

AN INTEGRAL BOUNDARY VALUE PROBLEM OF CONFORMABLE INTEGRO-DIFFERENTIAL EQUATIONS WITH A PARAMETER*

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Abstract In this article, we consider some properties of positive solutions for a new conformable integro-differential equation with integral boundary conditions and a parameter

$$\begin{cases} T_\alpha u(t) + \lambda f(t, u(t), I_\alpha u(t)) = 0, t \in [0, 1], \\ u(0) = 0, u(1) = \beta \int_0^1 u(t) dt, \beta \in [\frac{3}{2}, 2), \end{cases}$$

where $\alpha \in (1, 2]$, λ is a positive parameter, T_α is the usual conformable derivative and I_α is the conformable integral, $f : [0, 1] \times \mathbf{R}^+ \times \mathbf{R}^+ \rightarrow \mathbf{R}^+$ is a continuous function, where $\mathbf{R}^+ = [0, +\infty)$. We use a recent fixed point theorem for monotone operators in ordered Banach spaces, and then establish the existence and uniqueness of positive solutions for the boundary value problem. Further, we give an iterative sequence to approximate the unique positive solution and some good properties of positive solution about the parameter λ . A concrete example is given to better demonstrate our main result.

Keywords Positive solution, conformable derivative, integro-differential equations, fixed point theorem of generalized concave operators.

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1. Introduction

This paper is concerned with some properties of positive solutions for a new conformable integro-differential equation with an integral boundary condition and a parameter

$$\begin{cases} T_\alpha u(t) + \lambda f(t, u(t), I_\alpha u(t)) = 0, t \in [0, 1], \\ u(0) = 0, u(1) = \beta \int_0^1 u(t) dt, \beta \in [\frac{3}{2}, 2), \end{cases} \quad (1.1)$$

where $\alpha \in (1, 2]$, λ is a positive parameter, T_α is the usual conformable derivative and I_α is the conformable integral, $f : [0, 1] \times \mathbf{R}^+ \times \mathbf{R}^+ \rightarrow \mathbf{R}^+$ is a continuous

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function, where $\mathbf{R}^+ = [0, +\infty)$. We intend to show that, for any given parameter $\lambda > 0$, (1.1) has a unique positive solution u_λ^* in a special set P_h , and then we present some good properties of positive solution which include that the positive solution is continuous, strictly increasing with respect to the parameter.

Fractional calculus has made fast development and it has been applied to some fields which include mathematics, science, biophysics, viscoelasticity, engineering and so on. We refer the reader to a series of books and papers [1–9] and references cited therein. At present, there exist a number of definitions of fractional derivatives in the literature, the most popular of which are the Riemann-Liouville and Caputo fractional derivatives.

In [10], Khalil et al. introduced an interesting definition of local fractional derivative, called the conformable derivative, which modifies the basic limit expression of the derivative by inserting the multiple $t^{1-\alpha}$, $0 < \alpha < 1$ inside the definition. And we can find that it is different from the Riemann-Liouville and Caputo fractional derivatives. For some recent works on the basic theory properties and application generalizations of the conformable derivative, we can refer [11–13]. An account of results on initial boundary value problems of the conformable differential equations, we can refer to the literature [14–16].

Further, the existence of solutions to boundary value problems for some specific conformable differential equations have been paid much attention, see [14, 17–22, 24] for example. Their methods used are mainly the notion of tube solution and Schauder's fixed-point theorem ([14]), Krasnoselskii's fixed point theorem ([17]), the method of upper and lower solutions coupled with the monotone iterative technique ([19]), the fixed-point index theory, the barrier strips technique and a priori estimation ([21]), the topological transversality theorem ([22]) and so on.

Recently, the author [20] studied the existence of at least one positive solution for a conformable differential equation with integral boundary conditions

$$\begin{cases} T_\alpha u(t) + f(t, u(t)) = 0, t \in [0, 1], \\ u(0) = 0, u(1) = \beta \int_0^1 u(t) dt, \beta \in [0, 2), \end{cases}$$

where $\alpha \in (1, 2]$, T_α denotes the usual conformable derivative of order α , and $f : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$ is a continuous function.

Different from the papers mentioned above, motivated by the works [20] and [23], in this paper, we discuss the existence and uniqueness of positive solutions for problem (1.1), and then give some good properties which depend on the parameter. The method used here is a fixed point theorem of generalized concave operators in ordered Banach spaces. To the best of our knowledge, there are still very few papers devoted to the study of positive solutions for conformable integro-differential equations with integral boundary conditions. It should be pointed out that the method used here is new to the conformable boundary value problems. So it is worthwhile to investigate problem (1.1).

The rest of the paper is organized as follows. In Section 2, we recall some definitions, notations and known results. In Section 3, we provide our main results. The obtained result is well illustrated by an example in Section 4.

2. Preliminary results

Definition 2.1 (see [10, 12]). Let $\alpha \in (0, 1)$, the derivative of a function $f : [0, \infty) \rightarrow \mathbb{R}$ is defined by

$$T_\alpha f(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon},$$

is called the conformable derivative of order α . If $T_\alpha f(t)$ exists on $(0, b)$, implies $T_\alpha f(0) = \lim_{t \rightarrow 0} T_\alpha f(t)$.

Definition 2.2 (see [10, 12]). Let $\alpha \geq 1$, the derivative of a function $f : [0, \infty) \rightarrow \mathbb{R}$ is

$$T_\alpha f(t) = T_\beta f^n(t), \beta = \alpha - n,$$

where $n = [\alpha]$, $[\alpha]$ denotes the integer part of number α , is called the conformable derivative of order α .

Definition 2.3 (see [10, 12]). For $f : [0, \infty) \rightarrow \mathbb{R}$, the expression

$$I_\alpha f(t) = \frac{1}{n!} \int_0^t (t-s)^n s^{\alpha-n-1} f(s) ds,$$

where $n = [\alpha]$, is called the conformable integral of order α .

Lemma 2.1 (see [20]). Assume that $y \in C[0, 1]$, if $\beta \neq 2$ and $1 < \alpha \leq 2$, then the boundary value problem

$$\begin{cases} T_\alpha u(t) + y(t) = 0, 0 \leq t \leq 1, \\ u(0) = 0, u(1) = \beta \int_0^1 u(t) dt, \end{cases} \quad (2.1)$$

have a unique solution

$$u(t) = \int_0^1 G(t, s) y(s) ds, \quad (2.2)$$

where

$$G(t, s) = G_1(t, s) + G_2(t, s), \quad (2.3)$$

$$G_1(t, s) = \begin{cases} (1-t)s^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ ts^{\alpha-2}(1-s), & 0 < t \leq s \leq 1, \end{cases} \quad (2.4)$$

$$G_2(t, s) = \frac{2\beta t}{2-\beta} \int_0^1 G_1(\tau, s) d\tau. \quad (2.5)$$

By a simple calculus, we obtain

$$G_2(t, s) = \frac{2\beta t}{2-\beta} \int_0^1 G_1(\tau, s) d\tau$$

$$\begin{aligned}
&= \frac{2\beta t}{2-\beta} \left[\int_0^s G_1(\tau, s) d\tau + \int_s^1 G_1(\tau, s) d\tau \right] \\
&= \frac{2\beta t}{2-\beta} \left[\int_0^s \tau s^{\alpha-2} (1-s) d\tau + \int_s^1 (1-\tau) s^{\alpha-1} d\tau \right]. \\
&= \frac{\beta t s^{\alpha-1} (1-s)(s+1-s)}{2-\beta}. \\
&= \frac{\beta t s^{\alpha-1} (1-s)}{2-\beta}.
\end{aligned}$$

That is,

$$G_2(t, s) = \frac{\beta t s^{\alpha-1} (1-s)}{2-\beta}, t, s \in [0, 1]. \quad (2.6)$$

Lemma 2.2. *Let $G(t, s)$ be given in the statement of Lemma 2.1. Then we have the following properties:*

- (i) $G(t, s)$ is a continuous function on the unit square $(0, 1] \times [0, 1]$;
- (ii) $G(t, s) \geq 0$ for each $(t, s) \in (0, 1] \times [0, 1]$.

To discuss the existence of unique positive solution and properties of positive solutions for problem (1.1), we need the following abstract concepts and theorems.

Let $(E, \|\cdot\|)$ be a real Banach space, which is partially ordered by a cone $P \subset E$, i.e., $x \leq y$ if and only if $y - x \in P$. θ is the zero element of E . P is called normal if there is a constant $N > 0$ such that for all $x, y \in E$, $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$, N is called the normality constant of P . An operator $A : E \rightarrow E$ is increasing if $x \leq y$ implies $Ax \leq Ay$.

For $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leq y \leq \mu x$. We see that \sim is an equivalence relation. Given $h > \theta$ (i.e., $h \geq \theta$ and $h \neq \theta$), define $P_h := \{x \in E | x \sim h\}$. Clearly, $P_h \subset P$.

Lemma 2.3 (From Theorem 2.1 in [23]). *Let P be a normal cone in a real Banach space E , $h > \theta$, and let $A : P \rightarrow P$ be an increasing operator. In addition:*

- (i) there is $h_0 \in P_h$ such that $Ah_0 \in P_h$;
- (ii) for any $x \in P$ and $t \in (0, 1)$, there exists $\varphi(t) \in (t, 1)$ such that $A(tx) \geq \varphi(t)Ax$.

Then we have:

- (1) the operator equation $Ax = x$ has a unique solution x^* in P_h ;
- (2) for any initial value $x_0 \in P_h$, constructing successively the sequence $x_n = Ax_{n-1}$, $n = 1, 2, \dots$, we get $x_n \rightarrow x^*$ as $n \rightarrow \infty$.

Lemma 2.4 (From Theorem 2.1 in [23]). *Let all the conditions of Lemma 2.3 be satisfied and $x_\lambda (\lambda > 0)$ be the unique solution of operator equation $Ax = \lambda x$. Then the following conclusions hold:*

- (i) x_λ is strictly decreasing in λ , means that, $0 < \lambda_1 < \lambda_2$ implies $x_{\lambda_1} > x_{\lambda_2}$;
- (ii) if there exists $\gamma \in (0, 1)$ such that $\varphi(t) \geq t^\gamma$ for $t \in (0, 1)$, then x_λ is continuous in λ , that is, $\lambda \rightarrow \lambda_0 (\lambda_0 > 0)$ implies $\|x_\lambda - x_{\lambda_0}\| \rightarrow 0$;
- (iii) $\lim_{\lambda \rightarrow 0^+} \|x_\lambda\| = +\infty$, $\lim_{\lambda \rightarrow +\infty} \|x_\lambda\| = 0$.

3. Main results

For the convenience of the following discussion, we first consider a special function

$$h(t) = \int_0^1 G(t, s) ds = \int_0^1 [G_1(t, s) + G_2(t, s)] ds, t \in [0, 1], \quad (3.1)$$

where $G_1(t, s)$ and $G_2(t, s)$ are given in (2.4) and (2.6). Then we have

$$\begin{aligned} h(t) &= \int_0^t (1-t)s^{\alpha-1} ds + \int_t^1 ts^{\alpha-2}(1-s) ds + \int_0^1 \frac{\beta t(s^{\alpha-1} - s^\alpha)}{2-\beta} ds \\ &= \frac{t^\alpha(1-t)}{\alpha} + t \int_t^1 (s^{\alpha-2} - s^{\alpha-1}) ds + \frac{\beta t}{2-\beta} \int_0^1 (s^{\alpha-1} - s^\alpha) ds \\ &= \frac{t^\alpha(1-t)}{\alpha} + t \left(\frac{1-t^{\alpha-1}}{\alpha-1} - \frac{1-t^\alpha}{\alpha} \right) + \frac{\beta t}{2-\beta} \left(\frac{1}{\alpha} - \frac{1}{\alpha+1} \right) \\ &= \frac{(\alpha-1)(t^\alpha - t) + \alpha(t - t^\alpha)}{\alpha(\alpha-1)} + \frac{\beta t}{(2-\beta)\alpha(\alpha+1)} \\ &= \frac{1}{\alpha(\alpha-1)}(t - t^\alpha) + \frac{\beta}{(2-\beta)\alpha(\alpha+1)}t. \end{aligned}$$

Note that $1 < \alpha \leq 2$, $0 < \alpha - 1 \leq 1$, $\frac{3}{2} \leq \beta < 2$ and $t - t^\alpha = t(1 - t^{\alpha-1}) \geq 0$, we have $h(t) \geq 0$, $t \in [0, 1]$. Further, for all $t \in [0, 1]$, we have

$$\begin{aligned} h'(t) &= \frac{1}{\alpha(\alpha-1)}(1 - \alpha t^{\alpha-1}) + \frac{\beta}{(2-\beta)\alpha(\alpha+1)} \\ &= \frac{1}{\alpha} \left[\frac{1 - \alpha t^{\alpha-1}}{\alpha-1} + \frac{\beta}{(2-\beta)(\alpha+1)} \right] \\ &\geq \frac{1}{\alpha} \left[-1 + \frac{\beta}{(2-\beta)(\alpha+1)} \right] \\ &= \frac{2\beta - 2 - \alpha(2-\beta)}{\alpha(2-\beta)(\alpha+1)} \geq \frac{2\beta - 2 - 2(2-\beta)}{\alpha(2-\beta)(\alpha+1)} \\ &= \frac{2(2\beta - 3)}{\alpha(2-\beta)(\alpha+1)}. \end{aligned}$$

Let $\sigma_1 = (2-\beta)\alpha(\alpha+1)$, since $\frac{3}{2} \leq \beta < 2$, we have $\sigma_1 > 0$ and then

$$h'(t) = \frac{2(2\beta - 3)}{\sigma_1} \geq 0, h(t) = \frac{1}{\alpha(\alpha-1)}(t - t^\alpha) + \frac{\beta}{\sigma_1}t, t \in [0, 1]. \quad (3.2)$$

Consequently, we have the following conclusion:

Lemma 3.1. $h(t)$ is continuous, nonnegative and increasing on $[0, 1]$. Moreover,

$$0 = h(0) \leq h(t) \leq h(1) = \frac{\beta}{\sigma_1}, t \in [0, 1].$$

Lemma 3.2. Let $g(t) = I_\alpha h(t)$, then $g(t)$ is continuous, nonnegative and increasing on $t \in [0, 1]$. In addition,

$$0 = g(0) \leq g(t) \leq g(1) = \frac{\sigma_2 \sigma_3 \alpha (\alpha - 1)}{\sigma_1 (\alpha + 1)}, t \in [0, 1],$$

where $\sigma_1 = (2-\beta)\alpha(\alpha+1)$, $\sigma_2 = 6(\alpha+1) + \beta(\alpha-5)$, $\sigma_3 = \frac{1}{2\alpha^2(\alpha-1)(2\alpha-1)}$.

Proof. We get

$$\begin{aligned}
 g(t) &= I_\alpha h(t) = \int_0^t (t-s)s^{\alpha-2}h(s)ds \\
 &= \int_0^t (t-s)s^{\alpha-2} \left[\frac{1}{\alpha(\alpha-1)}(s-s^\alpha) + \frac{\beta}{(2-\beta)\alpha(\alpha+1)}s \right] ds \\
 &= \int_0^t \left[\frac{1}{\alpha(\alpha-1)}(t-s)(s^{\alpha-1}-s^{2\alpha-2}) + \frac{\beta}{(2-\beta)\alpha(\alpha+1)}(t-s)s^{\alpha-1} \right] ds \\
 &= \int_0^t \left[\frac{t}{\alpha(\alpha-1)}(s^{\alpha-1}-s^{2\alpha-2}) + \frac{\beta t}{(2-\beta)\alpha(\alpha+1)}s^{\alpha-1} \right. \\
 &\quad \left. - \frac{1}{\alpha(\alpha-1)}(s^\alpha-s^{2\alpha-1}) - \frac{\beta}{(2-\beta)\alpha(\alpha+1)}s^\alpha \right] ds \\
 &= \frac{t^{\alpha+1}}{\alpha^2(\alpha-1)} - \frac{t^{2\alpha}}{\alpha(\alpha-1)(2\alpha-1)} + \frac{\beta t^{\alpha+1}}{(2-\beta)\alpha^2(\alpha+1)} \\
 &\quad - \frac{t^{\alpha+1}}{\alpha(\alpha-1)(\alpha+1)} + \frac{t^{2\alpha}}{2\alpha^2(\alpha-1)} - \frac{\beta t^{\alpha+1}}{(2-\beta)\alpha(\alpha+1)^2} \\
 &= \frac{2(\alpha+1-\beta)}{(2-\beta)\alpha^2(\alpha+1)^2(\alpha-1)}t^{\alpha+1} - \frac{1}{2\alpha^2(\alpha-1)(2\alpha-1)}t^{2\alpha},
 \end{aligned}$$

and then

$$\begin{aligned}
 g'(t) &= \frac{2(\alpha+1-\beta)}{(2-\beta)\alpha^2(\alpha+1)(\alpha-1)}t^\alpha - \frac{1}{\alpha(\alpha-1)(2\alpha-1)}t^{2\alpha-1} \\
 &= \frac{t^\alpha}{\alpha(\alpha-1)(2\alpha-1)} \left[\frac{2(\alpha+1-\beta)(2\alpha-1)}{(2-\beta)\alpha(\alpha+1)} - t^{\alpha-1} \right] \\
 &= \frac{t^\alpha}{\alpha(\alpha-1)(2\alpha-1)} \left(\frac{\sigma_4}{\sigma_1} - t^{\alpha-1} \right),
 \end{aligned}$$

where

$$\sigma_1 = (2-\beta)\alpha(\alpha+1), \quad \sigma_4 = 2(\alpha+1-\beta)(2\alpha-1).$$

We have

$$\begin{aligned}
 \sigma_4 - \sigma_1 &= 2(\alpha+1-\beta)(2\alpha-1) - (2-\beta)\alpha(\alpha+1) \\
 &= 2\alpha^2 + 2\alpha - 3\alpha\beta + 2\beta - 2 + \alpha^2\beta - 2\alpha \\
 &= \beta(\alpha^2 - 3\alpha + 2) + 2(\alpha^2 - 1) \\
 &= (\alpha-1)[\beta(\alpha-2) + 2(\alpha+1)].
 \end{aligned}$$

Note that $1 < \alpha \leq 2$, that is $0 < \alpha - 1 \leq 1$ and $\frac{3}{2} \leq \beta < 2$, it is easy to check that $\sigma_1, \sigma_4 > 0$ and $\beta(\alpha - 2) + 2(\alpha + 1) > 0$. So, we get $\sigma_4 - \sigma_1 > 0$, $\sigma_4 > \sigma_1$, that is $\frac{\sigma_4}{\sigma_1} > 1$ and $0 \leq t^{\alpha-1} \leq 1$. Hence,

$$g'(t) = \frac{t^\alpha}{\alpha(\alpha-1)(2\alpha-1)} \left(\frac{\sigma_4}{\sigma_1} - t^{\alpha-1} \right) > 0.$$

Thus, we get $g(t)$ is an increasing function and $g(0) = 0$, that is $g(t)$ is a nonnegative function. Evidently, $g(t)$ is continuous. In addition,

$$\begin{aligned} g(1) &= I_\alpha h(1) = \frac{2(\alpha + 1 - \beta)}{(2 - \beta)\alpha^2(\alpha + 1)^2(\alpha - 1)} - \frac{1}{2\alpha^2(\alpha - 1)(2\alpha - 1)} \\ &= \frac{4(2\alpha - 1)(\alpha + 1 - \beta) - (2 - \beta)(\alpha + 1)^2}{2(2 - \beta)\alpha^2(\alpha + 1)^2(\alpha - 1)(2\alpha - 1)} \\ &= \frac{6(\alpha^2 - 1) + (\alpha^2 - 6\alpha + 5)\beta}{2(2 - \beta)\alpha^2(\alpha + 1)^2(\alpha - 1)(2\alpha - 1)} \\ &= \frac{6(\alpha + 1) + \beta(\alpha - 5)}{2(2 - \beta)\alpha^2(\alpha + 1)^2(2\alpha - 1)}. \end{aligned}$$

For convenience, let $\sigma_2 = 6(\alpha + 1) + \beta(\alpha - 5)$ and $\sigma_3 = \frac{1}{2\alpha^2(\alpha - 1)(2\alpha - 1)}$, we can easily show that $\sigma_2, \sigma_3 > 0$, and note that $\sigma_1 = (2 - \beta)\alpha(\alpha + 1)$. So we have $g(1) = \frac{\sigma_2\sigma_3\alpha(\alpha - 1)}{\sigma_1(\alpha + 1)}$. And we get $g(t) = \frac{2\sigma_3\sigma_4\alpha}{\sigma_1(\alpha + 1)}t^{\alpha+1} - \sigma_3t^{2\alpha}, t \in [0, 1]$. \square

Now we utilize the fixed point theorem of concave operators to study problem (1.1). We consider the Banach space $E = C[0, 1] = \{x : [0, 1] \rightarrow \mathbb{R} \text{ is continuous}\}$ with the standard norm $\|x\| = \sup\{|x(t)| : t \in [0, 1]\}$. Furthermore, we consider a cone $P = \{x \in E : x(t) \geq 0, t \in [0, 1]\}$. Then P is a normal cone.

Theorem 3.1. *Assume that*

(H₁) $f : [0, 1] \times [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$ is continuous with $f(t, 0, 0) \neq 0, t \in [0, 1]$;

(H₂) $f(t, u, v)$ is increasing with respect to the second and third variables for fixed $t \in [0, 1]$;

(H₃) for any $r \in (0, 1)$ and $u, v \geq 0$, there exists $\varphi(r) \in (r, 1)$ such that

$$f(t, ru, rv) \geq \varphi(r)f(t, u, v), t \in [0, 1], u, v \in [0, +\infty).$$

Then the following conclusions hold:

- (1) for any fixed $\lambda > 0$, problem (1.1) has a unique positive solution u_λ^* in P_h , where $h(t)$ is given as in (3.2). In addition, for any initial value $u_0 \in P_h$, constructing the iterative sequence

$$u_n(t) = \lambda \int_0^1 G(t, s)f(s, u_{n-1}(s), I_\alpha u_{n-1}(s))ds, \quad n = 1, 2, \dots,$$

we have $u_n(t) \rightarrow u_\lambda^*(t)$ as $n \rightarrow +\infty$, where $G(t, s)$ is given as in (2.3) and $I_\alpha u_{n-1}(s) = \int_0^s (s - \tau)\tau^{\alpha-2}u_{n-1}(\tau)d\tau$;

- (2) u_λ^* is strictly increasing in λ , namely $0 < \lambda_1 < \lambda_2$ implies $u_{\lambda_1}^* \leq u_{\lambda_2}^*$, $u_{\lambda_1}^* \neq u_{\lambda_2}^*$;
- (3) there exist $\gamma \in (0, 1)$ and a nonnegative function $\psi(t)$ such that $\varphi(t) = t^\gamma[1 + \psi(t)]$ for $t \in (0, 1)$, then u_λ^* is continuous in λ , namely, $\lambda \rightarrow \lambda_0 (\lambda_0 > 0)$ implies $\|u_\lambda^* - u_{\lambda_0}^*\| \rightarrow 0$;
- (4) $\lim_{\lambda \rightarrow 0^+} \|u_\lambda^*\| = 0$, $\lim_{\lambda \rightarrow +\infty} \|u_\lambda^*\| = +\infty$.

Proof. We define an operator $A : E \rightarrow E$ by

$$Au(t) = \int_0^1 G(t, s)f(s, u(s), I_\alpha u(s))ds,$$

where $G(t, s)$ is given as in (2.3) and $I_\alpha u(s) = \int_0^s (s - \tau)\tau^{\alpha-2}u(\tau)d\tau$.

Firstly, we show $A : P \rightarrow P$. For $u \in P$, then $u(t) \geq 0$, $t \in [0, 1]$, thus we get $I_\alpha u(t) = \int_0^t (t - s)s^{\alpha-2}u(s)ds \geq 0$, $t \in [0, 1]$. By Lemma 2.2 (i), (ii) and (H_1) , we have

$$Au(t) = \int_0^1 G(t, s)f(s, u(s), I_\alpha u(s))ds \geq 0,$$

which implies that $A : P \rightarrow P$.

By the monotonicity of f , we can easily prove that $A : P \rightarrow P$ is increasing.

In the sequel, we check that A satisfies other conditions of Lemma 2.3. Let $h(t)$ is given as in (3.2). From Lemma 3.1, it is clear that $h(t) \geq 0$ for $0 \leq t \leq 1$ and obviously, $h \in P$.

Next we mainly show that $Ah \in P_h$. Let

$$l_1 = \min_{t \in [0,1]} f(t, 0, 0), \quad l_2 = \max_{t \in [0,1]} f(t, h(1), g(1)) = \max_{t \in [0,1]} f\left(t, \frac{\beta}{\sigma_1}, \frac{\sigma_2 \sigma_3 \alpha (\alpha - 1)}{\sigma_1 (\alpha + 1)}\right).$$

From (H_1) , we know that $\min_{s \in [0,1]} f(s, 0, 0) > 0$ and thus $l_1 > 0$. Further, from (H_2) , we get $l_2 \geq l_1$. On the one hand, from Lemma 3.2 and (H_2) , we have

$$\begin{aligned} Ah(t) &= \int_0^1 G(t, s)f(s, h(s), I_\alpha h(s))ds = \int_0^1 G(t, s)f(s, h(s), g(s))ds \\ &\geq \int_0^1 G(t, s)f(s, h(0), g(0))ds = \int_0^1 G(t, s)f(s, 0, 0)ds \\ &\geq \min_{s \in [0,1]} f(s, 0, 0) \int_0^1 G(t, s)ds = l_1 h(t). \end{aligned}$$

On the other hand,

$$\begin{aligned} Ah(t) &= \int_0^1 G(t, s)f(s, h(s), I_\alpha h(s))ds = \int_0^1 G(t, s)f(s, h(s), g(s))ds \\ &\leq \int_0^1 G(t, s)f(s, h(1), g(1))ds = \int_0^1 G(t, s)f\left(s, \frac{\beta}{\sigma_1}, \frac{\sigma_2 \sigma_3 \alpha (\alpha - 1)}{\sigma_1 (\alpha + 1)}\right) ds \\ &\leq \max_{s \in [0,1]} f\left(s, \frac{\beta}{\sigma_1}, \frac{\sigma_2 \sigma_3 \alpha (\alpha - 1)}{\sigma_1 (\alpha + 1)}\right) \int_0^1 G(t, s)ds = l_2 h(t). \end{aligned}$$

Hence, $l_1 h(t) \leq Ah(t) \leq l_2 h(t)$, $t \in [0, 1]$. That is, $l_1 h \leq Ah \leq l_2 h$. So we have $Ah \in P_h$.

In the following, we indicate that the condition (ii) of Lemma 2.3 is also satisfied. For $r \in (0, 1)$, $u \in P$, from (H_3) , we have

$$A(ru)(t) = \int_0^1 G(t, s)f(s, ru(s), (I_\alpha ru)(s))ds$$

$$\geq \varphi(r) \int_0^1 G(t, s) f(s, u(s), I_\alpha u(s)) ds = \varphi(r) Au(t).$$

Therefore, $A(ru) \geq \varphi(r)Au$ for $r \in (0, 1)$ and $u \in P$. Hence A satisfies all the conditions of Lemma 2.3. Consequently, by Lemma 2.4, for $\lambda > 0$, there exists a unique $u_\lambda^* \in P_h$ such that $Au_\lambda^* = \frac{1}{\lambda}u_\lambda^*$, namely, $\lambda Au_\lambda^* = u_\lambda^*$, and then

$$u_\lambda^*(t) = \lambda \int_0^1 G(t, s) f(s, u_\lambda^*(s), I_\alpha u_\lambda^*(s)) ds.$$

From Lemma 2.1, u_λ^* is a unique positive solution of problem (1.1) for fixed $\lambda > 0$. Further, by Lemma 2.4 (i), we can claim that u_λ^* is strictly increasing in λ , that is $u_{\lambda_1}^* < u_{\lambda_2}^*$ for $0 < \lambda_1 < \lambda_2$; by Lemma 2.4(iii), one has $\lim_{\lambda \rightarrow 0^+} \|u_\lambda^*\| = 0$, $\lim_{\lambda \rightarrow +\infty} \|u_\lambda^*\| = +\infty$. Moreover, if $\varphi(t) = t^\gamma [1 + \psi(t)]$, then $\varphi(t) \geq t^\gamma$ for $t \in (0, 1)$, so Lemma 2.4(ii) implies that u_λ^* is continuous in λ , that is, $\lambda \rightarrow \lambda_0 (\lambda_0 > 0)$ implies $\|u_\lambda^* - u_{\lambda_0}^*\| \rightarrow 0$. and $u_n(t) \rightarrow u_\lambda^*(t)$ as $n \rightarrow +\infty$. \square

Corollary 3.1. *Assume that $(H_1) - (H_3)$ hold. Then the following conformable differential equation with integral boundary condition*

$$\begin{cases} T_\alpha u(t) + f(t, u(t)) = 0, t \in [0, 1], \\ u(0) = 0, u(1) = \beta \int_0^1 u(t) dt, \end{cases}$$

where $1 < \alpha \leq 2$, $\frac{3}{2} \leq \beta < 2$, has a unique positive solution u^* in P_h , where $h(t)$ is given as in (3.2). In addition, for any initial value $u_0 \in P_h$, constructing successively the sequence

$$u_n(t) = \int_0^1 G(t, s) f(s, u_{n-1}(s)) ds, \quad n = 1, 2, \dots,$$

we have $u_n(t) \rightarrow u^*(t)$ as $n \rightarrow +\infty$, where $G(t, s)$ is given as in (2.3).

Remark 3.1. If $\beta \in [0, 2)$, we can obtain the similar results to Theorem 3.1 and Corollary 3.1. In fact, noting that $h(t)$ is continuous and nonnegative on $[0, 1]$, then there exists a maximum value on $[0, 1]$ and setting $h_{\max} = \max\{h(t) : t \in [0, 1]\}$. Further, let

$$l_1 = \min_{t \in [0, 1]} f(t, 0, 0), \quad l_2 = \max_{t \in [0, 1]} f(t, h_{\max}, \frac{h_{\max}}{\alpha(\alpha - 1)}).$$

We can give the similar proof of Theorem 3.1.

4. Examples

In this section, we present an example to better illustrate our main result.

Example 4.1. Consider the following integro-differential equation boundary value problem with a parameter:

$$\begin{cases} T_{\frac{3}{2}} u(t) + \lambda(t^{\frac{3}{2}} + 1) \left\{ \frac{3}{2} + u^{\frac{1}{3}}(t) + \left[\int_0^t (t-s) s^{-\frac{1}{2}} u(s) ds \right]^{\frac{1}{4}} \right\} = 0, 0 \leq t \leq 1, \\ u(0) = 0, u(1) = \frac{7}{4} \int_0^1 u(t) dt. \end{cases} \quad (4.1)$$

Obviously, problem (4.1) fits the framework of (1.1) with $\alpha = \frac{3}{2}, \beta = \frac{7}{4}$. Let $v(t) = I_{\frac{3}{2}} u(t) = \int_0^t (t-s)s^{-\frac{1}{2}} u(s) ds$ and $f(t, u, v) = (t^{\frac{3}{2}} + 1)(\frac{3}{2} + u^{\frac{1}{3}} + v^{\frac{1}{4}})$. It is easy to see that $f : [0, 1] \times [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$ is continuous with $f(t, 0, 0) = \frac{3}{2}(t^{\frac{3}{2}} + 1) > 0, t \in [0, 1]$. Moreover, $f(t, u, v)$ is increasing in $u, v \in [0, +\infty)$ for fixed $t \in [0, 1]$. Set $\varphi(r) = r^{\frac{1}{3}}$, then $\varphi(r) \in (r, 1), r \in (0, 1)$. For $t \in (0, 1), u \geq 0$ and $v \geq 0$, we have

$$\begin{aligned} f(t, ru, rv) &= (t^{\frac{3}{2}} + 1) \left(\frac{3}{2} + r^{\frac{1}{3}} u^{\frac{1}{3}} + r^{\frac{1}{4}} v^{\frac{1}{4}} \right) \geq r^{\frac{1}{3}} (t^{\frac{3}{2}} + 1) \left(\frac{3}{2} r^{-\frac{1}{3}} + u^{\frac{1}{3}} + v^{\frac{1}{4}} \right) \\ &\geq r^{\frac{1}{3}} (t^{\frac{3}{2}} + 1) \left(\frac{3}{2} + u^{\frac{1}{3}} + v^{\frac{1}{4}} \right) = \varphi(r) f(t, u, v). \end{aligned}$$

In addition, $\varphi(t) = t^{\frac{1}{3}}[1 + \psi(t)]$, where $\psi(t) \equiv 0$. Hence, all the conditions of Theorem 3.1 are satisfied. Then the following conclusions hold:

- (1) for any fixed $\lambda > 0$, problem (2.3) has a unique positive solution u_λ^* in P_h , where $h(t) = \frac{4}{3}(t - t^{\frac{3}{2}}) + \frac{28}{15}t, t \in [0, 1]$. In addition, for any initial value $u_0 \in P_h$, constructing the iterative sequence

$$u_n(t) = \lambda \int_0^1 G(t, s) (s^{\frac{3}{2}} + 1) \left\{ \frac{3}{2} + u_{n-1}^{\frac{1}{3}}(s) + \left[\int_0^s (s-\tau)\tau^{-\frac{1}{2}} u_{n-1}(\tau) d\tau \right]^{\frac{1}{4}} \right\} ds,$$

$n = 1, 2, \dots$, then it satisfies that $u_n(t) \rightarrow u_\lambda^*(t)$, as $n \rightarrow +\infty$, where $G(t, s)$ is given as in (2.3) with $\alpha = \frac{3}{2}, \beta = \frac{7}{4}$;

- (2) u_λ^* is strictly increasing in λ , that is, $u_{\lambda_1}^* \leq u_{\lambda_2}^*, u_{\lambda_1}^* \neq u_{\lambda_2}^*$ for $0 < \lambda_1 < \lambda_2$;
 (3) u_λ^* is continuous in λ , that is $\|u_\lambda^* - u_{\lambda_0}^*\| \rightarrow 0$ as $\lambda \rightarrow \lambda_0 (\lambda_0 > 0)$;
 (4) $\lim_{\lambda \rightarrow 0^+} \|u_\lambda^*\| = 0, \lim_{\lambda \rightarrow +\infty} \|u_\lambda^*\| = +\infty$.

5. Conclusions

By using a recent fixed point theorem for concave operators, we discuss problem (1.1), a conformable fractional integro-differential equation with integral boundary conditions. For any fixed positive parameter λ , we present the existence and uniqueness of positive solutions, and give some good properties of positive solutions with respect to the parameter $\lambda > 0$. The method used here is new to conformable fractional differential equation boundary value problems and so the main results are new.

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