n(\gamma, K)$ and $\lambda_n(\alpha,\beta)$ and study their properties as functions of these parameters and to compute them numerically; see [14, 16], [26].

How are the eigenvalues determined by the separated boundary conditions (1.8) related to the eigenvalues determined by the coupled boundary conditions (1.7)? This is clearly not apparent from the representations (1.7) and (1.8). In 2000 Kong, Wu, Zettl [15] constructed a space of boundary conditions with a geometric structure and used this structure to study the relationship between eigenvalues determined by different boundary conditions. This was extended by Haertzen, Kong, et. al. [12] and by Cao, Kong, et. al. [6].

In this paper, we obtain a different canonical form representation of (1.5). This new self-adjoint boundary representation 'unifies' (1.7), (1.8). And we find a similar 'unified' canonical form for $n = 4$ where there are three types of self-adjoint conditions: separated, coupled, and mixed. (In the second order case there are no mixed self-adjoint conditions.)

Consider the equation

$$My = [(p_2y'')' - (p_1y')]' + p_0y = \lambda wy,\ \text{on } J = (a,b) \tag{1.9}$$

with coefficients satisfying

$$\frac{1}{p_2},\ p_1,\ p_0,\ w \in L^1(J,\mathbb{R}),\ p_2 > 0,\ w > 0 \text{ a.e. } J. \tag{1.10}$$

It is well known that (1.9) is a symmetric (formally self-adjoint) differential equation. For smooth coefficients the differential expression

$$M\,y = \sum_{j=0}^{2} (-1)^j (p_j y^{(j)})^{(j)}$$

is a closed form for the symmetric (formally self-adjoint) expressions of order four [7]. With hypothesis (1.10) the extra bracket [ ] is needed in (1.9), see [24].

It is well known that the self-adjointness characterization (1.5) extends to the fourth order case:

$$\mathrm{rank}(\mathrm{A} : \mathrm{B}) = 4,\ \mathrm{AE_4A^*} = \mathrm{BE_4B^*},\ \mathrm{E_4} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{1.11}$$

In [21] Wang et. al. proved that for fourth order problems there are *three* mutually exclusive classifications of the self-adjoint boundary conditions:

Theorem 1.1 (Wang-Sun-Zettl)**.** *Let* (1.9)*,* (1.10)*, and* (1.11) *hold. Then*

1.

$$2 \le \mathrm{rank(A)} \le 4,\ \ 2 \le \mathrm{rank(B)} \le 4. \tag{1.12}$$

2. Let $0 \le r \le 2$*. If* $\mathrm{rank(A)} = 2 + \mathrm{r}$*, then* $\mathrm{rank(B)} = 2 + \mathrm{r}$*. Assume that*

$$\mathrm{rank(A)} = 2 + \mathrm{r}. \tag{1.13}$$

Then the boundary conditions are separated when $r = 0$*, mixed when* $r = 1$*, and coupled when* $r = 2$*.*

Remark 1.1. This theorem gives a rigorous definition of the separated, coupled, and mixed self-adjoint boundary conditions. In [21] this theorem is proven for all even order problems $n = 2k,\ k > 1$. For each of these problems there are only three classifications of the self-adjoint boundary conditions: separated, coupled, and mixed.

For $n = 4$ canonical forms for all three classifications were found by Hao et. al. in [11] where it is shown that there are four types of coupled conditions, 16 types of mixed conditions, and 16 types of separated conditions. And for each of these three types there is a fundamental type in the sense that each of the conditions of that type can be transformed to the fundamental one with elementary matrix manipulations.

In this paper we find a new canonical form for $n = 2$ and a similar canonical form for $n = 4$. Both of these new forms unify the different types of conditions with each other. More specifically we:

1. Find a new canonical form for the second order problem. This unifies the coupled and separated forms (1.7), (1.8). It has the representations:

$$\begin{pmatrix} r_1 & 1 & c & 0 \\ -\bar{c} & 0 & r_2 & 1 \end{pmatrix}. \tag{1.14}$$

   - When $c \neq 0$ is a nonreal number, then (1.14) represents a canonical form for the nonreal coupled conditions.
   - When $c \neq 0$ is a real number, then (1.14) represents a canonical form for the real coupled conditions.
   - When $c = 0$, then (1.14) is a canonical form of the separated boundary conditions with the understanding that two 'special' separated conditions require letting r_1 and r_1 approach infinity. See Remarks 2.2 and 2.3 below for details.

2. Find a new canonical form for fourth order problems which has the block matrix representation:

$$(A : B) = \begin{pmatrix} R_1 & J_2 & C & 0 \\ -C^* & 0 & R_2 & J_2 \end{pmatrix}, \; J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; C \in M_2(\mathbb{C}). \tag{1.15}$$

   - When rank(C) $= 2$, then (1.15) represents a canonical form for the coupled conditions.
   - When rank(C) $= 1$, then (1.15) represents a canonical form for the mixed conditions.
   - When rank(C) $= 0$, then (1.15) represents a canonical form for the separated conditions.
   - Furthermore, R_1, R_2 and J_2 are symmetric and thus can be considered as playing the roles of r_1, r_2, and 1 in (1.14). The complex matrix C plays the role of the complex number c in (1.14).

In addition we find other equivalent canonical forms for the coupled conditions in both the second and fourth order cases. In the second order case this condition has the form:

$$(A : B) = \left(K_1 \begin{pmatrix} \frac{1}{c} & 0 \\ 0 & \bar{c} \end{pmatrix} K_2 : I_2 \right), \tag{1.16}$$

where $K_1 = \begin{pmatrix} 1 & 0 \\ -r_2 & 1 \end{pmatrix}$, $K_2 = \begin{pmatrix} r_1 & 1 \\ -1 & 0 \end{pmatrix}$.

For order four we get

$$(A : B) = \left(K_1 \begin{pmatrix} C^{-1} & 0_{2\times 2} \\ 0_{2\times 2} & C^* \end{pmatrix} K_2 : I_2 \right), \tag{1.17}$$

where

$$K_1 = \begin{pmatrix} I_2 & 0_{2\times 2} \\ -J_2R_2 & J_2 \end{pmatrix}, \quad K_2 = \begin{pmatrix} R_1 & J_2 \\ -I_2 & 0_{2\times 2} \end{pmatrix},$$

and $R_2,\ R_1,\ I_2,\ J_2$ are all symmetric matrices, and $|\det(K_1)| = |\det(K_2)| = 1$.

Note that there is a $1-1$ correspondence between (1.15) and (1.14) with $c \neq 0$ when $n = 2$, and between (1.17) and (1.15) with rank(C) $= 2$ when $n = 4$.

Our proof uses a delicate interplay between the theory of linear differential equations [10, 18, 27] and the theory of linear algebra.

This paper is organized as follows. In Section 2 we introduce a new method for studying the $n = 2$ case. In Section 3 we use some parts of this method of proof to establish the case $n = 4$. Examples to illustrate these results are given in Section 4. We plan to investigate the general case of $n = 2k$ for $k > 2$ in a subsequent paper.

2. A New Canonical Form For Order Two

In this section we develop a new method for studying the self-adjoint $n = 2$ problem. Let $\widetilde{A} = A$ and $\widetilde{B} = B$ and let

$$(\widetilde{A} : \widetilde{B}) = \begin{pmatrix} \widetilde{a}_{11} & \widetilde{a}_{12} & \widetilde{b}_{11} & \widetilde{b}_{12} \\ \widetilde{a}_{21} & \widetilde{a}_{22} & \widetilde{b}_{21} & \widetilde{b}_{22} \end{pmatrix}. \tag{2.1}$$

Then the linear submanifold:

$$D(S) = \left\{ y \in D_{max} : \widetilde{A}Y(a) + \widetilde{B}Y(b) = 0,\ Y = \begin{pmatrix} y^{[0]} \\ y^{[1]} \end{pmatrix} \right\}, \tag{2.2}$$

with $\widetilde{A},\ \widetilde{B}$ satisfying (1.5) is a self-adjoint domain, and all self-adjoint domains are generated in this way.

Next we derive a new canonical form for $n = 2$.

If rank($\widetilde{\mathrm{B}}$) $= 2$, then it follows from (1.5) that rank($\widetilde{\mathrm{A}}$) $= 2$ and therefore the boundary condition is coupled. So by elementary matrix transformation of rows, the matrix $\widetilde{B}$ can be transformed into the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Noting that the domain is invariant under the elementary matrix transformations of the rows of $(\widetilde{A} : \widetilde{B})$, the matrix $(\widetilde{A} : \widetilde{B})$ can be transformed into the following form if $\widetilde{a}_{11} \neq 0$:

$$\begin{aligned} (\widetilde{A} : \widetilde{B}) = \begin{pmatrix} \widetilde{a}_{11} & \widetilde{a}_{12} & \widetilde{b}_{11} & \widetilde{b}_{12} \\ \widetilde{a}_{21} & \widetilde{a}_{22} & \widetilde{b}_{21} & \widetilde{b}_{22} \end{pmatrix} &\xrightarrow[\text{rewrite}]{} \begin{pmatrix} \widetilde{a}_{11} & \widetilde{a}_{12} & 1 & 0 \\ \widetilde{a}_{21} & \widetilde{a}_{22} & 0 & 1 \end{pmatrix} \\ \to \begin{pmatrix} 1 & a_{12} & b_{11} & 0 \\ \widetilde{a}_{21} & \widetilde{a}_{22} & 0 & 1 \end{pmatrix} &\to \begin{pmatrix} 1 & a_{12} & b_{11} & 0 \\ 0 & a_{22} & b_{21} & 1 \end{pmatrix} = (A : B), \end{aligned} \tag{2.3}$$

where $a_{22} \neq 0,\ b_{11} \neq 0$.

Suppose $\widetilde{a}_{11} = 0$. Then from $\operatorname{rank}(\widetilde{\mathrm{A}}) = 2$, we have $\widetilde{a}_{12} \neq 0$, then $(\widetilde{A} : \widetilde{B})$ can be transformed to

$$(\widetilde{A} : \widetilde{B}) = \begin{pmatrix} \widetilde{a}_{11} & \widetilde{a}_{12} & \widetilde{b}_{11} & \widetilde{b}_{12} \\ \widetilde{a}_{21} & \widetilde{a}_{22} & \widetilde{b}_{21} & \widetilde{b}_{22} \end{pmatrix} \xrightarrow[\text{rewrite}]{} \begin{pmatrix} 0 & \widetilde{a}_{12} & 1 & 0 \\ \widetilde{a}_{21} & \widetilde{a}_{22} & 0 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 0 & 1 & b_{11} & 0 \\ \widetilde{a}_{21} & \widetilde{a}_{22} & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & b_{11} & 0 \\ a_{21} & 0 & b_{21} & 1 \end{pmatrix} = (A : B), \tag{2.4}$$

where $a_{21} \neq 0,\ b_{11} \neq 0$.

We have the following lemma:

Lemma 2.1. *Suppose* $\operatorname{rank}(\widetilde{\mathrm{A}} : \widetilde{\mathrm{B}}) = 2$ *and* $\operatorname{rank}(\widetilde{\mathrm{B}}) = 2$. *Then we obtain new coupled canonical forms:*

1.

$$(\widetilde{A} : \widetilde{B}) = \begin{pmatrix} \widetilde{a}_{11} & \widetilde{a}_{12} & \widetilde{b}_{11} & \widetilde{b}_{12} \\ \widetilde{a}_{21} & \widetilde{a}_{22} & \widetilde{b}_{21} & \widetilde{b}_{22} \end{pmatrix} = \begin{pmatrix} 1 & r_1 & c & 0 \\ 0 & \bar{c} & r_2 & 1 \end{pmatrix}, \tag{2.5}$$

where $r_1,\ r_2$ *are real numbers, i.e.* $a_{12} = \bar{a}_{12},\ b_{21} = \bar{b}_{21}$, *and* $b_{11} = \bar{a}_{22} = c$.

2. Or

$$(\widetilde{A} : \widetilde{B}) = \begin{pmatrix} \widetilde{a}_{11} & \widetilde{a}_{12} & \widetilde{b}_{11} & \widetilde{b}_{12} \\ \widetilde{a}_{21} & \widetilde{a}_{22} & \widetilde{b}_{21} & \widetilde{b}_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 & c & 0 \\ -\bar{c} & 0 & r & 1 \end{pmatrix}, \tag{2.6}$$

where r *is a real number, i.e.* $b_{21} = \bar{b}_{21}$, *and* $b_{11} = -\bar{a}_{21} = c,\ c \in \mathbb{C}$.

Proof. By using formula (1.5) and (2.3), we calculate that

$$\begin{pmatrix} 1 & a_{12} \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \bar{a}_{12} & \bar{a}_{22} \end{pmatrix} = \begin{pmatrix} b_{11} & 0 \\ b_{21} & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \bar{b}_{11} & \bar{b}_{21} \\ 0 & 1 \end{pmatrix},$$

so

$$\begin{pmatrix} a_{12} & -1 \\ a_{22} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \bar{a}_{12} & \bar{a}_{22} \end{pmatrix} = \begin{pmatrix} 0 & -b_{11} \\ 1 & -b_{21} \end{pmatrix} \begin{pmatrix} \bar{b}_{11} & \bar{b}_{21} \\ 0 & 1 \end{pmatrix}.$$

Therefore

$$\begin{pmatrix} a_{12} - \bar{a}_{12} & -\bar{a}_{22} \\ a_{22} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -b_{11} \\ \bar{b}_{11} & \bar{b}_{21} - b_{21} \end{pmatrix}.$$

This shows that

$$a_{12} = \bar{a}_{12},\ a_{22} = \bar{b}_{11} = \bar{c},\ b_{21} = \bar{b}_{21},\ c \in \mathbb{C},$$

and we obtain the form (2.5).

Similarly, by using (1.5) and (2.4), we obtain $b_{21} = \bar{b}_{21},\ a_{21} = -\bar{b}_{11} = -\bar{c}$, in (2.4), i.e., the canonical form (2.6) is established. □

Theorem 2.1. *The two canonical forms* (2.5) *and* (2.6) *in Lemma* 2.1 can be combined to obtain the following self-adjoint canonical form:

$$\begin{pmatrix} r_1 & 1 & c & 0 \\ -\bar{c} & 0 & r_2 & 1 \end{pmatrix}. \tag{2.7}$$

Proof. It is easy to verify that (2.7) is a self adjoint boundary condition. And:

- when $r_1 = 0$ in (2.7) it can be reduced to (2.6), where $r_2 = r$.
- when $r_1 \neq 0$ in (2.7) it can be transformed to

$$\begin{pmatrix} 1 & \frac{1}{r_1} & \frac{c}{r_1} & 0 \\ -\bar{c} & 0 & r_2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{1}{r_1} & \frac{c}{r_1} & 0 \\ 0 & \frac{\bar{c}}{r_1} & r_2 + \frac{c\bar{c}}{r_1} & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & \widetilde{r_1} & \widetilde{c} & 0 \\ 0 & \overline{\widetilde{c}} & \widetilde{r_2} & 1 \end{pmatrix} \ (\widetilde{r_1} = \frac{1}{r_1},\ \widetilde{c} = \frac{c}{r_1},\ \widetilde{r_2} = r_2 + \frac{c\bar{c}}{r_1})$$
$$\xrightarrow[\text{rewrite}]{} \begin{pmatrix} 1 & r_1 & c & 0 \\ 0 & \bar{c} & r_2 & 1 \end{pmatrix},$$

  where $r_1,\ r_2 \in \mathbb{R},\ c \in \mathbb{C}$. Thus, in this case, (2.7) can be transformed to the form (2.5). □

Remark 2.1. We can get separated self-adjoint boundary conditions from (2.7). In (2.7) if $c = 0$, then (2.7) is reduced to

$$(\widetilde{A} : \widetilde{B}) = \begin{pmatrix} r_1 & 1 & 0 & 0 \\ 0 & 0 & r_2 & 1 \end{pmatrix}. \tag{2.8}$$

This is a real separated boundary condition.

Remark 2.2. Further, we can derive the self-adjoint separated boundary conditions (1.8) from (2.8). In fact (2.8) can be transformed to

$$(\widetilde{A} : \widetilde{B}) = \begin{pmatrix} r_1 & 1 & 0 & 0 \\ 0 & 0 & r_2 & 1 \end{pmatrix}$$
$$\to \begin{pmatrix} -\frac{r}{\sqrt{r_1^2+1}} & -\frac{1}{\sqrt{r_1^2+1}} & 0 & 0 \\ 0 & 0 & -\frac{r_2}{\sqrt{r_2^2+1}} & -\frac{1}{\sqrt{r_2^2+1}} \end{pmatrix}. \tag{2.9}$$

Let

$$\cos(\alpha) = -\frac{r_1}{\sqrt{r_1^2+1}},\ -\sin(\alpha) = -\frac{1}{\sqrt{r_1^2+1}},\ \alpha \in (0,\pi),$$
$$\cos(\beta) = -\frac{r_2}{\sqrt{r_2^2+1}},\ -\sin(\beta) = -\frac{1}{\sqrt{r_2^2+1}},\ \beta \in (0,\pi),$$

then (2.9) can be transformed to

$$\rightarrow \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 & 0 \\ 0 & 0 & \cos(\beta) & -\sin(\beta) \end{pmatrix}. \tag{2.10}$$

This is the self-adjoint separated boundary condition (1.8). And

$$\begin{aligned}
&r_1 \to -\infty,\ \cos(\alpha) \to +1,\ \sin(\alpha) \to 0,\ \alpha \to 0,\\
&r_1 \to +\infty,\ \cos(\alpha) \to -1,\ \sin(\alpha) \to 0,\ \alpha \to \pi,\\
&r_1 = 0,\ \cos(\alpha) = 0,\ \sin(\alpha) = 1,\ \alpha = \frac{\pi}{2},\\
&r_2 \to -\infty,\ \cos(\beta) \to +1,\ \sin(\beta) \to 0,\ \beta \to 0,\\
&r_2 \to +\infty,\ \cos(\beta) \to -1,\ \sin(\beta) \to 0,\ \beta \to \pi,\\
&r_2 = 0,\ \cos(\beta) = 0,\ \sin(\beta) = 1,\ \beta = \frac{\pi}{2}.
\end{aligned}$$

Remark 2.3. Note that (2.7) is a new characterization of the second order self-adjoint domains, which is different from the well known canonical forms (1.7), (1.8). Also note that the separated conditions can be parameterized with $r_1 = -cot(\alpha),\ 0 < \alpha < \pi$ and $r_2 = -cot(\beta),\ 0 < \beta < \pi$ and the 'special' condition $(py')(a) = 0 = y(b)$ mentioned above corresponds to $\alpha = \frac{\pi}{2},\ \beta = \pi$.

Next we show that the coupled self-adjoint boundary forms described by (1.7) are equivalent to (2.7) in Theorem 2.1. Notice that the boundary conditions described in the domain $D(S)$ do not change when the matrix $(\widetilde{A} : \widetilde{B})$ is left multiplied by a nonsingular matrix B^{-1}.

When rank(A) = 2 = rank(B), we have $c \neq 0$ in (2.7) , then from (2.7) we consider the matrix product $\widetilde{B}^{-1}\widetilde{A}$, i.e.,

$$\widetilde{B}^{-1}\widetilde{A} = \begin{pmatrix} c & 0 \\ r_2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} r_1 & 1 \\ -\bar{c} & 0 \end{pmatrix} = \frac{1}{c}\begin{pmatrix} 1 & 0 \\ -r_2 & c \end{pmatrix}\begin{pmatrix} r_1 & 1 \\ -\bar{c} & 0 \end{pmatrix}$$

$$= \frac{1}{c}\begin{pmatrix} r_1 & 1 \\ -r_1r_2 - c\bar{c} & -r_2 \end{pmatrix} = \frac{\mid c \mid}{c}\begin{pmatrix} \frac{r_1}{|c|} & \frac{1}{|c|} \\ -\frac{r_1r_2+\bar{c}c}{|c|} & -\frac{r_2}{|c|} \end{pmatrix}.$$

Assume that $k_{11} = \frac{r_1}{|c|},\ k_{12} = \frac{1}{|c|},\ k_{21} = -\frac{r_1r_2+c\bar{c}}{|c|},\ k_{22} = -\frac{r_2}{|c|},\ \frac{|c|}{c} = e^{i\gamma}(-\pi < \gamma \leq \pi)$, and hence

$$\widetilde{B}^{-1}\widetilde{A} = e^{i\gamma}\begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} = e^{i\gamma}K, \tag{2.11}$$

where $K = (k_{ij})_{(i,j=1,2)}$ is a real matrix, and $\det(K) = 1$.

Eq. (2.11) shows that the second order coupled self-adjoint boundary conditions can be characterized as:

$$\widetilde{A}Y(a) + \widetilde{B}Y(b) = 0,$$

in which

$$\widetilde{B} = I_2,\ \widetilde{A} = e^{i\gamma}K,\ \gamma \in (-\pi, \pi]. \tag{2.12}$$

That is, the corresponding boundary matrix of the coupled canonical form is:

$$(\widetilde{A} : \widetilde{B}) = \left(e^{i\gamma}K : I_2\right), \tag{2.13}$$

where $K = (k_{ij})_{(i,j=1,2)}$ is a real matrix and $\det(K) = 1$.

Remark 2.4. The result (2.10) or (2.13) is well known, see [25]. Thus the boundary matrix form (2.7) gives a new description of the second order self-adjoint domains which unifies the different representations of the separated and coupled conditions. This should simplify the study of the relationship of the eigenvalues of coupled boundary conditions to the eigenvalues of nearby separated conditions.

Also notice that (2.11) can be rewritten as follows:

$$\begin{aligned}
\widetilde{B}^{-1}\widetilde{A} = e^{i\gamma}K &= e^{i\gamma}\begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \\
&= \begin{pmatrix} c & 0 \\ r_2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} r_1 & 1 \\ -\bar{c} & 0 \end{pmatrix} = \frac{1}{c}\begin{pmatrix} r_1 & 1 \\ -r_1r_2 - c\bar{c} & -r_2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ -r_2 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{c} & 0 \\ 0 & \bar{c} \end{pmatrix}\begin{pmatrix} r_1 & 1 \\ -1 & 0 \end{pmatrix} = K_1\begin{pmatrix} \frac{1}{c} & 0 \\ 0 & \bar{c} \end{pmatrix}K_2,
\end{aligned}$$

where $K_1 = \begin{pmatrix} 1 & 0 \\ -r_2 & 1 \end{pmatrix}$, $K_2 = \begin{pmatrix} r_1 & 1 \\ -1 & 0 \end{pmatrix}$.

Remark 2.5. That is, the boundary matrix forms (2.7) corresponding to the coupled self-adjoint canonical form for the second order differential operators can be transformed into:

$$(\widetilde{A} : \widetilde{B}) = \left(K_1\begin{pmatrix} \frac{1}{c} & 0 \\ 0 & \bar{c} \end{pmatrix}K_2 : I_2\right), \tag{2.14}$$

where the matrix $\widetilde{A} = K_1\begin{pmatrix} \frac{1}{c} & 0 \\ 0 & \bar{c} \end{pmatrix}K_2$ is the product of three matrices, the middle matrix is a diagonal matrix which is determined by $c = b_{11}$ and the absolute value of its determinant is 1; the other two matrices are real, and they are determined by r_1 and r_2 respectively (note that in the form (2.6) K_1 is determined by r and $K_2 = -E_2$) and $\det(K_1) = \det(K_2) = 1$ and K_1, K_2 are symplectic matrices [13]. And there is a one to one correspondence between the coupled self-adjoint canonical forms (2.7) and (2.14) (or (2.13)).

The canonical form of coupled self-adjoint conditions for $n = 4$ established below in Section 3 (Theorem 3.3, Remark 3.2, Theorem 3.4) are similar to the forms established here in Section 2 with the exception, of course, that for $n = 2$ there are only separated and coupled conditions but for $n = 4$ there are separated, coupled, and mixed conditions. The coupled self-adjoint canonical forms for $n = 4$ established below were motivated by our new form (2.7) for $n = 2$. Both new forms relate the different classifications with each other more than the previously known forms did, e.g. for $n = 2$ the forms (1.7) and (1.8) are not closely related. We expect that the more closely related forms will be useful tools to get more information about the relationships between the eigenvalues of separated and coupled boundary conditions.

3. Canonical Forms Of Self-Adjoint Boundary Conditions For Order Four

As mentioned above, in this section we obtain new canonical forms of self-adjoint boundary conditions for fourth order equations:

$$My = [(p_2y'')' - (p_1y')]' + p_0y = \lambda wy, \text{ on } J = (a, b), \tag{3.1}$$

with coefficients satisfying

$$\frac{1}{p_2},\ p_1,\ p_0,\ w \in L^1(J, \mathbb{R}),\ p_2 > 0,\ w > 0 \text{ a.e. } J. \tag{3.2}$$

Let

$$Q = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{p_2} & 0 \\ 0 & p_1 & 0 & -1 \\ p_0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.3}$$

and define quasi-derivatives by

$$y^{[0]} = y,\ y^{[1]} = y',\ y^{[2]} = p_2(y^{[1]})',\ y^{[3]} = p_1y^{[1]} - (y^{[2]})', \tag{3.4}$$

and let

$$My = y^{[4]} = p_0y^{[0]} - (y^{[3]})'. \tag{3.5}$$

Note that these quasi-derivatives $y^{[i]} = y_Q^{[i]}$, $i = 0, \cdots, 4$ depend on Q and (3.1) is given by

$$My = y^{[4]} = p_0y^{[0]} - (y^{[3]})' = \lambda wy.$$

For simplicity of exposition we omit the subscript Q. The domain of M, $D(M)$, consists of all complex valued functions y such that $y^{[i]} = y_Q^{[i]}$, $i = 0, \cdots, 4$ is absolutely continuous on each compact subinterval of $J = (a, b)$. Then

$$My = y^{[4]} = \lambda wy$$

is defined *a.e.* on J.

Consider the boundary conditions.

$$A\begin{pmatrix} y^{[0]}(a) \\ y^{[1]}(a) \\ y^{[2]}(a) \\ y^{[3]}(a) \end{pmatrix} + B\begin{pmatrix} y^{[0]}(b) \\ y^{[1]}(b) \\ y^{[2]}(b) \\ y^{[3]}(b) \end{pmatrix} = 0,\ A,\ B \in M_4(\mathbb{C}), \tag{3.6}$$

where $y^{[i]} = y_Q^{[i]}$, $i = 0, \cdots, 4$.

Fundamental to the study of boundary value problems is the Lagrange identity. The next Lemma and Theorem establish this identity.

Lemma 3.1. *Let Q be given by* (3.3)*. Then $Q = -F_4^{-1}Q^*F_4$, where*

$$F_4 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0_{2\times 2} & -J_2 \\ J_2 & 0_{2\times 2} \end{pmatrix}, \tag{3.7}$$

$0_{2\times 2}$ *is a* 2 *by* 2 *zero matrix and* $J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Proof. Note that

$$F_4^* = -F_4 = F_4^{-1}. \tag{3.8}$$

By a direct calculation, we have

$$\begin{aligned} -F_4^{-1}Q^*F_4 &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & p_0 \\ 1 & 0 & p_1 & 0 \\ 0 & \frac{1}{p_2} & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -\frac{1}{p_2} & 0 & 0 \\ 1 & 0 & p_1 & 0 \\ 0 & 0 & 0 & p_0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{p_2} & 0 \\ 0 & p_1 & 0 & -1 \\ p_0 & 0 & 0 & 0 \end{pmatrix} = Q. \end{aligned}$$

$\square$

Theorem 3.1 (Lagrange Identity)**.** *Let Q be given by* (3.3) *and let $M = M_Q = y^{[4]}$. For any $y,\ z \in D(M)$ we have*

$$[y, z] = \sum_{k=1}^{2}\{y^{[k-1]}\overline{z^{[4-k]}} - y^{[4-k]}\overline{z^{[k-1]}}\} = Z^*F_4Y, \tag{3.9}$$

where

$$Z = \left(z^{[0]}\ z^{[1]}\ z^{[2]}\ z^{[3]} \right)^T,\ Y = \left(y^{[0]}\ y^{[1]}\ y^{[2]}\ y^{[3]} \right)^T, \tag{3.10}$$

and $y^{[i]}$, $z^{[i]}$, $i = 0, 1, \cdots, 3$ *are defined by* (3.4).

Proof. By a direct caculation, we have

$$\begin{aligned}
[y,z]_a^b &= \int_a^b \overline{z} My dx - \int_a^b \overline{Mz} y dx = -\int_a^b (y^{[3]})' \overline{z} dx + \int_a^b \overline{(z^{[3]})'} y dx \\
&= -y^{[3]}\overline{z} + \int_a^b y^{[3]} \overline{z}' dx + \overline{z^{[3]}} y - \int_a^b \overline{z^{[3]}} y' dx \\
&= -y^{[3]}\overline{z} + \int_a^b \{p_1 y^{[1]} - (y^{[2]})'\} \overline{z^{[1]}} dx + \overline{z^{[3]}} y - \int_a^b \overline{\{p_1 z^{[1]} - (z^{[2]})'\}} y^{[1]} dx \\
&= -y^{[3]}\overline{z} - \int_a^b (y^{[2]})' \overline{z^{[1]}} dx + \overline{z^{[3]}} y + \int_a^b \overline{(z^{[2]})'} y^{[1]} dx \\
&= -y^{[3]}\overline{z} - y^{[2]}\overline{z^{[1]}} + \overline{z^{[3]}} y + \overline{z^{[2]}} y^{[1]} \\
&= \left(\overline{z^{[3]}}, \overline{z^{[2]}}, -\overline{z^{[1]}}, -\overline{z^{[0]}} \right) Y = Z^* F_4 Y.
\end{aligned}$$

□

The above Lagrange identity is based on the representation $M = M_Q$. The next Theorem establishes a different (but, of course, equivalent) characterization of the self-adjoint boundary conditions (1.11). The following Theorem, with the help of Lemma 3.1 and Theorem 3.1 above, will establish a new canonical form for the fourth order self-adjoint boundary conditions which is similar to the new second order canonical form established in Section 2 above.

Theorem 3.2. *Let* Q *be given by* (3.3) *and let* $M = M_Q$ *and let the quasi-derivatives* $y^{[i]} = y_Q^{[i]}$ $i = 0, 1, 2, 3$ *be defined by* (3.4). *Let* F_4 *be given by* (3.7) *and assume the matrices* A, $B \in M_4(\mathbb{C})$ *satisfy*

$$AF_4A^* = BF_4B^*,\ \mathrm{rank(A : B)} = 4. \tag{3.11}$$

Define a linear submanifold $D(S)$ *of* $D_{\max}$ *by*

$$D(S) = \left\{ y \in D_{max} :\ AY(a) + BY(b) = 0,\ Y = \left(y^{[0]}\ y^{[1]}\ y^{[2]}\ y^{[3]} \right)^T \right\}. \tag{3.12}$$

Then $D(S)$ *is the domain of a self-adjoint extension* S *of* $S_{\min}$ *, i.e.,*

$$S_{\min} \subset S = S^* \subset S_{\max}, \tag{3.13}$$

and every self-adjoint extension of $S_{\min}$ *is determined this way.*

In particular, if rank(A) = rank(B) = 4, *i.e.* $r = 2$ *in Theorem* 1.1, then the boundary condition (3.12) is coupled and every coupled self-adjoint boundary condition is generated in this way.

Proof. This follows from the Lagrange identity. By Theorem 3.1, we obtain

$$\int_a^b \overline{z} My dx - \int_a^b \overline{Mz} y dx = [y,z]_a^b = Z^*(b)F_4Y(b) - Z^*(a)F_4Y(a) = 0,$$

then

$$D(S)=\left\{y\in D_{max}:AY(a)+BY(b)=0,\ Y=\begin{pmatrix}y^{[0]}\ y^{[1]}\ y^{[2]}\ y^{[3]}\end{pmatrix}^{T}\right\}$$

is self-adjoint domain if and only if

$$AF_4A^*=BF_4B^*. \qquad \square$$

Based on Theorem 3.2, we construct the canonical forms of the self-adjoint boundary conditions. For this it is convenient to use the following block form of $(A:B)$:

$$(A:B)=\begin{pmatrix}A_1 & A_2 & B_1 & B_2\\ A_3 & A_4 & B_3 & B_4\end{pmatrix},\ A_i,\ B_i\in M_2(\mathbb{C}),\ i=1,2,3,4. \tag{3.14}$$

If $rank(A)=4$, then $\mathrm{rank}(\mathrm{A}_2:\mathrm{A}_4)^{\mathrm{T}}=2$. So the matrix $(A_2:A_4)^T$ can be transformed into the following form by elementary matrix transformations of rows:

$$\begin{pmatrix}A_2\\ A_4\end{pmatrix}=\begin{pmatrix}0&1\\1&0\\0&0\\0&0\end{pmatrix}. \tag{3.15}$$

Since $AF_4A^*=BF_4B^*$, $\mathrm{rankB}=\mathrm{rankA}=4$. We have:

Lemma 3.2 ([11]). *Noticing that when* $\mathrm{rank(B)}=\mathrm{rank(A)}=4$, *we have* $rank(B_3:B_4)=2$. *By row transformations,* $(B_3:B_4)$ *can be transformed into the following six forms:*

(1) $\begin{pmatrix}b_{31}&b_{32}&0&1\\b_{41}&b_{42}&1&0\end{pmatrix}$, *(2)* $\begin{pmatrix}b_{31}&0&b_{33}&1\\b_{41}&1&0&0\end{pmatrix}$, *(3)* $\begin{pmatrix}0&b_{32}&1&0\\1&0&0&0\end{pmatrix}$,

(4) $\begin{pmatrix}0&1&0&0\\1&0&0&0\end{pmatrix}$, *(5)* $\begin{pmatrix}0&b_{32}&b_{33}&1\\1&0&0&0\end{pmatrix}$, *(6)* $\begin{pmatrix}b_{31}&0&1&0\\b_{41}&1&0&0\end{pmatrix}$.

Since the boundary conditions (3.12) are invariant under elementary matrix transformation of rows of $(A:B)$, in case (1) of Lemma 3.1 the matrix $(A:B)$ can be transformed into the following form:

$$(A:B)=\begin{pmatrix}A_1&A_2&B_1&B_2\\A_3&A_4&B_3&B_4\end{pmatrix}\rightarrow\begin{pmatrix}\widetilde{A}_1&J_2&\widetilde{B}_1&\widetilde{B}_2\\A_3&A_4&B_3&B_4\end{pmatrix}\rightarrow\begin{pmatrix}\widetilde{A}_1&J_2&\widetilde{B}_1&\widetilde{B}_2\\\widetilde{A}_3&0_{2\times2}&\widetilde{B}_3&\widetilde{B}_4\end{pmatrix} \tag{3.16}$$

$$\xrightarrow[\text{rewrite}]{}\begin{pmatrix}\widetilde{A}_1&J_2&\widetilde{B}_1&\widetilde{B}_2\\\widetilde{A}_3&0_{2\times2}&\widetilde{B}_3&J_2\end{pmatrix}\xrightarrow[\text{rewrite}]{}\begin{pmatrix}\widetilde{A}_1&J_2&\widetilde{B}_1&0_{2\times2}\\\widetilde{A}_3&0_{2\times2}&\widetilde{B}_3&J_2\end{pmatrix}=(\widetilde{A}:\widetilde{B}),$$

where A_3 and B_1 are nonsingular matrices. Then we have the following Theorem:

Theorem 3.3. *If* $\mathrm{rank(A : B)} = 4$, $(A_2 : A_4)^T$ *can be transformed into* (3.15), *with* $\mathrm{rank(B_4)} = 2$, *then the matrix* $(A : B)$ *satisfying* $AF_4A^* = BF_4B^*$ *can be transformed into the following canonical form:*

$$(A : B) = \begin{pmatrix} A_1 & A_2 & B_1 & B_2 \\ A_3 & A_4 & B_3 & B_4 \end{pmatrix} = \begin{pmatrix} R_1 & J_2 & C & 0_{2\times2} \\ -C^* & 0_{2\times2} & R_2 & J_2 \end{pmatrix}, \tag{3.17}$$

where $J_2,\ R_1,\ R_2$ *are symmetric matrices, and the canonical form is determined by* $R_1,\ R_2,\ C$.

Proof. From (3.16), the matrix $(A : B)$ can be transformed into

$$(A : B) = \begin{pmatrix} \widetilde{A}_1 & J_2 & \widetilde{B}_1 & 0_{2\times2} \\ \widetilde{A}_3 & 0_{2\times2} & \widetilde{B}_3 & J_2 \end{pmatrix}.$$

From $AF_4A^* = BF_4B^*$ we have

$$\begin{pmatrix} \widetilde{A}_1 & J_2 \\ \widetilde{A}_3 & 0_{2\times2} \end{pmatrix} \begin{pmatrix} 0_{2\times2} & -J_2 \\ J_2 & 0_{2\times2} \end{pmatrix} \begin{pmatrix} \widetilde{A}_1^* & \widetilde{A}_3^* \\ J_2^* & 0_{2\times2} \end{pmatrix} = \begin{pmatrix} \widetilde{B}_1 & 0_{2\times2} \\ \widetilde{B}_3 & J_2 \end{pmatrix} \begin{pmatrix} 0_{2\times2} & -J_2 \\ J_2 & 0_{2\times2} \end{pmatrix} \begin{pmatrix} \widetilde{B}_1^* & \widetilde{B}_3^* \\ 0_{2\times2} & J_2^* \end{pmatrix}.$$

Hence

$$\begin{pmatrix} J_2J_2 & -\widetilde{A}_1J_2 \\ 0_{2\times2} & -\widetilde{A}_3J_2 \end{pmatrix} \begin{pmatrix} \widetilde{A}_1^* & \widetilde{A}_3^* \\ J_2^* & 0_{2\times2} \end{pmatrix} = \begin{pmatrix} 0_{2\times2} & -\widetilde{B}_1J_2 \\ J_2J_2 & -\widetilde{B}_3J_2 \end{pmatrix} \begin{pmatrix} \widetilde{B}_1^* & \widetilde{B}_3^* \\ 0_{2\times2} & J_2^* \end{pmatrix}.$$

Since $J_2^{-1} = J_2 = J_2^*$, we have

$$\begin{pmatrix} \widetilde{A}_1^* - \widetilde{A}_1 & A_3^* \\ -\widetilde{A}_3 & 0_{2\times2} \end{pmatrix} = \begin{pmatrix} 0_{2\times2} & -\widetilde{B}_1 \\ \widetilde{B}_1^* & \widetilde{B}_3^* - \widetilde{B}_3 \end{pmatrix}.$$

From this it follows that

$$\widetilde{A}_1 = \widetilde{A}_1^*,\quad \widetilde{A}_3 = -\widetilde{B}_1^*,\quad \widetilde{B}_3 = \widetilde{B}_3^*.$$

Therefore we have the self-adjoint boundary condition

$$(A : B) = \begin{pmatrix} R_1 & J_2 & C & 0_{2\times2} \\ -C^* & 0_{2\times2} & R_2 & J_2 \end{pmatrix},$$

where $R_1,\ R_2$ are symmetric, and $C \in M_2(\mathbb{C})$, completing the proof. □

Remark 3.1. Note that if the boundary matrices $A,\ B$ in (3.14) satisfy $AF_4A^* = BF_4B^*$ and $\mathrm{rank(B_4)} = 2$, then a canonical form for coupled self-adjoint boundary conditions of order four is:

$$(A : B) = \begin{pmatrix} r_1 & \bar{a}_{21} & 0 & 1 & b_{11} & b_{12} & 0 & 0 \\ a_{21} & r_2 & 1 & 0 & b_{21} & b_{22} & 0 & 0 \\ -\bar{b}_{11} & -\bar{b}_{21} & 0 & 0 & r_3 & \bar{b}_{41} & 0 & 1 \\ -\bar{b}_{12} & -\bar{b}_{22} & 0 & 0 & b_{41} & r_4 & 1 & 0 \end{pmatrix}, \tag{3.18}$$

where $r_1,\ r_2,\ r_3,\ r_4$ are real numbers, i.e., R_1, R_2 are symmetric matrices, and $A_3 = -B_1^* = -C^*$.

Remark 3.2. Here in (3.17) the symmetric matrices $R_1,\ R_2$ correspond to the real numbers $r_1,\ r_2$ in (2.7), $A_3 = -B_1^*$ corresponds to the complex conjugate $a_{21} = -\overline{b}_{11}$ in (2.7).

Remark 3.3. From (3.17), we see that if $C = 0$, then (3.17) represents the separated conditions, if the rank(C) = 1, then (3.17) represents the mixed conditions. In this way the canonical forms (3.17) unify the coupled, separated, and mixed canonical forms into one single form. We believe this will become an important tool for studying the dependence of the eigenvalues on the boundary conditions.

Next we show that for coupled self-adjoint boundary conditions the self-adjoint domains characterized by (3.17) are quite similar to those of the second order case. This may lead to similar methods for studying the dependence of the eigenvalues on the boundary conditions for fourth order problems.

For coupled self-adjoint boundary conditions (3.17), rank(B) = 4, rank(C) = 2. Since the boundary conditions are invariant under left multiplication, we have:

$$\begin{aligned} B^{-1}A &= \begin{pmatrix} C & 0_{2\times 2} \\ R_2 & J_2 \end{pmatrix}^{-1} \begin{pmatrix} R_1 & J_2 \\ -C^* & 0_{2\times 2} \end{pmatrix} = \begin{pmatrix} C^{-1} & 0_{2\times 2} \\ -J_2R_2C^{-1} & J_2 \end{pmatrix} \begin{pmatrix} R_1 & J_2 \\ -C^* & 0_{2\times 2} \end{pmatrix} \\ &= K_1 \begin{pmatrix} C^{-1} & 0_{2\times 2} \\ 0_{2\times 2} & C^* \end{pmatrix} K_2, \end{aligned}$$

where

$$K_1 = \begin{pmatrix} I_2 & 0_{2\times 2} \\ -J_2R_2 & J_2 \end{pmatrix},\ K_2 = \begin{pmatrix} R_1 & J_2 \\ -I_2 & 0_{2\times 2} \end{pmatrix}.$$

Remark 3.4. That is, the boundary matrix corresponding to the coupled canonical form for the fourth order differential operators can be transformed into:

$$(A : B) = \left(K_1 \begin{pmatrix} C^{-1} & 0_{2\times 2} \\ 0_{2\times 2} & C^* \end{pmatrix} K_2 : I_4 \right), \tag{3.19}$$

where $A = K_1 \begin{pmatrix} C^{-1} & 0_{2\times 2} \\ 0_{2\times 2} & C^* \end{pmatrix} K_2$ is the product of three matrices, the second matrix is a diagonal matrix, which is determined by C, and the determinant of A has absolute value equal to 1; the other two matrices consist of four symmetric block matrices, which are determined by R_2 and R_1 respectively, and $|\det(K_1)| = |\det(K_2)| = 1$, I_4 denotes the 4 by 4 identity matrix. And there is a 1-1 correspondence between the coupled self-adjoint canonical forms (3.17) and (3.19). The examples in the next section will illustrate this point further.

Remark 3.5. Compare these four formulas (2.7), (2.14), (3.17), (3.19), the coupled self-adjoint canonical forms for the fourth order differential operators have very similar forms with the second order case.

According to the above analysis, we have:

Theorem 3.4. *Coupled self-adjoint domains $D(S)$ of fourth order differential operators* (3.1) *are determined by two point boundary conditions*

$$D(S) = \left\{ y \in D_{max} : AY(a) + Y(b) = 0,\, Y = \begin{pmatrix} y^{[0]} & y^{[1]} & y^{[2]} & y^{[3]} \end{pmatrix}^T \right\}, \tag{3.20}$$

with matrices $A,\ B \in M_4(\mathbb{C})$ satisfying

$$A = \begin{pmatrix} I_2 & 0_{2\times 2} \\ -J_2R_2 & J_2 \end{pmatrix} \begin{pmatrix} C^{-1} & 0_{2\times 2} \\ 0_{2\times 2} & C^* \end{pmatrix} \begin{pmatrix} R_1 & J_2 \\ -I_2 & 0_{2\times 2} \end{pmatrix} = K_1 \begin{pmatrix} C^{-1} & 0_{2\times 2} \\ 0_{2\times 2} & C^* \end{pmatrix} K_2,$$

where $I_2,\ J_2,\ R_1,\ R_2$ are symmetric matrices, and $\mathrm{rank(C)} = 2$.

Theorem 3.5. *For the boundary matrices satisfying* (3.11)*, according to the classification of $(B_3 : B_4)$ in Lemma* 3.1, we have:

- If $(B_3 : B_4) = \begin{pmatrix} b_{31} & 0 & b_{33} & 1 \\ b_{41} & 1 & 0 & 0 \end{pmatrix}$, the canonical forms follows below:

$$(A : B) = \begin{pmatrix} A_1 & A_2 & B_1 & B_2 \\ A_3 & A_4 & B_3 & B_4 \end{pmatrix} = \begin{pmatrix} r_1 & \bar{a}_{21} & 0 & 1 & b_{11} & 0 & b_{13} & 0 \\ a_{21} & r_2 & 1 & 0 & b_{21} & 0 & b_{23} & 0 \\ -\bar{b}_{11} & -\bar{b}_{21} & 0 & 0 & r_3 & 0 & b_{33} & 1 \\ \bar{b}_{13} & \bar{b}_{23} & 0 & 0 & -\bar{b}_{33} & 1 & 0 & 0 \end{pmatrix}, \tag{3.21}$$

  where $r_1,\ r_2,\ r_3 \in \mathbb{R}$.

- If $(B_3 : B_4) = \begin{pmatrix} 0 & b_{32} & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, the canonical forms are as below:

$$(A : B) = \begin{pmatrix} A_1 & A_2 & B_1 & B_2 \\ A_3 & A_4 & B_3 & B_4 \end{pmatrix} = \begin{pmatrix} r_1 & \bar{a}_{21} & 0 & 1 & 0 & b_{12} & 0 & b_{14} \\ a_{21} & r_2 & 1 & 0 & 0 & b_{22} & 0 & b_{24} \\ -\bar{b}_{12} & -\bar{b}_{22} & 0 & 0 & 0 & r_3 & 1 & 0 \\ \bar{b}_{14} & \bar{b}_{24} & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \tag{3.22}$$

  where $r_1,\ r_2,\ r_3 \in \mathbb{R}$.

- If $(B_3 : B_4) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, the canonical forms for coupled conditions are:

$$(A : B) = \begin{pmatrix} A_1 & A_2 & B_1 & B_2 \\ A_3 & A_4 & B_3 & B_4 \end{pmatrix} = \begin{pmatrix} r_1 & \bar{a}_{21} & 0 & 1 & 0 & 0 & b_{13} & b_{14} \\ a_{21} & r_2 & 1 & 0 & 0 & 0 & b_{23} & b_{24} \\ \bar{b}_{13} & \bar{b}_{23} & 0 & 0 & 0 & 1 & 0 & 0 \\ \bar{b}_{14} & \bar{b}_{24} & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \tag{3.23}$$

where $r_1,\ r_2 \in \mathbb{R}$.

- For

$$(B_3 : B_4) = \begin{pmatrix} 0 & b_{32} & b_{33} & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

and

$$(B_3 : B_4) = \begin{pmatrix} b_{31} & 0 & 1 & 0 \\ b_{41} & 1 & 0 & 0 \end{pmatrix},$$

there are no canonical forms for coupled self-adjoint boundary conditions of order four.

Proof. These observations follow from the above results. □

Remark 3.6. Comparing (3.17) and Theorem 3.5, we see the latter should be transformed by the former (3.17) by elementary column transformations.

We call (3.17) (or (3.18)) as the "fundamental canonical form" of the self-adjoint boundary conditions. Theorem 3.5 greatly simplifies the canonical forms in [11]. See the next Remark.

Remark 3.7. The results in [11] are based on the well known characterization (1.11) based on the matrix E_4. The characterization given by Theorem 3.5 is based on the matrix F_4. The quasi-derivatives used in [11] are somewhat different from those used here. We find it remarkable that such an apparently minor change from E_4 to F_4 has a major consequence in the construction of canonical forms.

4. Examples

In this section we give some simple examples to illustrate our main results.

Example 4.1. Let $r_1,\ r_2 \in \mathbb{R},\ b_{11} = i$, in (2.7), then we have the canonical form of self-adjoint differential operators of order two as following form:

$$(A : B) = \begin{pmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ b_{21} & b_{22} & b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} r_1 & 1 & i & 0 \\ i & 0 & r_2 & 1 \end{pmatrix}.$$

From (2.14), we know a canonical form for the coupled self-adjoint second order differential operators also can be written as below:

$$(A : B) = \left(K_1 \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} K_2 : I_2 \right) = \begin{pmatrix} -ir_1 & -i & 1 & 0 \\ ir_1r_2 - i & ir_2 & 0 & 1 \end{pmatrix}, \tag{4.1}$$

where $K_1 = \begin{pmatrix} 1 & 0 \\ -r_2 & 1 \end{pmatrix}$, $K_2 = \begin{pmatrix} r_1 & 1 \\ -1 & 0 \end{pmatrix}$, $\det(K_1) = \det(K_2) = 1$, and $\det(A) = -1$.

Remark 4.1. By (2.8), $A = e^{i\gamma}K$, here

$$A = \begin{pmatrix} -ir_1 & -i \\ ir_1r_2 - i & ir_2 \end{pmatrix} = i \begin{pmatrix} -r_1 & -1 \\ r_1r_2 - 1 & r_2 \end{pmatrix} = e^{\frac{\pi}{2}i}K,$$

where K is a real matrix and $\det(K) = 1$.

If we change b_{11} to 0, then the above canonical form will become a separated self-adjoint boundary condition.

Next we give some examples for canonical forms of self-adjoint differential operators of order four:

Example 4.2. Let $A_1 = I_2,\ B_1 = E_2,\ B_3 = J_2$, in terms of (3.17), then we have the following canonical form of self-adjoint domains:

$$(A : B) = \begin{pmatrix} A_1 & A_2 & B_1 & B_2 \\ A_3 & A_4 & B_3 & B_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}. \tag{4.2}$$

From (3.19), we get the coupled self-adjoint canonical form:

$$(A : B) = \left(K_1 \begin{pmatrix} -E_2 & 0_{2\times 2} \\ 0_{2\times 2} & -E_2 \end{pmatrix} K_2 : I_4 \right), \tag{4.3}$$

where $K_1 = \begin{pmatrix} I_2 & 0_{2\times 2} \\ -I_2 & J_2 \end{pmatrix}$, $K_2 = \begin{pmatrix} I_2 & J_2 \\ -I_2 & 0_{2\times 2} \end{pmatrix}$ and $\det(K_1) = \det(K_2) = -1$, i.e.:

$$(A : B) = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.4}$$

where $\det(A) = 1$.

If we let $B_1 = 0_{2\times 2}$, the boundary condition (4.2) becomes a separated self-adjoint boundary condition. If we let $B_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, the boundary condition (4.2) becomes a mixed self-adjoint boundary condition.

Example 4.3. Assume that $A_1 = B_3 = iE_2,\ B_1 = iJ_2,$ from (3.17), we obtain the canonical form of self-adjoint domains as follows:

$$(A:B)=\begin{pmatrix} A_1 & A_2 & B_1 & B_2 \\ A_3 & A_4 & B_3 & B_4 \end{pmatrix}=\begin{pmatrix} 0 & -i & 0 & 1 & 0 & i & 0 & 0 \\ i & 0 & 1 & 0 & i & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & -i & 0 & 1 \\ i & 0 & 0 & 0 & i & 0 & 1 & 0 \end{pmatrix},$$

where $A_1,\ B_3$ are complex symmetric matrices.

From (3.19), we obtain the coupled self-adjoint canonical form:

$$(A:B)=\left(K_1\begin{pmatrix} -iJ_2 & 0_{2\times 2} \\ 0_{2\times 2} & -iJ_2 \end{pmatrix}K_2:I_4\right)=\begin{pmatrix} 1 & 0 & -i & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -i & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.5}$$

where $K_1=\begin{pmatrix} I_2 & 0_{2\times 2} \\ -iJ_2E_2 & J_2 \end{pmatrix}$, $K_2=\begin{pmatrix} iE_2 & J_2 \\ -I_2 & 0_{2\times 2} \end{pmatrix}$ and $\det(K_1)=\det(K_2)=-1$. Furthermore, we have $\det(A)=1$.

Example 4.4. Assume that $A_1=I_2,\ B_3=J_2,\ B_1=\begin{pmatrix} 1+i & 0 \\ 0 & 3+i \end{pmatrix}$ in terms of (3.17), we obtain canonical forms of self-adjoint domains having the following form:

$$(A:B)=\begin{pmatrix} I_2 & J_2 & B_1 & 0_{2\times 2} \\ -B_1^* & 0_{2\times 2} & J_2 & J_2 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 & 1 & 1+i & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 3+i & 0 & 0 \\ -1+i & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & -3+i & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix},$$

where $A_1,\ B_3$ are symmetric matrices.

From (3.19), we obtain the coupled self-adjoint canonical form:

$$(A:B)=\left(K_1\begin{pmatrix} B_1^{-1} & 0_{2\times 2} \\ 0_{2\times 2} & B_1^* \end{pmatrix}K_2:I_4\right)=\begin{pmatrix} \frac{1-i}{2} & 0 & 0 & \frac{1-i}{2} & 1 & 0 & 0 & 0 \\ 0 & \frac{3-i}{10} & \frac{3-i}{10} & 0 & 0 & 1 & 0 & 0 \\ -\frac{1-i}{2} & -3+i & 0 & -\frac{1-i}{2} & 0 & 0 & 1 & 0 \\ -1+i & -\frac{3-i}{10} & -\frac{3-i}{10} & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.6}$$

where $K_1=\begin{pmatrix} I_2 & 0_{2\times 2} \\ -I_2 & J_2 \end{pmatrix}$, $K_2=\begin{pmatrix} I_2 & J_2 \\ -I_2 & 0_{2\times 2} \end{pmatrix}$, $B_1^*=\begin{pmatrix} 1-i & 0 \\ 0 & 3-i \end{pmatrix}$, $B_1^{-1}=$

$\begin{pmatrix} \frac{1-i}{2} & 0 \\ 0 & \frac{3-i}{10} \end{pmatrix}$, and $|\det(A)| = |\frac{[(1-i)(3-i)]^2}{20}| = 1$.

Example 4.5. Let $(B_3 : B_4) = \begin{pmatrix} 0 & 0 & i & 1 \\ i & 1 & 0 & 0 \end{pmatrix}$. Suppose $A_1 = A_3 = iE_2$ in terms of (3.20) in Theorem 3.5, we obtain the coupled canonical form of self-adjoint domains having the following form:

$$(A:B) = \begin{pmatrix} A_1 & A_2 & B_1 & B_2 \\ A_3 & A_4 & B_3 & B_4 \end{pmatrix} = \begin{pmatrix} 0 & -i & 0 & 1 & 0 & 0 & -i & 0 \\ i & 0 & 1 & 0 & -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & i & 1 \\ i & 0 & 0 & 0 & i & 1 & 0 & 0 \end{pmatrix},$$

where $A_1 = A_3$ are all complex symmetric matrices.

Example 4.6. Let $(B_3 : B_4) = \begin{pmatrix} 0 & i & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$. Suppose $A_1 = iE_2$, $A_3 = iJ_2$ in terms of (3.22) in Theorem 3.5, we obtain the coupled canonical form of self-adjoint domains having the following form:

$$(A:B) = \begin{pmatrix} A_1 & A_2 & B_1 & B_2 \\ A_3 & A_4 & B_3 & B_4 \end{pmatrix} = \begin{pmatrix} 0 & -i & 0 & 1 & 0 & 0 & 0 & -i \\ i & 0 & 1 & 0 & 0 & i & 0 & 0 \\ 0 & i & 0 & 0 & 0 & i & 1 & 0 \\ i & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

where A_1 is a complex symmetric matrix.

Example 4.7. Let $(B_3 : B_4) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$. Suppose $A_1 = I_2$, $A_3 = iJ_2$ in terms of (3.23) in Theorem 3.5, we obtain the coupled canonical form of self-adjoint domains having the following form:

$$(A:B) = \begin{pmatrix} A_1 & A_2 & B_1 & B_2 \\ A_3 & A_4 & B_3 & B_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & -i \\ 0 & 1 & 1 & 0 & 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 & 0 & 1 & 0 & 0 \\ i & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

where A_1 is a real symmetric matrix.

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