Journal of Applied Analysis and Computation
Volume 9, Number 4, August 2019, 1558–1570
Website:http://jaac.ijournal.cn
DOI:10.11948/2156-907X.20180340

A NON-RADIALLY SYMMETRIC SOLUTION TO A CLASS OF ELLIPTIC EQUATION WITH KIRCHHOFF TERM*

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Abstract We consider the following equation with Kirchhoff term $-(a + b\int_{\mathbb{R}^3} |\nabla u|^2dx)\,\Delta u+u = |u|^{p-2}u$, $u \in H^1(\mathbb{R}^3)$, where a, b are positive constants and $2 < p < 6$. By deducing a variant variational identity and a constraint set, we are able to prove the existence of a non-radially symmetric solution $u(x_1, x_2, x_3)$ for the full range of $p \in (2, 6)$. Moreover this solution $u(x_1, x_2, x_3)$ is radially symmetric with respect to (x_1, x_2) and odd with respect to x_3.

Keywords Equation with Kirchhoff term, non-radially symmetric solution, variant variational identity.

MSC(2010) 35J20.

1. Introduction and main results

This paper is concerned with the existence of non-radially symmetric solutions to the following equation with Kirchhoff term

$$\begin{cases} -\left(a+b\displaystyle\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u+u=|u|^{p-2}u, \\ u:=u(x),\ x\in\mathbb{R}^3,\ u\in H^1(\mathbb{R}^3), \end{cases} \tag{1.1}$$

where a, b are positive constants and $2 < p < 6$. Equation (1.1) is a model of the following

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u+V(x)u=f(x,u), \tag{1.2}$$

where $a > 0$, $b \geq 0$, $V:\ \mathbb{R}^3\times\mathbb{R}\to\mathbb{R}$ and $f \in C(\mathbb{R}^3\times\mathbb{R},\mathbb{R})$. The $\int_{\mathbb{R}^3}|\nabla u|^2dx\Delta u$ is usually called Kirchhoff term.

In the past ten years, many researchers have been devoted to finding solutions to (1.2), see e. g. [11, 13, 14, 16, 23]. In these papers, critical point theorems are applied to the functional

$$\Phi(u)=\frac{1}{2}\int_{\mathbb{R}^3}\left(a|\nabla u|^2+V(x)|u|^2\right)dx+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^2-\int_{\mathbb{R}^3}F(x,u)dx$$

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*The authors were supported by National Natural Science Foundation of China (Nos. 11871152, 11671085 and 11501107).

defined on $E := \{u \in H^1(\mathbb{R}^3)\ :\ \int_{\mathbb{R}^3} V(x)|u|^2dx < \infty\}$ with $F(x,u) = \int_0^u f(x,s)ds$. In the process of finding critical points of Φ, there are some difficulties. The first is lack of compactness embedding from E into $L^q(\mathbb{R}^3)$ for $2 < q < 6$. To overcome this, one may assume that both $V(x)$ and $f(x,u)$ are radially symmetric on x and then restrict Φ on the subspace of E which contains only radially symmetric functions. Or assume that $V(x) \in C(\mathbb{R}^3, \mathbb{R})$ and satisfies suitable compactness condition such that the embedding from E into $L^q(\mathbb{R}^3)$ $(2 < q < 6)$ is compact, see e.g. [9–11,15,22].

The second difficulty is the "geometry condition". Comparing with the case of $b = 0$, one usually assume that the growth of $F(x,u)$ on u is faster than $|u|^4$, see e. g. [5,8,11,22], where the authors assume $f(x,u)$ is 4−superlinear at infinity in the sense that

$$\lim_{|u|\to+\infty} \frac{F(x,u)}{|u|^4} = +\infty \quad \text{uniformly in} \quad x \in \mathbb{R}^3.$$

Observing the results mentioned above, a typical case of nonlinear function $f(x,u) = |u|^{p-2}u$ is not covered when $2 < p < 4$. Recently, by using monotonicity trick, the authors in [12] proved that the following equation

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\Delta u + V(x)u = |u|^{p-2}u, \tag{1.3}$$

has a positive ground state solution in E for any $p \in (3,6)$.

For (1.1), Wu [22] has essentially proven the existence of radially symmetric solution when $4 < p < 6$. Also for $p \in (4,6)$, Sun and Zhang [21] proved that the positive ground state solution to (1.1) is unique and radially symmetric. Recently, such kind of results has been extended to fractional Kirchhoff type equation or p−Kirchhoff equations, see e. g. [1,3,6,17,18] as well as the references therein. But we do not see any results about the existence of non-radial solutions to (1.1) for $2 < p < 4$. The purpose of the present paper is to prove that (1.1) admits at least one non-radially symmetric solution for the full range of $p \in (2,6)$. Our main result is the following theorem.

Theorem 1.1. *Assume that $a, b > 0$ and $2 < p < 6$. Then (1.1) admits a non-radially symmetric solution $u \in H^1(\mathbb{R}^3)$. Moreover if denoting $x = (x_1, x_2, x_3)$ and $u := u(x_1, x_2, x_3)$, then u is radially symmetric with respect to (x_1, x_2) and odd with respect to x_3.*

The proof of Theorem 1.1 is by variational methods. Our idea is inspired from the paper of Ruiz [20] where the author constructed a kind of Nehari-Pohozaev type identity and studied a class of Schrodinger-Poisson system. Our strategy is to deduce a variant variational identity and define a subset $\mathcal{M}$ (see Section 3) of $H^1(\mathbb{R}^3)$. On the set $\mathcal{M}$ we minimize the following functional

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(a|\nabla u|^2 + u^2\right) dx + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2 - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p dx$$

and prove that the minimum can be achieved.

This paper is organized as follows. In Section 2, we give some preliminaries about group action on $\mathbb{R}^3$. In Section 3, we deduce a variant variational identity and define a subset $\mathcal{M}$; then we prove that a minimizer of $I_{|\mathcal{M}}$ is a critical point of I on $H^1(\mathbb{R}^3)$. We emphasize that with the help of this construction, we manage to prove the existence of solution for the full range of $p \in (2,6)$, which is a complement of several previous works. In Section 4, we finish the proof of Theorem 1.1.

Notation. Throughout this paper, all integrals are taken over $\mathbb{R}^3$ unless specified. C_n $(n = 1,\ 2,\ \cdots)$ denotes a positive constant whose exact value is not important. $L^q(\mathbb{R}^3)$ $(1 \le q < +\infty)$ is the usual Lebesgue space with the standard norm $|u|_q$. For $a > 0$, we introduce an equivalent norm on $H^1(\mathbb{R}^3)$: $\|u\|^2 := \int \left(a|\nabla u|^2 + u^2\right) dx$ with the corresponding inner product $(u,v) := \int (a\nabla u \nabla v + uv)\, dx$.

2. Preliminaries

In this section, we introduce a group action on $\mathbb{R}^3$ which is originated from [4]. For every $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, we define a map on $H^1(\mathbb{R}^3)$ as

$$(T_\theta u)(x) := -u(g_\theta x), \quad \text{where} \quad g_\theta := \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Then, $T_\theta : u \mapsto T_\theta u$ is a linear operator from $H^1(\mathbb{R}^3)$ to $H^1(\mathbb{R}^3)$. And T_θ satisfies the following properties.

Proposition 2.1 ([4]). *For any $\theta_1,\ \theta_2 \in \mathbb{R}/2\pi\mathbb{Z}$ and $u, v \in H^1(\mathbb{R}^3)$*

$$T_{\theta_1}T_{\theta_2} = T_{\theta_1+\theta_2}T_0 = T_0 T_{\theta_1+\theta_2} = T_{\theta_2}T_{\theta_1}, \tag{2.1}$$

$$T_0 T_0 = Id, \tag{2.2}$$

$$(T_\theta u, T_\theta v) = (u,v). \tag{2.3}$$

Next, we introduce a set of fixed points in $H^1(\mathbb{R}^3)$:

$$\tilde{H} := \left\{u \in H^1(\mathbb{R}^3)\ :\ \text{for any } \theta \in \mathbb{R}/2\pi\mathbb{Z},\ T_\theta u = u\right\}.$$

Remark 2.1. (1)**.** The $\tilde{H}$ is closed and weakly closed in $H^1(\mathbb{R}^3)$.

(2)**.** For every $u \in \tilde{H}$, u is radially symmetric with respect to (x_1, x_2) and odd with respect to x_3. Indeed, for a. e. $x := (x_1, x_2, x_3) \in \mathbb{R}^3$

$$u(x_1, x_2, -x_3) = u(g_0 x) = -u(x_1, x_2, x_3)$$

and

$$u(r_\theta(x_1,x_2), x_3) = u(g_\theta(x_1, x_2, -x_3)) = -u(x_1, x_2, -x_3) = u(x_1, x_2, x_3),$$

where

$$r_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Lemma 2.1. *If $u \in \tilde{H}$ is a critical point of $I|_{\tilde{H}}$, then u is a critical point of I.*

Proof. From (2.3), we get that for every $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ and $u, v \in H^1(\mathbb{R}^3)$

$$\begin{aligned} I(T_\theta u) &= \frac{1}{2}\int \left(a|\nabla(T_\theta u)|^2 + |T_\theta u|^2\right) dx + \frac{b}{4}\left(\int |\nabla(T_\theta u)|^2 dx\right)^2 - \frac{1}{p}\int (|T_\theta u|^2)^{p/2} dx \\ &= \frac{1}{2}\int \left(a|\nabla u|^2 + |u|^2\right) dx + \frac{b}{4}\left(\int |\nabla u|^2 dx\right)^2 - \frac{1}{p}\int |u|^p dx \\ &= I(u), \end{aligned}$$

and

$$\begin{aligned} \langle I'(u), T_\theta v\rangle &= \frac{d}{d\lambda} I(u + \lambda T_\theta v)\Big|_{\lambda=0} = \frac{d}{d\lambda} I(T_\theta(T_{-\theta} u + \lambda v))\Big|_{\lambda=0} \\ &= \frac{d}{d\lambda} I(T_{-\theta} u + \lambda v)\Big|_{\lambda=0} = \langle I'(T_{-\theta} u), v\rangle. \end{aligned}$$

Since the gradient of I at u is defined by, for every $v \in H^1(\mathbb{R}^3)$,

$$(\nabla I(u), v) = \langle I'(u), v\rangle,$$

we have that for every $u \in \tilde{H}$ and $v \in H^1(\mathbb{R}^3)$

$$\begin{aligned} (T_\theta(\nabla I(u)), v) &= (T_\theta(\nabla I(u)), T_\theta T_{-\theta} v) = (\nabla I(u), T_{-\theta} v) \\ &= \langle I'(u), T_{-\theta} v\rangle = \langle I'(T_\theta u), v\rangle \\ &= (\nabla I(T_\theta u), v) = (\nabla I(u), v), \end{aligned}$$

which implies that

$$\nabla I(u) \in \tilde{H}.$$

Since $\tilde{H}$ is closed in $H^1(\mathbb{R}^3)$, denoted with $\tilde{H}^\perp$ its orthogonal, we write

$$H^1(\mathbb{R}^3) = \tilde{H} + \tilde{H}^\perp.$$

If $u \in \tilde{H}$ is a critical point of $I|_{\tilde{H}}$, for every $v \in H^1(\mathbb{R}^3)$ as the sum of $v_1 \in \tilde{H}$ and $v_2 \in \tilde{H}^\perp$

$$\begin{aligned} \langle I'(u), v\rangle &= \langle I'(u), v_1\rangle + \langle I'(u), v_2\rangle \\ &= \langle (I|_{\tilde{H}})'(u), v_1\rangle + (\nabla I(u), v_2) \\ &= 0. \end{aligned}$$

□

3. Variant variational identity and a constraint set

The aim of this section is to construct a suitable constraint set, on which we can define a minimization problem. The construction is based on a variant variational identity ($G(u) = 0$, see Remark 3.1). Keeping the definition of the functional I in mind, we observe that due to the presence of Kirchhoff term, when $p \in (2, 4)$, it is not easy to see if the functional I is not bounded from below. We begin with the following proposition.

Proposition 3.1. *Let $a > 0$, $b > 0$ and $p \in (2, 6)$. The functional I is not bounded below on $\tilde{H}$.*

Proof. For any $u \in \tilde{H}$ and any $t > 0$, denote $w(x) := t^{\frac{1}{4}} u(t^{-\frac{1}{2}} x)$. Then by direct computations, we have that

$$\int |\nabla w|^2 dx = t \int |\nabla u|^2 dx, \quad \int |w|^2 dx = t^2 \int |u|^2 dx, \quad \int |w|^p dx = t^{\frac{p+6}{4}} \int |u|^p dx,$$

and therefore

$$I(w) = \frac{1}{2} t \int a|\nabla u|^2 dx + \frac{1}{2} t^2 \int |u|^2 dx + \frac{b}{4} t^2 \left(\int |\nabla u|^2 dx \right)^2 - \frac{1}{p} t^{\frac{p+6}{4}} \int |u|^p dx.$$

Since $\frac{p+6}{4} > 2$, we deduce that $I(w) \to -\infty$ as $t \to +\infty$. □

Lemma 3.1. *Let $\alpha, \beta, \gamma, \delta$ be positive constants and $p \in (2, 6)$. For $t \geq 0$, we define $f(t) := \alpha t + \beta t^2 + \gamma t^2 - \delta t^{\frac{p+6}{4}}$. Then f has a unique critical point which corresponds to its maximum.*

Proof. For $t \geq 0$, we compute directly that

$$f'(t) = \alpha + 2\beta t + 2\gamma t - \frac{p+6}{4} \delta t^{\frac{p+2}{4}},$$
$$f''(t) = 2\beta + 2\gamma - \frac{p+6}{4} \frac{p+2}{4} \delta t^{\frac{p-2}{4}}.$$

Since f'' is strictly decreasing with respect to $t > 0$ and $f''(0) = 2\beta + 2\gamma > 0$, there exists $t_1 > 0$ such that $f''(t_1) = 0$ and $f''(t)(t_1 - t) > 0$ for $t \neq t_2$.

Since $f'(0) = \alpha > 0$ and f' is increasing for $t < t_1$, f' takes positive values at least for $t \in [0, t_1]$. For $t > t_1$, f' decreases, and goes to $-\infty$. Then there exists $t_0 > t_1$ such that $f'(t_0) = 0$ and $f'(t)(t_0 - t) > 0$ for $t \neq t_0$.

Taking a conclusion, t_0 is the unique critical point of f and corresponds to its maximum as $\frac{p+6}{4} > 2$. □

We are now in a position to construct a manifold on which we can define a minimization problem. Our idea is to establish a variant variational identity and use this identity to construct a set. Then we prove that this set is a manifold and share some properties similar to Nehari manifold. More precisely, we will construct a manifold such that for any $u \neq 0$, there is a curve passing u and acrossing the manifold only at one point. Moreover, along this curve the functional I achieves its maximum at a unique point. To attain this goal, for $w(x) := t^{\frac{1}{4}} u(t^{-\frac{1}{2}} x)$ defined as above, we consider

$$I(w) = \frac{1}{2} t \int a|\nabla u|^2 dx + \frac{1}{2} t^2 \int |u|^2 dx + \frac{b}{4} t^2 \left(\int |\nabla u|^2 dx \right)^2 - \frac{1}{p} t^{\frac{p+6}{4}} \int |u|^p dx.$$

Then for u fixed, $I(w)$ is positive for small t and tends to $-\infty$ as $t \to +\infty$. Choosing $f(t) := I(w)$, from Lemma 3.1, we know that $f(t)$ has a unique critical point, corresponding to its maximum. Define the functional $G : \tilde{H} \longrightarrow \mathbb{R}$ as

$$G(u) := \frac{1}{2} \int a|\nabla u|^2 dx + \int u^2 dx + \frac{b}{2} \left(\int |\nabla u|^2 dx \right)^2 - \frac{p+6}{4p} \int |u|^p dx$$

and set

$$\mathcal{M} = \{u \in \tilde{H} \backslash \{0\} : G(u) = 0\}.$$

Proposition 3.2. *Let $p \in (2,6)$. For any $u \in \tilde{H}\backslash\{0\}$, there is a unique $t_0 := t_0(u) > 0$ such that $t_0^{\frac{1}{4}} u(t_0^{-\frac{1}{2}} x) \in \mathcal{M}$. Moreover if $G(u) < 0$, then $t_0 \in (0,1)$.*

Proof. Firstly, for any $u \in \tilde{H}\backslash\{0\}$ and any $t > 0$, we choose $f(t) := I(w)$ with $w(x) := t^{\frac{1}{4}} u(t^{-\frac{1}{2}} x)$. Then from the proof of Lemma 3.1, $f(t)$ has a unique critical point $t_0 := t_0(u)$(here $t_0(u)$ means t_0 depends on u), corresponding to its maximum. Therefore

$$f'(t_0) = \frac{1}{2}\int a|\nabla u|^2 dx + t_0 \int |u|^2 dx + \frac{b}{2} t_0 \left(\int |\nabla u|^2 dx\right)^2 - \frac{p+6}{4p} t_0^{\frac{p+2}{4}} \int |u|^p dx = 0.$$

Denoting $w_0(x) := t_0^{\frac{1}{4}} u(t_0^{-\frac{1}{2}} x)$, then $w_0 \neq 0$ and we have that

$$\begin{aligned} G(w_0) =& \frac{1}{2}\int a|\nabla w_0|^2 dx + \int |w_0|^2 dx + \frac{b}{2}\left(\int |\nabla w_0|^2 dx\right)^2 - \frac{p+6}{4p}\int |w_0|^p dx \\ =& \frac{1}{2} t_0 \int a|\nabla u|^2 dx + t_0^2 \int |u|^2 dx + \frac{b}{2} t_0^2 \left(\int |\nabla u|^2 dx\right)^2 - \frac{p+6}{4p} t_0^{\frac{p+6}{4}} \int |u|^p dx \\ =& t_0 f'(t_0) = 0. \end{aligned}$$

Hence $w_0(x) := t_0^{\frac{1}{4}} u(t_0^{-\frac{1}{2}} x) \in \mathcal{M}$.

Secondly, if $G(u) < 0$, then from

$$G(u) = \frac{1}{2}\int a|\nabla u|^2 dx + \int u|^2 dx + \frac{b}{2}\left(\int |\nabla u|^2 dx\right)^2 - \frac{p+6}{4p}\int |u|^p dx < 0$$

and

$$G(w_0) = \frac{1}{2} t_0 \int a|\nabla u|^2 dx + t_0^2 \int |u|^2 dx + \frac{b}{2} t_0^2 \left(\int |\nabla u|^2 dx\right)^2 - \frac{p+6}{4p} t_0^{\frac{p+6}{4}} \int |u|^p dx = 0,$$

we obtain that

$$\frac{1}{2}\left(t_0^{\frac{p+6}{4}} - t_0\right)\int a|\nabla u|^2 dx + \left(t_0^{\frac{p+6}{4}} - t_0^2\right)\int |u|^2 dx + \frac{b}{2}\left(t_0^{\frac{p+6}{4}} - t_0^2\right)\left(\int |\nabla u|^2 dx\right)^2 < 0,$$

which implies $t_0 < 1$. Therefore $t_0 \in (0,1)$. □

Remark 3.1. If $v \neq 0$ is a weak solution of (1.1), then by the calculation of the Pohozaev [19] identity of equation (1.1), $P(v) = 0$, where

$$P(v) := \frac{1}{2}\int a|\nabla v|^2 dx + \frac{3}{2}\int |v|^2 dx + \frac{b}{2}\left(\int |\nabla v|^2 dx\right)^2 - \frac{3}{p}\int |v|^p dx.$$

Moreover, for this v, according to the proof of Proposition 3.2, there is a unique $t_0(v) > 0$ such that $(t_0(v))^{\frac{1}{4}} u((t_0(v))^{-\frac{1}{2}} x) \in \mathcal{M}$. We claim that $t_0(v) = 1$. To see this, one only notices that $\frac{1}{4}\langle I'(v), v\rangle = 0$, $\frac{1}{4}P(v) = 0$ and $G(v) = \frac{1}{4}\langle I'(v), v\rangle + \frac{1}{2}P(v) = 0$.

Lemma 3.2. *Let $a > 0$, $b > 0$ and $p \in (2,6)$. Then $\mathcal{M}$ is bounded away from zero.*

Proof. For any $u \in \mathcal{M}$, we deduce from $G(u) = 0$ and the Sobolev inequality that

$$\frac{1}{2}\int a|\nabla u|^2dx + \int |u|^2dx \leq \frac{1}{2}\int a|\nabla u|^2dx + \int |u|^2dx + \frac{b}{2}\left(\int |\nabla u|^2dx\right)^2 = \frac{p+6}{4p}\int |u|^pdx \leq C_1\|u\|^p.$$

Which implies that there is a $C_2 > 0$ such that $\|u\|^{p-2} \geq C_2$. □

Lemma 3.3. *Let $a,\ b > 0$ and $p \in (2,6)$. Then $\mathcal{M}$ is a nature C^1-constraint in the sense that a critical point of $I|_{\mathcal{M}}$ is also a critical point of I in $\tilde{H}$.*

Proof. The proof can be sketched as following 2 steps.

Step 1. We prove that for every $u \in \mathcal{M}$, $G'(u) \neq 0$. Then $\mathcal{M}$ is a C^1-manifold.

Suppose that there is $u \in \mathcal{M}$ such that $G'(u) = 0$. We denote

$$i := \int |\nabla u|^2dx, \quad j := \int |u|^2dx \text{ and } k := \int |u|^pdx.$$

Next, in a weak sense, the equation $G'(u) = 0$ can be written as

$$-\left(a + 2b\int |\nabla u|^2dx\right)\Delta u + 2u = \frac{p+6}{4}|u|^{p-2}u. \tag{3.1}$$

Then we have the following relations:

$$\begin{cases} ai + 2j + bi^2 - \dfrac{p+6}{2p}k = 0, \\ ai + 2j + 2bi^2 - \dfrac{p+6}{4}k = 0, \\ \dfrac{1}{2}ai + 3j + bi^2 - \dfrac{3(p+6)}{4p}k = 0, \\ \dfrac{1}{4}ai + \dfrac{p-2}{8p}k = I(u), \end{cases}$$

where the first one is from $2G(u) = 0$; the second one comes from multiplying (3.1) by u and integrating by parts; the third one is the Pohozaev equality of (3.1) and the fourth one is due to the definition of $I(u)$ and $G(u) = 0$.

Now solving these equations as the following: combining the second one with the third one and the first one respectively, we obtain that

$$\begin{cases} 4j = \dfrac{(p+6)(6-p)}{4p}k, \\ ai + 2j = \dfrac{(p+6)(4-p)}{4p}k. \end{cases} \tag{3.2}$$

From (3.2) and $p \in (2,6)$, we deduce that

$$ai = \frac{(p+6)(2-p)}{8p}k < 0,$$

which is a contradiction. This proves the Step 1.

Step 2. We will prove: if u is a critical point of $I|_{\mathcal{M}}$, then $I'(u)=0$.

If u is a critical point of I restricted on the manifold $\mathcal{M}$, then there exists a Lagrange multiplier $\lambda\in\mathbb{R}$ such that $I'(u)=\lambda G'(u)$. Denote $I_0:=I(u)$. Our aim is to prove $\lambda=0$.

Firstly, from $I'(u)=\lambda G'(u)$, in a weak sense, we have that

$$\begin{aligned}&-\left(a+b\int|\nabla u|^2dx\right)\Delta u+u-|u|^{p-2}u\\ =&\lambda\left(-\left(a+2b\int|\nabla u|^2dx\right)\Delta u+2u-\frac{p+6}{4}|u|^{p-2}u\right).\end{aligned}$$

Rewrite the above equation as

$$-\left((\lambda-1)a+(2\lambda-1)b\int|\nabla u|^2dx\right)\Delta u+(2\lambda-1)u=\left(\frac{p+6}{4}\lambda-1\right)|u|^{p-2}u. \tag{3.3}$$

Secondly, using the notations i, j and k as in the previous step, we obtain that

$$\frac{1}{4}ai+\frac{p-2}{8p}k=I_0, \tag{3.4}$$

$$ai+2j+bi^2-\frac{p+6}{2p}k=0, \tag{3.5}$$

$$(\lambda-1)ai+(2\lambda-1)j+(2\lambda-1)bi^2-\left(\frac{p+6}{4}\lambda-1\right)k=0 \tag{3.6}$$

and

$$\frac{1}{2}(\lambda-1)ai+\frac{3}{2}(2\lambda-1)j+\frac{1}{2}(2\lambda-1)bi^2-\frac{3}{p}\left(\frac{p+6}{4}\lambda-1\right)k=0, \tag{3.7}$$

where (3.4) is from $I_0:=I(u)$ and $G(u)=0$; (3.5) is due to $G(u)=0$; (3.6) comes from multiplying (3.3) by u and integrating by parts; (3.7) is the Pohozaev identity of (3.3).

For the linear system (3.4)-(3.7), taking elementary transformation to the coefficient matrix A

$$A=\begin{pmatrix}\frac{1}{4} & 0 & 0 & \frac{p-2}{8p}\\ 1 & 2 & 1 & -\frac{p+6}{2p}\\ \lambda-1 & 2\lambda-1 & 2\lambda-1 & 1-\frac{p+6}{4}\lambda\\ \frac{1}{2}(\lambda-1) & \frac{3}{2}(2\lambda-1) & \frac{1}{2}(2\lambda-1) & \frac{3}{p}\left(1-\frac{p+6}{4}\lambda\right)\end{pmatrix}$$

and computing its determinant, we obtain that

$$\det A=\frac{(p+2)(2-p)}{32p}\lambda(2\lambda-1).$$

If $\det A\neq 0$, then by Cramer rule, we know the linear system (3.4)–(3.7) has a unique solution and

$$k=\frac{I_0}{\det A}\lambda(2\lambda-1)=\frac{32pI_0}{(p+2)(2-p)}. \tag{3.8}$$

Notice that $I_0 := I(u)$. Then we deduce from $G(u) = 0$ that

$$\begin{aligned} I_0 := I(u) =& \frac{1}{2}(ai + j) + \frac{b}{4}i^2 - \frac{k}{p} \\ =& \frac{1}{2}(ai + j) + \frac{b}{4}i^2 - \frac{4}{p+6}\left(\frac{1}{2}ai + j + \frac{b}{2}i^2\right) \\ =& \frac{p+2}{2(p+6)}ai + \frac{p-2}{2(p+6)}j + \frac{(p-2)b}{4(p+6)}i^2 > 0. \end{aligned}$$

Combining this with $p > 2$, we know that the right hand side of (3.8) is negative. This contradicts to the definition of k.

Therefore $\det A = 0$. This means that

$$\lambda = 0 \quad \text{or} \quad \lambda = \frac{1}{2}.$$

Suppose that $\lambda = \frac{1}{2}$. Then (3.6) becomes

$$-\frac{1}{2}ai - \frac{p-2}{8}k = 0,$$

which is also a contradiction since $p > 2$, $a > 0$, $i > 0$ and $k > 0$. Therefore $\lambda = 0$. Hence we deduce that $I'(u) = 0$.

In sum, we finish the proof of Lemma 3.3. □

4. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1. Our strategy is to prove the following minimization problem

$$h := \inf\{I(u) \ : \ u \in \mathcal{M}\} \tag{4.1}$$

is achieved by an element in $\tilde{H}$ which is a non-radially symmetric solution as required. We start with proving the following lemma.

Lemma 4.1. *Let $\{u_n\} \subset \mathcal{M}$ be such that $u_n \rightharpoonup u$ weakly in $\tilde{H}$. Denote $v_n := u_n - u$. Then for n large enough,*

$$o(1) + G(u) + G(v_n) \le 0.$$

Proof. By Brézis-Lieb Lemma [2], for n large enough,

$$|u_n|_p^p = |u|_p^p + |v_n|_p^p + o(1)$$

and

$$\|u_n\|^2 = \|u\|^2 + \|v_n\|^2 + o(1).$$

As $u_n \in \mathcal{M}$, we obtain that

$$\begin{aligned} 0 =& G(u_n) = \frac{a}{2}|\nabla u_n|_2^2 + |u_n|_2^2 + \frac{b}{2}|\nabla u_n|_2^4 - \frac{p+6}{4p}|u_n|_p^p \\ =& \frac{a}{2}|\nabla u|_2^2 + \frac{a}{2}|\nabla v_n|_2^2 + |u|_2^2 + |v_n|_2^2 + \frac{b}{2}|\nabla u|_2^4 + \frac{b}{2}|\nabla v_n|_2^4 \\ & + b|\nabla u|_2^2|\nabla v_n|_2^2 - \frac{p+6}{4p}|u|_p^p - \frac{p+6}{4p}|v_n|_p^p + o(1) \\ \ge& G(u) + G(v_n) + o(1). \end{aligned}$$

This proves the lemma. □

Proof of Theorem 1.1. Firstly, from Lemma 3.2 we know that the h defined by (4.1) satisfies $h > 0$. Let $\{u_n\} \subset \mathcal{M}$ be a minimizing sequence of I on $\mathcal{M}$, i.e.

$$\lim_{n\to\infty} I(u_n) = h \quad \text{and} \quad G(u_n) = 0.$$

Define

$$i_n := \int |\nabla u_n|^2, \quad j_n := \int u_n^2 dx, \quad k_n := \int |u_n|^p dx.$$

Obviously, i_n, j_n, k_n are positive and

$$\begin{cases} \frac{1}{2} a i_n + \frac{1}{2} j_n + \frac{1}{4} b i_n^2 - \frac{1}{p} k_n = h + o(1), \\ \frac{1}{2} a i_n + j_n + \frac{1}{2} b i_n^2 - \frac{p+6}{4p} k_n = 0. \end{cases}$$

Combining the two equations, one has that

$$\frac{p+2}{8} a i_n + \frac{p-2}{8} j_n + \frac{p-2}{16} b i_n^2 = \frac{p+6}{4} h + o(1),$$

which implies that, for n large enough,

$$a i_n + j_n < \frac{2(p+6)}{p-2} h + 1.$$

Therefore, $\{u_n\}$ is bounded in $\tilde{H}$.

In the following, we will prove that $\{u_n\}$ contains a convergent subsequence which converges to a minimizer of the minimum h defined in (4.1). We manage to do this by three steps.

Step (i). Up to a subsequence, still denoted by $\{u_n\}$, we may assume $u_n \rightharpoonup u_0$ in $\tilde{H}$. The $G(u_n) = 0$ and Lemma 3.2 imply that there is $d_0 > 0$ such that $\int_{\mathbb{R}^3} |u_n|^p dx \ge d_0 > 0$. We set $D_m := \mathbb{R}^2 \times (m, m+1)$ for every $m \in \mathbb{Z}$. Then

$$\begin{aligned} d_0 &\le \int_{\mathbb{R}^2\times\mathbb{R}} |u_n|^p dx = \sum_{m\in\mathbb{Z}} \int_{D_m} |u_n|^{p-2} |u_n|^2 dx \\ &\le \sum_{m\in\mathbb{Z}} \left(\int_{D_m} |u_n|^p dx \right)^{\frac{p-2}{p}} \left(\int_{D_m} |u_n|^p dx \right)^{\frac{2}{p}} \\ &\le \sup_{m\in\mathbb{Z}} \left(\int_{D_m} |u_n|^p dx \right)^{\frac{p-2}{p}} \sum_{m\in\mathbb{Z}} \left(\int_{D_m} |u_n|^p dx \right)^{\frac{2}{p}} \\ &\le C_3 \sup_{m\in\mathbb{Z}} \left(\int_{D_m} |u_n|^p dx \right)^{\frac{p-2}{p}} \sum_{m\in\mathbb{Z}} \|u_n\|^2_{H^1(D_m)} \\ &\le C_4 \sup_{m\in\mathbb{Z}} \left(\int_{D_m} |u_n|^p dx \right)^{\frac{p-2}{p}}. \end{aligned}$$

Then $\sup_{m\in\mathbb{Z}} \left(\int_{D_m} |u_n|^p dx \right)^{\frac{p-2}{p}}$ is bounded away from zero uniformly with respect to n. By Esteban and Lions' Theorem [7, P.381, Theorem 1], we have that $u_0 \neq 0$.

Step (ii). We will prove that $G(u_0) = 0$.

Arguing by a contradiction, we firstly assume that $G(u_0) < 0$. Then by Proposition 3.2, there exists $t_0 := t_0(u_0) \in (0,1)$ such that $w(x) := t_0^{\frac{1}{4}} u(t_0^{-\frac{1}{2}} x) \in \mathcal{M}$. As $\{u_n\}$ is a minimizing sequence, we find that

$$\begin{aligned} h + o(1) = & I(u_n) = \frac{1}{4}\int a|\nabla u_n|^2 dx + \frac{p-2}{8p}\int |u_n|^p dx \\ \geq & \frac{1}{4}\int a|\nabla u_0|^2 dx + \frac{p-2}{8p}\int |u_0|^p dx \\ > & \frac{1}{4} t\int a|\nabla u_0|^2 dx + \frac{p-2}{8p} t^{\frac{p+6}{4}}\int |u_0|^p dx \\ = & \frac{1}{4}\int a|\nabla w|^2 dx + \frac{p-2}{8p}\int |w|^p dx = I(w), \end{aligned}$$

which is a contradiction as $w \in \mathcal{M}$.

If $G(u_0) > 0$, then by Lemma 4.1, we deduce that $\limsup_{n\to\infty} G(v_n) < 0$. By Proposition 3.2, there exists $t_n := t_n(v_n) \in (0,1)$ such that $w_n(x) := t_n^{\frac{1}{4}} v_n(t_n^{-\frac{1}{2}} x) \in \mathcal{M}$. Furthermore, we have that $\limsup_{n\to\infty} t_n < 1$. In fact, up to a subsequence, assuming that $t_n \to 1$, then

$$\begin{aligned} G(v_n) = & \frac{1}{2}\int a|\nabla v_n|^2 dx + \int v_n^2 dx + \frac{b}{2}\left(\int |\nabla v_n|^2 dx\right)^2 - \frac{p+6}{4p}\int |v_n|^p dx \\ = & \frac{1}{2} t_n \int a|\nabla v_n|^2 dx + t_n^2 \int v_n^2 dx + \frac{b}{2} t_n^2 \left(\int |\nabla v_n|^2 dx\right)^2 \\ & - \frac{p+6}{4p} t_n^{\frac{p+6}{4}} \int |v_n|^p dx + o(1) \\ = & G(w_n) + o(1) = o(1), \end{aligned}$$

which is a contradiction. With a similar argument, we find that

$$\begin{aligned} & h + o(1) = I(u_n) = \frac{1}{4}\int a|\nabla u_n|^2 dx + \frac{p-2}{8p}\int |u_n|^p dx \\ & \geq \frac{1}{4}\int a|\nabla v_n|^2 dx + \frac{p-2}{8p}\int |v_n|^p dx + \frac{1}{4}\int a|\nabla u_0|^2 dx + \frac{p-2}{8p}\int |u_0|^p dx \\ & > \frac{1}{4} t_n \int a|\nabla v_n|^2 dx + \frac{p-2}{8p} t_n^{\frac{p+6}{4}} \int |v_n|^p dx + \frac{1}{4}\int a|\nabla u_0|^2 dx + \frac{p-2}{8p}\int |u_0|^p dx \\ & = I(w_n) + \frac{1}{4}\int a|\nabla u_0|^2 dx + \frac{p-2}{8p}\int |u_0|^p dx, \end{aligned}$$

which is a contradiction as $w_n \in \mathcal{M}$.

Hence we have proven that $G(u_0) = 0$. Therefore $u_0 \in \mathcal{M}$.

Step 3. We prove that $\lim_{n\to\infty} \|v_n\| = 0$.

From $G(u_n) = 0$, for n large enough, we deduce that

$$h + o(1) = I(u_n) = \frac{1}{2}\int a|\nabla u_n|^2 dx + \frac{1}{2}\int u_n^2 dx + \frac{b}{4}\left(\int |\nabla u_n|^2 dx\right)^2 - \frac{1}{p}\int |u_n|^p dx$$

$$
\begin{aligned}
&= \frac{p+2}{2(p+6)}\int a|\nabla u_n|^2dx + \frac{p-2}{2(p+6)}\int u_n^2dx + \frac{(p-2)b}{4(p+6)}\left(\int |\nabla u_n|^2dx\right)^2 \\
&\geq \frac{p+2}{2(p+6)}\int a|\nabla v_n|^2dx + \frac{p-2}{2(p+6)}\int v_n^2dx + \frac{(p-2)b}{4(p+6)}\left(\int |\nabla v_n|^2dx\right)^2 \\
&\quad + \frac{p+2}{2(p+6)}\int a|\nabla u_0|^2dx + \frac{p-2}{2(p+6)}\int u_0^2dx + \frac{(p-2)b}{4(p+6)}\left(\int |\nabla u_0|^2dx\right)^2 + o(1) \\
&\geq I(u_0) + \frac{p+2}{2(p+6)}\int a|\nabla v_n|^2dx + \frac{p-2}{2(p+6)}\int v_n^2dx + 0(1) \\
&\geq h + \frac{p+2}{2(p+6)}\int a|\nabla v_n|^2dx + \frac{p-2}{2(p+6)}\int v_n^2dx, \ \text{(since } u_0 \in \mathcal{M})
\end{aligned}
$$

which implies that $\lim_{n\to\infty} \|v_n\| = 0$.

Hence we have proven that $u_n \to u_0$ strongly in $\tilde{H}$. Therefore, $\inf I|_{\mathcal{M}}$ is achieved at u_0. By Lemma 3.3, u_0 is a critical point of I. Combining the definition of $\tilde{H}$, we know that u_0 is non-radially symmetric and satisfies the properties as required. The proof of Theorem 1.1 is complete. □

Acknowledgements. The authors sincerely thank the unknown referees for valuable comments.

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