A NON-RADIAL SYMMETRIC SOLUTION TO A CLASS OF ELLIPTIC EQUATION WITH KIRCHHOFF TERM

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Abstract We consider the following equation with Kirchhoff term
\[-(a + b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx) \Delta u + u = |u|^{p-2}u, \quad u \in H^1(\mathbb{R}^3),\]
where $a, b$ are positive constants and $2 < p < 6$. By deducing a variant variational identity and a constraint set, we are able to prove the existence of a non-radially symmetric solution $u(x_1, x_2, x_3)$ for the full range of $p \in (2, 6)$. Moreover this solution $u(x_1, x_2, x_3)$ is radially symmetric with respect to $(x_1, x_2)$ and odd with respect to $x_3$.

Keywords Equation with Kirchhoff term, non-radially symmetric solution, variant variational identity.


1. Introduction and main results

This paper is concerned with the existence of non-radially symmetric solutions to the following equation with Kirchhoff term
\[
\begin{cases}
-\left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right) \Delta u + u = |u|^{p-2}u, \\
u := u(x), \quad x \in \mathbb{R}^3, \quad u \in H^1(\mathbb{R}^3),
\end{cases}
\]
where $a, b$ are positive constants and $2 < p < 6$. Equation (1.1) is a model of the following
\[-\left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right) \Delta u + V(x)u = f(x, u),
\]
where $a > 0$, $b \geq 0$, $V : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ and $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$. The $\int_{\mathbb{R}^3} |\nabla u|^2 \, dx \Delta u$ is usually called Kirchhoff term.

In the past ten years, many researchers have been devoted to finding solutions to (1.2), see e. g. [11, 13, 14, 16, 23]. In these papers, critical point theorems are applied to the functional
\[
\Phi(u) = \frac{1}{2} \int_{\mathbb{R}^3} (a|\nabla u|^2 + V(x)|u|^2) \, dx + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right)^2 - \int_{\mathbb{R}^3} F(x, u) \, dx
\]
defined on $E := \{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)|u|^2 dx < \infty \}$ with $F(x, u) = \int_0^u f(x, s)ds$. In the process of finding critical points of $\Phi$, there are some difficulties. The first is lack of compactness embedding from $E$ into $L^q(\mathbb{R}^3)$ for $2 < q < 6$. To overcome this, one may assume that both $V(x)$ and $f(x, u)$ are radially symmetric on $x$ and then restrict $\Phi$ on the subspace of $E$ which contains only radially symmetric functions. Or assume that $V(x) \in C(\mathbb{R}^3, \mathbb{R})$ and satisfies suitable compactness condition such that the embedding from $E$ into $L^q(\mathbb{R}^3)$ ($2 < q < 6$) is compact, see e.g. [9–11,15,22].

The second difficulty is the “geometry condition”. Comparing with the case of $b = 0$, one usually assume that the growth of $F(x, u)$ on $u$ is faster than $|u|^4$, see e. g. [5,8,11,22], where the authors assume $f(x, u)$ is $4-$superlinear at infinity in the sense that

$$\lim_{|u| \to +\infty} \frac{F(x, u)}{|u|^4} = +\infty$$ uniformly in $x \in \mathbb{R}^3$.

Observing the results mentioned above, a typical case of nonlinear function $f(x, u) = |u|^{p-2}u$ is not covered when $2 < p < 4$. Recently, by using monotonicity trick, the authors in [12] proved that the following equation

$$- \left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx \right) \Delta u + V(x)u = |u|^{p-2}u,$$

has a positive ground state solution in $E$ for any $p \in (3, 6)$.

For (1.1), Wu [22] has essentially proven the existence of radially symmetric solution when $4 < p < 6$. Also for $p \in (4, 6)$, Sun and Zhang [21] proved that the positive ground state solution to (1.1) is unique and radially symmetric. Recently, such kind of results has been extended to fractional Kirchhoff type equation or $p-$Kirchhoff equations, see e. g. [1,3,6,17,18] as well as the references therein. But we do not see any results about the existence of non-radial solutions to (1.1) for $2 < p < 4$. The purpose of the present paper is to prove that (1.1) admits at least one non-radially symmetric solution for the full range of $p \in (2, 6)$. Our main result is the following theorem.

**Theorem 1.1.** Assume that $a, b > 0$ and $2 < p < 6$. Then (1.1) admits a non-radially symmetric solution $u \in H^1(\mathbb{R}^3)$. Moreover if denoting $x = (x_1, x_2, x_3)$ and $u := u(x_1, x_2, x_3)$, then $u$ is radially symmetric with respect to $(x_1, x_2)$ and odd with respect to $x_3$.

The proof of Theorem 1.1 is by variational methods. Our idea is inspired from the paper of Ruiz [20] where the author constructed a kind of Nehari-Pohozaev type identity and studied a class of Schrodinger-Poisson system. Our strategy is to deduce a variant variational identity and define a subset $\mathcal{M}$ (see Section 3 ) of $H^1(\mathbb{R}^3)$. On the set $\mathcal{M}$ we minimize the following functional

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^3} (a|\nabla u|^2 + u^2) dx + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p dx$$

and prove that the minimum can be achieved.

This paper is organized as follows. In Section 2, we give some preliminaries about group action on $\mathbb{R}^3$. In Section 3, we deduce a variant variational identity and define a subset $\mathcal{M}$; then we prove that a minimizer of $I_{\mathcal{M}}$ is a critical point of $I$ on $H^1(\mathbb{R}^3)$. We emphasize that with the help of this construction, we manage to
prove the existence of solution for the full range of $p \in (2, 6)$, which is a complement of several previous works. In Section 4, we finish the proof of Theorem 1.1.

**Notation.** Throughout this paper, all integrals are taken over $\mathbb{R}^3$ unless specified. $C_n$ ($n = 1, 2, \cdots$) denotes a positive constant whose exact value is not important. $L^q(\mathbb{R}^3)$ ($1 \leq q < +\infty$) is the usual Lebesgue space with the standard norm $\|u\|_q$. For $a > 0$, we introduce an equivalent norm on $H^1(\mathbb{R}^3)$: $\|u\|_2 := \int (a|\nabla u|^2 + u^2) \, dx$ with the corresponding inner product $(u, v) := \int (a\nabla u \nabla v + uv) \, dx$.

2. Preliminaries

In this section, we introduce a group action on $\mathbb{R}^3$ which is originated from [4]. For every $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, we define a map on $H^1(\mathbb{R}^3)$ as

$$(T_\theta u)(x) := -u(g_\theta x), \quad \text{where} \quad g_\theta := \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Then, $T_\theta : u \mapsto T_\theta u$ is a linear operator from $H^1(\mathbb{R}^3)$ to $H^1(\mathbb{R}^3)$. And $T_\theta$ satisfies the following properties.

**Proposition 2.1** ([4]). For any $\theta_1, \theta_2 \in \mathbb{R}/2\pi\mathbb{Z}$ and $u, v \in H^1(\mathbb{R}^3)$

$$T_{\theta_1}T_{\theta_2} = T_{\theta_1+\theta_2}T_0 = T_0T_{\theta_1+\theta_2} = T_{\theta_2}T_{\theta_1}, \quad (2.1)$$

$$T_0T_0 = \text{Id}, \quad (2.2)$$

$$(T_{\theta}u, T_{\theta}v) = (u, v). \quad (2.3)$$

Next, we introduce a set of fixed points in $H^1(\mathbb{R}^3)$:

$$\tilde{H} := \{ u \in H^1(\mathbb{R}^3) : \text{for any } \theta \in \mathbb{R}/2\pi\mathbb{Z}, \ T_\theta u = u \}.$$

**Remark 2.1.**

1. The $\tilde{H}$ is closed and weakly closed in $H^1(\mathbb{R}^3)$.
2. For every $u \in \tilde{H}$, $u$ is radially symmetric with respect to $(x_1, x_2)$ and odd with respect to $x_3$. Indeed, for a. e. $x := (x_1, x_2, x_3) \in \mathbb{R}^3$

$$u(x_1, x_2, -x_3) = u(g_\theta x) = -u(x_1, x_2, x_3)$$

and

$$u(r_\theta(x_1, x_2), x_3) = u(g_\theta(x_1, x_2, -x_3)) = -u(x_1, x_2, -x_3) = u(x_1, x_2, x_3),$$

where

$$r_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

**Lemma 2.1.** If $u \in \tilde{H}$ is a critical point of $I|_{\tilde{H}}$, then $u$ is a critical point of $I$. 
Proof. From (2.3), we get that for every $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ and $u, v \in H^1(\mathbb{R}^3)$

$$I(T_\theta u) = \frac{1}{2} \int (a|\nabla (T_\theta u)|^2 + |T_\theta u|^2) \, dx + \frac{b}{4} \left( \int |\nabla (T_\theta u)|^2 \, dx \right)^2 - \frac{1}{p} \int |T_\theta u|^{p/2} \, dx$$

$$= \frac{1}{2} \int (a|\nabla u|^2 + |u|^2) \, dx + \frac{b}{4} \left( \int |\nabla u|^2 \, dx \right)^2 - \frac{1}{p} \int |u|^p \, dx$$

$$= I(u),$$

and

$$\langle I'(u), T_\theta v \rangle = \frac{d}{d\lambda} I(u + \lambda T_\theta v) \bigg|_{\lambda=0} = \frac{d}{d\lambda} I(T_\theta (T_\theta u + \lambda v)) \bigg|_{\lambda=0}$$

$$= \frac{d}{d\lambda} I(T_\theta u + \lambda v) \bigg|_{\lambda=0} = \langle I'(T_\theta u), v \rangle.$$

Since the gradient of $I$ at $u$ is defined by, for every $v \in H^1(\mathbb{R}^3)$,

$$\langle \nabla I(u), v \rangle = \langle I'(u), v \rangle,$$

we have that for every $u \in \tilde{H}$ and $v \in H^1(\mathbb{R}^3)$

$$\langle T_\theta (\nabla I(u)), v \rangle = \langle T_\theta (\nabla I(u)), T_\theta T_\theta v \rangle = \langle \nabla I(u), T_\theta v \rangle$$

$$= \langle I'(u), T_\theta v \rangle = \langle I'(T_\theta u), v \rangle$$

$$= \langle \nabla I(T_\theta u), v \rangle = \langle \nabla I(u), v \rangle,$$

which implies that

$$\nabla I(u) \in \tilde{H}.$$ 

Since $\tilde{H}$ is closed in $H^1(\mathbb{R}^3)$, denoted with $\tilde{H}^\perp$ its orthogonal, we write

$$H^1(\mathbb{R}^3) = \tilde{H} + \tilde{H}^\perp.$$ 

If $u \in \tilde{H}$ is a critical point of $I|_{\tilde{H}}$, for every $v \in H^1(\mathbb{R}^3)$ as the sum of $v_1 \in \tilde{H}$ and $v_2 \in \tilde{H}^\perp$

$$\langle I'(u), v \rangle = \langle I'(u), v_1 \rangle + \langle I'(u), v_2 \rangle$$

$$= \langle (I|_{\tilde{H}})'(u), v_1 \rangle + \langle \nabla I(u), v_2 \rangle$$

$$= 0.$$

\[\square\]

3. Variant variational identity and a constraint set

The aim of this section is to construct a suitable constraint set, on which we can define a minimization problem. The construction is based on a variant variational identity ($G(u) = 0$, see Remark 3.1). Keeping the definition of the functional $I$ in mind, we observe that due to the presence of Kirchhoff term, when $p \in (2, 4)$, it is not easy to see if the functional $I$ is not bounded from below. We begin with the following proposition.

**Proposition 3.1.** Let $a > 0$, $b > 0$ and $p \in (2, 6)$. The functional $I$ is not bounded below on $\tilde{H}$.
Proof. For any \( u \in \tilde{H} \) and any \( t > 0 \), denote \( w(x) := t^4 u(t^{-\frac{1}{2}} x) \). Then by direct computations, we have that
\[
\int |\nabla w|^2 dx = t \int |\nabla u|^2 dx, \quad \int |w|^2 dx = t^2 \int |u|^2 dx, \quad \int |w|^p dx = t^{\frac{p+6}{p}} \int |u|^p dx,
\]
and therefore
\[
I(w) = \frac{1}{2} t \int a|\nabla u|^2 dx + \frac{1}{2} t^2 \int |u|^2 dx + \frac{b}{4} t^2 \left( \int |\nabla u|^2 dx \right)^2 - \frac{1}{p} t^{\frac{p+6}{p}} \int |u|^p dx.
\]
Since \( \frac{p+6}{p} > 2 \), we deduce that \( I(w) \to -\infty \) as \( t \to +\infty \).

Lemma 3.1. Let \( \alpha, \beta, \gamma, \delta \) be positive constants and \( p \in (2, 6) \). For \( t \geq 0 \), we define \( f(t) := \alpha t + \beta t^2 + \gamma t^3 - \delta t^{\frac{p+6}{4}} \). Then \( f \) has a unique critical point which corresponds to its maximum.

Proof. For \( t > 0 \), we compute directly that
\[
f'(t) = \alpha + 2\beta t + 2\gamma t - \frac{p+6}{4} \delta t^{\frac{p+2}{4}},
\]
\[
f''(t) = 2\beta + 2\gamma - \frac{p+6}{4} \delta t^{-\frac{p-2}{4}}.
\]
Since \( f'' \) is strictly decreasing with respect to \( t > 0 \) and \( f''(0) = 2\beta + 2\gamma > 0 \), there exists \( t_1 > 0 \) such that \( f''(t_1) = 0 \) and \( f''(t_1 - t) > 0 \) for \( t \neq t_2 \).

Since \( f'(0) = \alpha > 0 \) and \( f' \) is increasing for \( t < t_1 \), \( f' \) takes positive values at least for \( t \in [0, t_1] \). For \( t > t_1 \), \( f' \) decreases, and goes to \(-\infty \). Then there exists \( t_0 > t_1 \) such that \( f'(t_0) = 0 \) and \( f'(t_0 - t) > 0 \) for \( t \neq t_0 \).

Taking a conclusion, \( t_0 \) is the unique critical point of \( f \) and corresponds to its maximum as \( \frac{p+6}{4} > 2 \).

We are now in a position to construct a manifold on which we can define a minimization problem. Our idea is to establish a variant variational identity and use this identity to construct a set. Then we prove that this set is a manifold and share some properties similar to Nehari manifold. More precisely, we will construct a manifold such that for any \( u \neq 0 \), there is a curve passing \( u \) and crossing the manifold only at one point. Moreover, along this curve the functional \( I \) achieves its maximum at a unique point. To attain this goal, for \( w(x) := t^4 u(t^{-\frac{1}{2}} x) \) defined as above, we consider
\[
I(w) = \frac{1}{2} t \int a|\nabla u|^2 dx + \frac{1}{2} t^2 \int |u|^2 dx + \frac{b}{4} t^2 \left( \int |\nabla u|^2 dx \right)^2 - \frac{1}{p} t^{\frac{p+6}{p}} \int |u|^p dx.
\]
Then for \( u \) fixed, \( I(w) \) is positive for small \( t \) and tends to \(-\infty \) as \( t \to +\infty \). Choosing \( f(t) := I(w) \), from Lemma 3.1, we know that \( f(t) \) has a unique critical point, corresponding to its maximum. Define the functional \( G: \tilde{H} \to \mathbb{R} \) as
\[
G(u) := \frac{1}{2} \int a|\nabla u|^2 dx + \int u^2 dx + \frac{b}{2} \left( \int |\nabla u|^2 dx \right)^2 - \frac{p+6}{4p} \int |u|^p dx
\]
and set
\[
\mathcal{M} = \{ u \in \tilde{H} \setminus \{ 0 \} : G(u) = 0 \}.
\]
Proposition 3.2. Let \( p \in (2, 6) \). For any \( u \in \dot{H} \setminus \{0\} \), there is a unique \( t_0 := t_0(u) > 0 \) such that \( t_0^2 u(t_0^{-2} x) \in \mathcal{M} \). Moreover if \( G(u) < 0 \), then \( t_0 \in (0, 1) \).

Proof. Firstly, for any \( u \in \dot{H} \setminus \{0\} \) and any \( t > 0 \), we choose \( f(t) := I(w) \) with \( w(x) := t^\frac{4}{p} u(t^{-\frac{4}{p}} x) \). Then from the proof of Lemma 3.1, \( f(t) \) has a unique critical point \( t_0 := t_0(u) \) (here \( t_0(u) \) means \( t_0 \) depends on \( u \)), corresponding to its maximum. Therefore

\[
f'(t_0) = \frac{1}{2} \int a|\nabla u|^2 dx + t_0 \int |u|^2 dx + \frac{b}{2} t_0 \left( \int |\nabla u|^2 dx \right)^2 - \frac{p + 6}{4p} t_0^2 \int |u|^p dx = 0.
\]

Denoting \( w_0(x) := t_0^\frac{4}{p} u(t_0^{-\frac{4}{p}} x) \), then \( w_0 \neq 0 \) and we have that

\[
G(w_0) = \frac{1}{2} \int a|\nabla w_0|^2 dx + \int |w_0|^2 dx + \frac{b}{2} \left( \int |\nabla w_0|^2 dx \right)^2 - \frac{p + 6}{4p} \int |w_0|^p dx
\]

\[= \frac{1}{2} t_0 \int a|\nabla u|^2 dx + t_0^2 \int |u|^2 dx + \frac{b}{2} t_0 \left( \int |\nabla u|^2 dx \right)^2 - \frac{p + 6}{4p} t_0^2 \int |u|^p dx
\]

\[= t_0 f'(t_0) = 0.
\]

Hence \( w_0(x) := t_0^\frac{4}{p} u(t_0^{-\frac{4}{p}} x) \in \mathcal{M} \).

Secondly, if \( G(u) < 0 \), then from

\[
G(u) = \frac{1}{2} \int a|\nabla u|^2 dx + \int |u|^2 dx + \frac{b}{2} \left( \int |\nabla u|^2 dx \right)^2 - \frac{p + 6}{4p} \int |u|^p dx < 0
\]

and

\[
G(w_0) = \frac{1}{2} t_0 \int a|\nabla u|^2 dx + t_0^2 \int |u|^2 dx + \frac{b}{2} t_0 \left( \int |\nabla u|^2 dx \right)^2 - \frac{p + 6}{4p} t_0^2 \int |u|^p dx = 0,
\]

we obtain that

\[
\frac{1}{2} \left( t_0^\frac{4}{p} - t_0 \right) \int a|\nabla u|^2 dx + \left( t_0^\frac{4}{p} - t_0^2 \right) \int |u|^2 dx + \frac{b}{2} \left( t_0^\frac{4}{p} - t_0^2 \right) \left( \int |\nabla u|^2 dx \right)^2 < 0,
\]

which implies \( t_0 < 1 \). Therefore \( t_0 \in (0, 1) \). \( \square \)

Remark 3.1. If \( v \neq 0 \) is a weak solution of (1.1), then by the calculation of the Pohozaev [19] identity of equation (1.1), \( P(v) = 0 \), where

\[
P(v) := \frac{1}{2} \int a|\nabla v|^2 dx + \frac{3}{2} \int |v|^2 dx + \frac{b}{2} \left( \int |\nabla v|^2 dx \right)^2 - \frac{3}{p} \int |v|^p dx.
\]

Moreover, for this \( v \), according to the proof of Proposition 3.2, there is a unique \( t_0(v) > 0 \) such that \( t_0(v)^\frac{4}{p} u(t_0(v)^{-\frac{4}{p}} x) \in \mathcal{M} \). We claim that \( t_0(v) = 1 \). To see this, one only notices that \( \frac{1}{4} \langle P'(v), v \rangle = 0 \), \( \frac{1}{4} P(v) = 0 \) and \( G(v) = \frac{1}{4} \langle P'(v), v \rangle + \frac{1}{2} P(v) = 0 \).

Lemma 3.2. Let \( a > 0 \), \( b > 0 \) and \( p \in (2, 6) \). Then \( \mathcal{M} \) is bounded away from zero.
Proof. For any $u \in \mathcal{M}$, we deduce from $G(u) = 0$ and the Sobolev inequality that

\[
\frac{1}{2} \int a|\nabla u|^2 \, dx + \int |u|^2 \, dx \leq \frac{1}{2} \int a|\nabla u|^2 \, dx + \int |u|^2 \, dx + \frac{b}{2} \left( \int |\nabla u|^2 \, dx \right)^2 = \frac{p+6}{4p} \int |u|^p \, dx \leq C_1 ||u||^p.
\]

Which implies that there is a $C_2 > 0$ such that $||u||^{p-2} \geq C_2$.

Lemma 3.3. Let $a, b > 0$ and $p \in (2, 6)$. Then $\mathcal{M}$ is a nature $C^1$-constraint in the sense that a critical point of $I|_{\mathcal{M}}$ is also a critical point of $I$ in $\mathcal{H}$.

Proof. The proof can be sketched as following 2 steps.

Step 1. We prove that for every $u \in \mathcal{M}$, $G'(u) \neq 0$. Then $\mathcal{M}$ is a $C^1$-manifold.

Suppose that there is $u \in \mathcal{M}$ such that $G'(u) = 0$. We denote

\[
i := \int |\nabla u|^2 \, dx, \quad j := \int |u|^2 \, dx \text{ and } k := \int |u|^p \, dx.
\]

Next, in a weak sense, the equation $G'(u) = 0$ can be written as

\[
- \left( a + 2b \int |\nabla u|^2 \, dx \right) \Delta u + 2u = \frac{p+6}{4} |u|^{p-2} u.
\]

Then we have the following relations:

\[
\begin{aligned}
ai + 2j + bi^2 - \frac{p+6}{2p} k &= 0, \\
ai + 2j + 2bi^2 - \frac{p+6}{4} k &= 0, \\
\frac{1}{2} ai + 3j + bi^2 - \frac{3(p+6)}{4p} k &= 0, \\
\frac{1}{4} ai + \frac{p-2}{8p} k &= I(u),
\end{aligned}
\]

where the first one is from $2G(u) = 0$; the second one comes from multiplying (3.1) by $u$ and integrating by parts; the third one is the Pohozaev equality of (3.1) and the fourth one is due to the definition of $I(u)$ and $G(u) = 0$.

Now solving these equations as the following: combining the second one with the third one and the first one respectively, we obtain that

\[
\begin{aligned}
4j &= \frac{(p+6)(6-p)}{4p} k, \\
ai + 2j &= \frac{(p+6)(4-p)}{4p} k.
\end{aligned}
\]

From (3.2) and $p \in (2, 6)$, we deduce that

\[
ai = \frac{(p+6)(2-p)}{8p} k < 0,
\]

which is a contradiction. This proves the Step 1.
Step 2. We will prove: if \( u \) is a critical point of \( I|_M \), then \( I'(u) = 0 \).

If \( u \) is a critical point of \( I \) restricted on the manifold \( M \), then there exists a Lagrange multiplier \( \lambda \in \mathbb{R} \) such that \( I'(u) = \lambda G'(u) \). Denote \( I_0 := I(u) \). Our aim is to prove \( \lambda = 0 \).

Firstly, from \( I'(u) = \lambda G'(u) \), in a weak sense, we have that
\[
- \left( a + b \int |\nabla u|^2 \right) \Delta u + u - |u|^{p-2}u = \lambda \left( - \left( a + 2b \int |\nabla u|^2 \right) \Delta u + 2u - \frac{p+6}{4} |u|^{p-2}u \right).
\]
Rewrite the above equation as
\[
- \left( (\lambda - 1)a + (2\lambda - 1)b \int |\nabla u|^2 \right) \Delta u + (2\lambda - 1)u = \left( \frac{p+6}{4} \lambda - 1 \right) |u|^{p-2}u. \tag{3.3}
\]
Secondly, using the notations \( i, j \) and \( k \) as in the previous step, we obtain that
\[
\begin{align*}
\frac{1}{4} ai + \frac{p-2}{8p} k &= I_0, \tag{3.4} \\
ai + 2j + bi^2 - \frac{p+6}{2p} k &= 0, \tag{3.5} \\
(\lambda - 1)ai + (2\lambda - 1)j + (2\lambda - 1)bi^2 - \left( \frac{p+6}{4} \lambda - 1 \right) k &= 0 \tag{3.6}
\end{align*}
\]
and
\[
\frac{1}{2}(\lambda - 1)ai + \frac{3}{2}(2\lambda - 1)j + \frac{1}{2}(2\lambda - 1)bi^2 - \frac{3}{p} \left( \frac{p+6}{4} \lambda - 1 \right) k &= 0, \tag{3.7}
\]
where (3.4) is from \( I_0 := I(u) \) and \( G(u) = 0 \); (3.5) is due to \( G(u) = 0 \); (3.6) comes from multiplying (3.3) by \( u \) and integrating by parts; (3.7) is the Pohozaev identity of (3.3).

For the linear system (3.4)-(3.7), taking elementary transformation to the coefficient matrix \( A \)
\[
A = \begin{pmatrix}
\frac{1}{4} & 0 & 0 & \frac{p-2}{8p} \\
1 & 2 & 1 & -\frac{p+6}{2p} \\
\lambda - 1 & 2\lambda - 1 & 2\lambda - 1 & 1 - \frac{p+6}{4} \lambda \\
\frac{1}{2}(\lambda - 1) & \frac{3}{2}(2\lambda - 1) & \frac{1}{2}(2\lambda - 1) & \frac{3}{p} \left( 1 - \frac{p+6}{4} \lambda \right)
\end{pmatrix}
\]
and computing its determinant, we obtain that
\[
\det A = \frac{(p+2)(2-p)}{32p} \lambda(2\lambda - 1).
\]
If \( \det A \neq 0 \), then by Cramer rule, we know the linear system (3.4)–(3.7) has a unique solution and
\[
k = \frac{I_0}{\det A} \lambda(2\lambda - 1) = \frac{32p I_0}{(p+2)(2-p)}. \tag{3.8}
\]
Notice that $I_0 := I(u)$. Then we deduce from $G(u) = 0$ that

$$I_0 := I(u) = \frac{1}{2} (ai + j) + \frac{b}{4} i^2 - \frac{k}{p}$$

$$= \frac{1}{2} (ai + j) + \frac{b}{4} i^2 - \frac{4}{p+6} \left( \frac{1}{2} ai + j + \frac{b}{2} i^2 \right)$$

$$= \frac{p+2}{2(p+6)} ai + \frac{p-2}{2(p+6)} j + \frac{(p-2)b}{4(p+6)} i^2 > 0.$$  

Combining this with $p > 2$, we know that the right hand side of (3.8) is negative. This contradicts to the definition of $k$.

Therefore $\det A = 0$. This means that

$$\lambda = 0 \quad \text{or} \quad \lambda = \frac{1}{2}.$$

Suppose that $\lambda = \frac{1}{2}$. Then (3.6) becomes

$$-\frac{1}{2} ai - \frac{p-2}{8} k = 0,$$

which is also a contradiction since $p > 2$, $a > 0$, $i > 0$ and $k > 0$. Therefore $\lambda = 0$.

Hence we deduce that $I'(u) = 0$.

In sum, we finish the proof of Lemma 3.3. \hfill \Box

4. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1. Our strategy is to prove the following minimization problem

$$h := \inf \{I(u) : u \in \mathcal{M} \}$$

(4.1)

is achieved by an element in $\tilde{H}$ which is a non-radially symmetric solution as required. We start with proving the following lemma.

Lemma 4.1. Let $\{u_n\} \subset \mathcal{M}$ be such that $u_n \rightharpoonup u$ weakly in $\tilde{H}$. Denote $v_n := u_n - u$. Then for $n$ large enough,

$$o(1) + G(u) + G(v_n) \leq 0.$$

Proof. By Brézis-Lieb Lemma [2], for $n$ large enough,

$$|u_n|^p = |u|^p + |v_n|^p + o(1)$$

and

$$\|u_n\|^2 = \|u\|^2 + \|v_n\|^2 + o(1).$$

As $u_n \in \mathcal{M}$, we obtain that

$$0 = G(u_n) = \frac{a}{2} |\nabla u_n|^2 + |u_n|^2 + \frac{b}{2} |\nabla u_n|^2 - \frac{p+6}{4p} |u_n|^p$$

$$= \frac{a}{2} |\nabla u|^2 + \frac{a}{2} |\nabla v_n|^2 + |u|^2 + |v_n|^2 + \frac{b}{2} |\nabla u|^2 + \frac{b}{2} |\nabla v_n|^2$$

$$+ b|\nabla u|^2 |\nabla v_n|^2 - \frac{p+6}{4p} |u|^p - \frac{p+6}{4p} |v_n|^p + o(1)$$

$$\geq G(u) + G(v_n) + o(1).$$
This proves the lemma.

**Proof of Theorem 1.1.** Firstly, from Lemma 3.2 we know that the $h$ defined by (4.1) satisfies $h > 0$. Let $\{u_n\} \subset \mathcal{M}$ be a minimizing sequence of $I$ on $\mathcal{M}$, i.e.

$$\lim_{n \to \infty} I(u_n) = h \quad \text{and} \quad G(u_n) = 0.$$ 

Define

$$i_n := \int |\nabla u_n|^2, \quad j_n := \int u_n^2 dx, \quad k_n := \int |u_n|^p dx.$$ 

Obviously, $i_n, j_n, k_n$ are positive and

$$\begin{align*}
\frac{1}{2} a i_n + \frac{1}{2} j_n + \frac{1}{4} b i_n^2 - \frac{1}{p} k_n &= h + o(1), \\
\frac{1}{2} a i_n + j_n + \frac{1}{2} b i_n^2 - \frac{p + 6}{4p} k_n &= 0.
\end{align*}$$

Combining the two equations, one has that

$$\frac{p + 2}{8} a i_n + \frac{p - 2}{8} j_n + \frac{p - 2}{16} b i_n^2 = \frac{p + 6}{4} h + o(1),$$

which implies that, for $n$ large enough,

$$a i_n + j_n < \frac{2(p + 6)}{p - 2} h + 1.$$ 

Therefore, $\{u_n\}$ is bounded in $\tilde{H}$.

In the following, we will prove that $\{u_n\}$ contains a convergent subsequence which converges to a minimizer of the minimum $h$ defined in (4.1). We manage to do this by three steps.

**Step (i).** Up to a subsequence, still denoted by $\{u_n\}$, we may assume $u_n \rightharpoonup u_0$ in $\tilde{H}$. The $G(u_n) = 0$ and Lemma 3.2 imply that there is $d_0 > 0$ such that

$$\int_{\mathbb{R}^3} |u_n|^p dx \geq d_0 > 0.$$ 

We set $D_m := \mathbb{R}^2 \times (m, m + 1)$ for every $m \in \mathbb{Z}$. Then

$$d_0 \leq \int_{\mathbb{R}^3} |u_n|^p dx = \sum_{m \in \mathbb{Z}} \int_{D_m} |u_n|^p dx = \sum_{m \in \mathbb{Z}} \left( \int_{D_m} |u_n|^p dx \right) \frac{p - 2}{p} |u_n|^2 dx$$

$$\leq \sum_{m \in \mathbb{Z}} \left( \int_{D_m} |u_n|^p dx \right) \frac{p - 2}{p} \left( \int_{D_m} |u_n|^p dx \right)^{\frac{2}{p}}$$

$$\leq \sup_{m \in \mathbb{Z}} \left( \int_{D_m} |u_n|^p dx \right) \frac{p - 2}{p} \sum_{m \in \mathbb{Z}} \left( \int_{D_m} |u_n|^p dx \right)^{\frac{2}{p}}$$

$$\leq C_3 \sup_{m \in \mathbb{Z}} \left( \int_{D_m} |u_n|^p dx \right) \frac{p - 2}{p} \sum_{m \in \mathbb{Z}} \|u_n\|^2_{L^1(D_m)}$$

$$\leq C_4 \sup_{m \in \mathbb{Z}} \left( \int_{D_m} |u_n|^p dx \right) \frac{p - 2}{p}.$$ 

Then $\sup_{m \in \mathbb{Z}} \left( \int_{D_m} |u_n|^p dx \right) \frac{p - 2}{p}$ is bounded away from zero uniformly with respect to $n$. By Esteban and Lions’ Theorem [7, P.381, Theorem 1], we have that $u_0 \neq 0$. 

\begin{thebibliography}{9}

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**Step (ii).** We will prove that $G(u_0) = 0$.

Arguing by a contradiction, we firstly assume that $G(u_0) < 0$. Then by Proposition 3.2, there exists $t_0 := t_0(u_0) \in (0, 1)$ such that $w(x) := \frac{1}{t_0^2} u(t_0^{-\frac{2}{3}} x) \in \mathcal{M}$. As \{u_n\} is a minimizing sequence, we find that

$$h + o(1) = I(u_n) = \frac{1}{4} \int a|\nabla u_n|^2 dx + \frac{p - 2}{8p} \int |u_n|^p dx$$

$$\geq \frac{1}{4} \int a|\nabla w|^2 dx + \frac{p - 2}{8p} \int |w|^p dx$$

$$> \frac{1}{4} t_n \int a|\nabla u_0|^2 dx + \frac{p - 2}{8p} t_n^{\frac{p + 2}{4}} \int |u_0|^p dx$$

$$= \frac{1}{4} \int a|\nabla w|^2 dx + \frac{p - 2}{8p} \int |w|^p dx = I(w),$$

which is a contradiction as $w \in \mathcal{M}$.

If $G(u_0) > 0$, then by Lemma 4.1, we deduce that $\limsup_{n \to \infty} G(v_n) < 0$. By Proposition 3.2, there exists $t_n := t_n(v_n) \in (0, 1)$ such that $w_n(x) := \frac{1}{t_n^2} v_n(t_n^{-\frac{2}{3}} x) \in \mathcal{M}$. Furthermore, we have that $\limsup_{n \to \infty} t_n < 1$. In fact, up to a subsequence, assuming that $t_n \to 1$, then

$$G(v_n) = \frac{1}{2} \int a|\nabla v_n|^2 dx + \int v_n^2 dx + \frac{b}{2} \left( \int |\nabla v_n|^2 dx \right)^2 - \frac{p + 6}{4p} \int |v_n|^p dx$$

$$= \frac{1}{2} t_n \int a|\nabla v_n|^2 dx + t_n^2 \int v_n^2 dx + \frac{b}{2} t_n^2 \left( \int |\nabla v_n|^2 dx \right)^2$$

$$- \frac{p + 6}{4p} t_n^{p + 2} \int |v_n|^p dx + o(1)$$

$$= G(w_n) + o(1) = o(1),$$

which is a contradiction. With a similar argument, we find that

$$h + o(1) = I(u_n) = \frac{1}{4} \int a|\nabla u_n|^2 dx + \frac{p - 2}{8p} \int |u_n|^p dx$$

$$\geq \frac{1}{4} \int a|\nabla w|^2 dx + \frac{p - 2}{8p} \int |w|^p dx + \frac{1}{4} \int a|\nabla u_0|^2 dx + \frac{p - 2}{8p} \int |u_0|^p dx$$

$$> \frac{1}{4} t_n \int a|\nabla u_n|^2 dx + \frac{p - 2}{8p} t_n^{\frac{p + 2}{4}} \int |u_n|^p dx + \frac{1}{4} \int a|\nabla u_0|^2 dx + \frac{p - 2}{8p} \int |u_0|^p dx$$

$$= I(w_n) + \frac{1}{4} \int a|\nabla u_0|^2 dx + \frac{p - 2}{8p} \int |u_0|^p dx,$$

which is a contradiction as $w_n \in \mathcal{M}$.

Hence we have proven that $G(u_0) = 0$. Therefore $u_0 \in \mathcal{M}$.

**Step 3.** We prove that $\lim_{n \to \infty} \|v_n\| = 0$.

From $G(u_n) = 0$, for $n$ large enough, we deduce that

$$h + o(1) = I(u_n) = \frac{1}{2} \int a|\nabla u_n|^2 dx + \frac{1}{2} \int u_n^2 dx + \frac{b}{4} \left( \int |\nabla u_n|^2 dx \right)^2 - \frac{1}{p} \int |u_n|^p dx$$
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\[
\begin{align*}
&\quad = \frac{p+2}{2(p+6)} \int a|\nabla u_n|^2 \, dx + \frac{p-2}{2(p+6)} \int u_n^2 \, dx + \frac{(p-2)b}{4(p+6)} \left( \int |\nabla u_n|^2 \, dx \right)^2 \\
&\quad \geq \frac{p+2}{2(p+6)} \int a|\nabla v_n|^2 \, dx + \frac{p-2}{2(p+6)} \int v_n^2 \, dx + \frac{(p-2)b}{4(p+6)} \left( \int |\nabla v_n|^2 \, dx \right)^2 \\
&\quad + \frac{p+2}{2(p+6)} \int a|\nabla u_0|^2 \, dx + \frac{p-2}{2(p+6)} \int u_0^2 \, dx + \frac{(p-2)b}{4(p+6)} \left( \int |\nabla u_0|^2 \, dx \right)^2 + o(1) \\
&\quad \geq I(u_0) + \frac{p+2}{2(p+6)} \int a|\nabla v_n|^2 \, dx + \frac{p-2}{2(p+6)} \int v_n^2 \, dx + o(1) \\
&\quad \geq h + \frac{p+2}{2(p+6)} \int a|\nabla v_n|^2 \, dx + \frac{p-2}{2(p+6)} \int v_n^2 \, dx, \text{ (since } u_0 \in M) \\
\end{align*}
\]

which implies that \( \lim_{n \to \infty} \|v_n\| = 0. \)

Hence we have proven that \( u_n \to u_0 \) strongly in \( \hat{H} \). Therefore, \( \inf I|_M \) is achieved at \( u_0 \). By Lemma 3.3, \( u_0 \) is a critical point of \( I \). Combining the definition of \( \hat{H} \), we know that \( u_0 \) is non-radially symmetric and satisfies the properties as required. The proof of Theorem 1.1 is complete. \( \square \)

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References


