DYNAMICAL ANALYSIS OF A LOTKA-VOLTERRA LEARNING-PROCESS MODEL*

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Abstract A Lotka-Volterra learning-process model was proposed by Monteiro and Notargiacomo in [*Commum. Nonlinear Sci. Numer. Simulat.* **47**(2017), 416-420] to approach learning process as an interplay between understanding and doubt. They studied the stability of the boundary equilibria and gave some numerical simulations but no further discussion for bifurcations. In this paper, we study the qualitative properties of the interior equilibria and a singular line segment completely. Moreover, we discuss their bifurcations such as transcritical, pitchfork, Hopf bifurcation on isolated equilibria and transcritical bifurcation without parameters on non-isolated equilibria. Finally, we also demonstrate these analytical theory by numerical simulations.

Keywords Saddle-node, transcritical bifurcation, picthfork bifurcation, Hopf bifurcation, Singular line segment.

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1. Introduction

In applied science and engineering, many problems can be modelled by differential equations and dynamical systems, see [3,8,11] and the references therein.

As said in [11], to study educational issues, many mathematical models were founded, such as social learning using internet [1], academic performance [9], student dropout [10]. A model on the learning process was proposed by [11]

$$\begin{cases} \dot{U} = a\{U(U-1)(\alpha - U) - fUD\}\{1 - (U+D)\}, \\ \dot{D} = b\{D(\beta - D) + gUD\}\{1 - (U+D)\} \end{cases} \tag{1.1}$$

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in which $0 < \alpha < 1$ and $0 < \beta < 1$ present the minimum background required to learn about a subject and the maximum level of doubt that you can have about a subject that you did not learn about it, respectively. Notice that system (1.1) is of Lotka-Volterra type. The parameters a and b, related to the speed of the learning process, are positive. The parameters f and g describe the interaction between $U(t)$ and $D(t)$ and $fg > 0$. If doubt destroys comprehension, then $f > 0$ and $g > 0$; if doubt drives comprehension, then $f < 0$ and $g < 0$. The term $1 - (U + D)$ restricts the dynamics to the right triangle domain given by $0 \leq U \leq 1$, $0 \leq D \leq 1$, and $0 \leq U + D \leq 1$.

In [11], they discussed the stability of the boundary equilibria completely and gave some numerical simulations to illustrate these analytical results. However, the complete qualitative properties of the interior equilibria and a singular line segment and their local bifurcations of equilibria and the singular line segment are still unknown. In this work, we want to solve them completely.

This paper is organized as follows. Next section is to demonstrate all possibilities of the qualitative properties and existence of the boundary, interior equilibria and singular line segment of system (1.1). In section 3, we study the bifurcations of isolated equilibria and the singular line segement. In section 4, our results are illustrated by numerical simulations. Section 5 includes the conclusions.

2. Equilibria and Singular Line

First of all, we make a time rescaling $d\tau = adt$ to reduce system (1.1) to the form

$$
\begin{cases}
\frac{dU}{d\tau} = \{U(U-1)(\alpha - U) - fUD\}\{1 - (U+D)\}, \\[2mm]
\frac{dD}{d\tau} = r\{D(\beta - D) + gUD\}\{1 - (U+D)\},
\end{cases}
\tag{2.1}
$$

where $r = b/a > 0$. The number of parameters is reduced from 6 to 5.

In view of the physical sense, we only consider equilibria of system (2.1) in the closure $\mathcal{Q} := \{(U, D) : 0 \leq U \leq 1, \ \ 0 \leq D \leq 1 \ \ 0 \leq U + D \leq 1\}$ for all possibilities of $(\alpha, \beta, f, g, r) \in \mathcal{P} := \{0 < \alpha < 1, \ \ 0 < \beta < 1, \ \ fg > 0, \ \ r > 0\}$. We will partition the space $\mathcal{P}$ of the parameters for various cases of equilibria.

Theorem 2.1. *Equilibria of system* (2.1) *have the following qualitative properties:*

(i) *The origin $O : (0,0)$ is a saddle.*

(ii) *$E_1 : (0, \beta)$ is either a stable node if $\alpha + \beta f > 0$ or a saddle if $\alpha + \beta f < 0$. When $\alpha + \beta f = 0$, E_1 is degenerate.*

(iii) *$E_2 : (\alpha, 0)$ is either a saddle if $\alpha g + \beta < 0$ or an unstbale node if $\alpha g + \beta > 0$. When $\alpha g + \beta = 0$, E_2 is degenerate.*

(iv) *For $\Delta > 0$ and either $\max\{m_1, m_2, 0\} < f < (1+\alpha)/g, g > 0$ or $\max\{-\alpha/\beta, (\alpha+1)/g < f < \min\{0, m_3\}$, $-\beta/g < \alpha < 1$, $g < -\beta$, $0 < \beta < 1$, exactly two interior equilibria $E_{\pm} : (U_{\pm}, gU_{\pm} + \beta)$ exist, $U_{\pm} := \{\alpha + 1 - fg \pm \sqrt{\Delta}\}/2$, $\Delta := (\alpha + 1 - fg)^2 - 4(\alpha + f\beta)$, $m_1 := (1-\beta)/(g+1) - \alpha$, $m_2 := \alpha/g + (2\beta + g - 1)/g(g+1)$ and $m_3 := \alpha/g + (2\beta + g)/g^2$. For $-\beta/g < \alpha < \min\{1, -\beta f\}$, $g < -\beta$, $0 < \beta < 1$ and $f < 0$, only one interior equilibrium E_+ exists. Only one interior equilibrium E_- exists if and only if one of*

the following conditions satisfies:

(N1) $0 < f < m_1$, $g > 0$, $0 < \alpha, \beta < 1$;

(N2) $f = m_1$, $g > 0$, $0 < \alpha < (1 - 2\beta - g\beta)/(g + 1)^2$, $0 < \beta < 1/(g + 2)$;

(N3) $-\alpha/\beta < f < \min\{0, m_1\}$, $-\beta < g < 0$;

(N4) $-\alpha/\beta < f < 0$, $0 < \alpha < -\beta/g$, $g \leq -\beta$ *and* $0 < \beta < 1$.

Then, E_- is a saddle and E_+ is either a stable node or focus if and only if $r > (fg - \sqrt{\Delta})U_+/(gU_+ + \beta)$ or is an unstable node of focus if and only if $r < (fg - \sqrt{\Delta})U_+/(gU_+ + \beta)$. When $r = (fg - \sqrt{\Delta})U_+/(gU_+ + \beta)$, E_+ is of center-focus type.

(v)   *The singular line segment*

$$\mathcal{L} := \{(U, D) : U + D = 1, 0 \leq U \leq 1, \ \ 0 \leq D \leq 1\}.$$

Proof.   Equilibria of system (2.1) are determined by the polynomial system

$$\begin{cases} \{U(U - 1)(\alpha - U) - fUD\}\{1 - (U + D)\} = 0, \\ r\{D(\beta - D) + gUD\}\{1 - (U + D)\} = 0. \end{cases}$$

We have a singular line segment

$$\mathcal{L} := \{(U, D) : U + D = 1, 0 \leq U \leq 1, \ \ 0 \leq D \leq 1\},$$

on which infinite equilibria exist. Obviously, $O : (0, 0)$, $E_1 : (0, \beta)$ and $E_2 : (\alpha, 0)$ are boundary equilibria, i.e., equilibria on the boundary $\partial \mathcal{Q}$.

Interior equilibria of system (2.1) are $E_\pm : (U_\pm, D_\pm)$, where

$$U_\pm := \frac{1}{2}\{\alpha + 1 - fg \pm \sqrt{(\alpha + 1 - fg)^2 - 4(\alpha + f\beta)}\},$$
$$D_\pm := gU_\pm + \beta,$$

which are determined by the polynomial system

$$\begin{cases} (U - 1)(\alpha - U) - fD = 0, \\ (\beta - D) + gU = 0. \end{cases} \tag{2.2}$$

Considering whether $E_\pm$ lie in the closure $\mathcal{Q}$, we solve the inequalities $U_\pm > 0, D_\pm > 0$ and $U_\pm + D_\pm < 1$. Then we consider the following two cases: (C1) $f > 0$, $g > 0$ and (C2) $f < 0$, $g < 0$. Substituting $D = gU + \beta$ into the first equation (2.2), we get

$$F(U) = U^2 + (fg - \alpha - 1)U + f\beta + \alpha.$$

In the case (C1), from the inequalities we have $0 < U_\pm < (1 - \beta)/(1 + g)$. Since $F(0) > 0$ and $F(1) > 0$, system (2.1) has two isolated interior equilibria $E_\pm$ lie in the closure $\mathcal{Q}$ if and only if $\Delta := (\alpha + 1 - fg)^2 - 4(\alpha + f\beta) > 0$, $0 < (\alpha + 1 - fg)/2 < (1 - \beta)/(1 + g)$ and $F((1 - \beta)/(1 + g)) > 0$; system (2.1) has only interior equilibrium E_- lies in the closure $\mathcal{Q}$ if and only if either $\Delta \geq 0$, $0 < \alpha + 1 - fg < 2(1 - \beta)/(1 + g) < 2$ and $F((1 - \beta)/(1 + g)) < 0$ or $\Delta \geq 0$, $2(1 - \beta)/(1 + g) \leq \alpha + 1 - fg < 2$ and $F((1 - \beta)/(1 + g)) \leq 0$; otherwise no equilibrium lies in the closure $\mathcal{Q}$. Thus, in this case system (2.1) has two interior $E_\pm$ when $\Delta > 0$ and $\max\{0, m_1, m_2\} < f <$

$(1+\alpha)/g$ and has only one interior E_- when either $0 < f < m_1$, $g > 0$, $0 < \alpha, \beta < 1$ or $f = m_1$, $g > 0$, $0 < \alpha < (1 - 2\beta - g\beta)/(g + 1)^2$, $0 < \beta < 1/(g + 2)$, where m_1 and m_2 are given in Theorem 2.1. In the case (C2), from the inequalities we have $0 < U_\pm < -\beta/g$, $(1 + g)U_\pm < 1 - \beta$. Hence, we consider the following 3 subcases: (C21) $g < -\beta$, (C22) $g = -\beta$ and (C23) $-\beta < g < 0$. In subcase (C21), we need consider the real roots of $F(U)$ when $0 < U < -\beta/g$. we need to check $F(U)$ at the endpoints $U = 0$ and $U = -\beta/g$ are

$$F(0) = \alpha + f\beta, \quad F(-\beta/g) = (g + \beta)(\alpha g + \beta)/g^2$$

respectively. Then, we obtain that in the closure $\mathcal{Q}$ system (2.1) has only one interior equilibrium E_- if and only if $-\beta f < \alpha < -\beta/g$, $g < -\beta$, $f < 0$ and $0 < \beta < 1$; only one interior equilibrium E_+ if and only if $-\beta/g < \alpha < -\beta f$, $g < -\beta$, $f < 0$ and $0 < \beta < 1$; exactly two interior equilibria $E_\pm$ if and only if $\Delta > 0$, $\max\{-\alpha/\beta, (\alpha + 1)/g\} < f < \min\{0, m_3\}$, $-\beta/g < \alpha < 1$, $g < -\beta$ and $0 < \beta < 1$. In subcase (C22), $F(U) = 0$ has two real roots 1 and $\alpha + \beta f$. Thus, system (2.1) has only interior equilibrium E_- lies in the closure $\mathcal{Q}$ if and only if $0 < \alpha + \beta f < 1$, $g = -\beta$ and $f < 0$. In subcase (C23), similar analysis to (C1), we discuss the real roots of $F(U)$ when $0 < U < (1 - \beta)/(1 + g)$. Then, system (2.1) has only interior equilibrium E_- if and only if $-\alpha/\beta < f < m_1$, $-\beta < g < 0$ and $f < 0$. Summarily, the above discussion shows that in the closure $\mathcal{Q}$ system (2.1) has exactly two interior equilibria $E_\pm$ if and only if $\Delta > 0$ and either $\max\{m_1, m_2, 0\} < f < (1 + \alpha)/g, g > 0$ or

$$\max\{-\alpha/\beta, (\alpha + 1)/g < f < \min\{0, m_3\},$$
$$-\beta/g < \alpha < 1, g < -\beta, 0 < \beta < 1;$$

only one interior equilibrium E_+ if and only if $-\beta/g < \alpha < \min\{1, -\beta f\}$, $g < -\beta$, $0 < \beta < 1$ and $f < 0$; only one interior equilibrium E_- if and only if one of the following conditions satisfies: (N1) $0 < f < m_1$, $g > 0$, $0 < \alpha, \beta < 1$; (N2) $f = m_1$, $g > 0$, $0 < \alpha < (1 - 2\beta - g\beta)/(g + 1)^2$, $0 < \beta < 1/(g + 2)$; (N3) $-\alpha/\beta < f < \min\{0, m_1\}$, $-\beta < g < 0$; (N4) $-\alpha/\beta < f < 0$, $0 < \alpha < -\beta/g$, $g \leq -\beta$ and $0 < \beta < 1$.

Compute the Jacobian matrix of the vector field (2.1)

$$A := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where

$a_{11} := 4U^3 + (-6 - 3\alpha + 3D)U^2 + 2((f - 1 - \alpha)D + 2\alpha + 1)U + fD^2 + (\alpha - f)D - \alpha,$

$a_{12} := U^3 + (f - 1 - \alpha)U^2 + (2fD + \alpha - f)U,$

$a_{21} := -2rgUD - r(g - 1)D^2 - r(\beta - g)D,$

$a_{22} := -rgU^2 + (-2r(g - 1)D - r(\beta - g))U + 3rD^2 - 2r(\beta + 1)D + r\beta.$

System (2.1) at O has eigenvalues $-\alpha$ and $r\beta$. Then the origin O is a saddle. At E_1, system (2.1) has eigenvalues $r\beta(\beta - 1) < 0$ and $(-1 + \beta)(\beta f + \alpha)$. It is easy to see that E_1 is either a stable node if $\alpha + \beta f > 0$ or a saddle if $\alpha + \beta f < 0$. When $\alpha + \beta f = 0$, E_1 is degenerate. At E_2, system (2.1) has eigenvalues $\alpha(\alpha - 1)^2 > 0$

and $r(1-\alpha)(\alpha g + \beta)$. It is easy to see that E_2 is either a saddle if $\alpha g + \beta < 0$ or an unstbale node if $\alpha g + \beta > 0$. When $\alpha g + \beta = 0$, E_2 is degenerate.

The characteristic polynomial at E_+ is

$$\Phi_+(\lambda) := \lambda^2 + \Lambda_1^+ \lambda + \Lambda_0^+,$$

and at E_- is

$$\Phi_-(\lambda) := \lambda^2 + \Lambda_1^- \lambda + \Lambda_0^-$$

respectively, where

$$\Lambda_1^+ := (1 - U_+ - D_+)\{(\sqrt{\Delta} - fg)U_+ + rD_+\},$$
$$\Lambda_0^+ := rU_+ D_+(1 - U_+ - D_+)^2\sqrt{\Delta} > 0,$$
$$\Lambda_1^- := (1 - U_- - D_-)\{-(\sqrt{\Delta} + fg)U_- + rD_-\},$$
$$\Lambda_0^- := -rU_- D_-(1 - U_- - D_-)^2\sqrt{\Delta} < 0.$$

It is clear that E_- is a saddle. Moreover, we obtain that E_+ is a stable node or a stable focus if and only if $r > (fg - \sqrt{\Delta})U_+/D_+$ and is an unstable node or an unstable focus if and only if $r < (fg - \sqrt{\Delta})U_+/D_+$. When $r = (fg - \sqrt{\Delta})U_+/D_+$, i.e., $\Lambda_1^+ = 0$, there is a pair of purely imaginary conjugate eigenvalues at E_+. $\qquad\square$

Those nonhyperbolic cases mentioned in Theorem 2.1 need a further discussion for their qualitative properties and bifurcations: E_1 is degenerate for $\alpha + \beta f = 0$; E_2 is degenerate for $\alpha g + \beta = 0$, and will be discussed in Section 3.1; E_+ is of center-focus type for $r = (fg - \sqrt{\Delta})U_+/D_+$ and will be discussed in Section 3.2. Bifurcations on singular line segment $\mathcal{L}$ will be discussed in Section 3.3.

3. Bifurcations

3.1. Bifurcation of boundary equilibria

Theorem 2.1 shows that system (2.1) has degenerate equilibria E_1 and E_2 if $\alpha + \beta f = 0$ and $\alpha g + \beta = 0$, respectively. For simplicity, let

$$\mu_1 := f + \alpha/\beta, \qquad \mu_2 := g + \beta/\alpha.$$

When $\mu_1 = 0$, i.e., $f = -\alpha/\beta$, the Jacobian matrix at E_1 has eigenvalues 0 and $r\beta(\beta - 1) < 0$. When $\mu_2 = 0$, i.e., $f = -\beta/\alpha$, the Jacobian matrix at E_2 has eigenvalues 0 and $\alpha(\alpha - 1)^2 > 0$.

Theorem 3.1. (i) *For $f = -\alpha/\beta$, E_1 is a saddle-node of system (2.1). Moreover, in the case of $g \neq -\beta(\alpha + 1)/\alpha$, as f crosses $-\alpha/\beta$, a transcritical bifurcation happens at E_1 such that a saddle E_1 changes into a stable node E_1 and a saddle E_-. In the case of $g = -\beta(\alpha + 1)/\alpha$, as f crosses $-\alpha/\beta$, a pitchfork bifurcation happens at E_1 such that a saddle E_1 and E_+ changes into a stable node at E_1.*
(ii) *For $g = -\beta/\alpha$, E_2 is a saddle-node of system (2.1). Moreover, in the case of $f \neq \alpha(\alpha - 1)/\beta$, as g crosses $-\beta/\alpha$, a transcritical bifurcation happens at E_2 such that a saddle E_2 change into an unstable node E_2. In the case of $f = \alpha(\alpha - 1)/\beta$, as g crosses $-\beta/\alpha$, a pitchfork bifurcation happens at E_2 such that a saddle E_2 and E_- change into an unstable node E_2.*

Proof. For sufficiently small $|\mu_1|$ and $|\mu_2|$, by translating E_1 and E_2 to the origin O and diagonalizing the linear part of system (2.1) in the case that $\mu_1 = 0$ and $\mu_2 = 0$, we can transform system (2.1) into form

$$
\begin{cases}
\frac{dx}{d\tau_1} = -x + g_1(x,y), \\[2mm]
\frac{dy}{d\tau_1} = \frac{\mu_1}{r}y + g_2(x,y)
\end{cases}
\tag{3.1}
$$

and

$$
\begin{cases}
\frac{dx}{d\tau_2} = x + \tilde{g}_1(x,y), \\[2mm]
\frac{dy}{d\tau_2} = \frac{r\mu_2}{1-\alpha}y + \tilde{g}_2(x,y)
\end{cases}
\tag{3.2}
$$

respectively, where functions $g_i(x,y)$, $\tilde{g}_i(x,y)(i=1,2)$ are given in Appendix, $d\tau_1 = r\beta(1-\beta)dt$ and $d\tau_2 = \alpha(\alpha-1)^2 dt$. Suspended with the parameters μ_1 and μ_2 respectively, systems (3.1) and (3.2) can be regarded as 3-dimensional ones. The center manifold theory (see [2]) shows that the suspended systems of (3.1) and (3.2) have smooth 2-dimensional center manifolds

$$
\mathcal{W}_{\mu_1}^c = \{(x,y,\mu_1) \,|\, x = h_1(y,\mu_1), \ h_1(0,0) = 0, \ Dh_1(0,0) = 0\}
$$

and

$$
\mathcal{W}_{\mu_2}^c = \{(x,y,\mu_2) \,|\, x = h_2(y,\mu_2), \ h_2(0,0) = 0, \ Dh_2(0,0) = 0\}
$$

near the origin respectively, the smooth functions h_1 and h_2 can be approximated as $h_1(y,\mu_1) := \phi_{21}(y,\mu_1) + O(\|(y,\mu_1)\|^3)$ and $h_2(y,\mu_2) := \phi_{22}(y,\mu_2) + O(\|(y,\mu_2)\|^3)$, where the second order approximations ϕ_{21} and ϕ_{22}, by Theorem 3 in [2], satisfies

$$
(M\phi_{21})(y,\mu_1) := \frac{\partial\phi_{21}}{\partial y}\left(\frac{\mu_1}{r}y + g_2(x,y)\right) + x - g_1(x,y) = O(\|(y,\mu_1)\|^3) \tag{3.3}
$$

and

$$
(M\phi_{22})(y,\mu_2) := \frac{\partial\phi_{22}}{\partial y}\left(\frac{r\mu_2}{1-\alpha}y + \tilde{g}_2(x,y)\right) - x - \tilde{g}_1(x,y) = O(\|(y,\mu_1)\|^3). \tag{3.4}
$$

Comparing the coefficients in (3.3) and (3.4), we obtain

$$
\phi_{21}(y,\mu_1) = -\frac{\alpha\beta + \alpha g + \beta}{\beta^2 rg}y^2, \quad \phi_{22}(y,\mu_2) = -\frac{f(\alpha^2 f + \alpha^2 r - \beta fr - \alpha r)}{\alpha^2(\alpha-1)^3}y^2.
$$

Thus we obtain the restricted equation of systems (3.1) and (3.2) to the center manifold $\mathcal{W}_{\mu_1}^c$ and $\mathcal{W}_{\mu_1}^c$ respectively i.e.,

$$
\frac{dy}{d\tau_1} = -\frac{\mu_1}{r}y + G_2(\mu_1)y^2 + G_3(\mu_1)y^3 + O(|y,\mu_1|^4), \tag{3.5}
$$

and

$$
\frac{dy}{d\tau_2} = \frac{r\mu_2}{1-\alpha}y + \tilde{G}_2(\mu_2)y^2 + \tilde{G}_3(\mu_2)y^3 + O(|y,\mu_2|^4), \tag{3.6}
$$

where

$$G_2(\mu_1) = \frac{\alpha\beta + \alpha g + \beta}{\beta^2 rg} + O(|\mu_1|),$$

$$G_3(\mu_1) = \frac{1}{r^2 g^2 \beta^4 (-1+\beta)}\{\beta^2(\alpha\beta g + \alpha g^2 + \alpha\beta + \alpha g - \beta^2 + \beta g + 2\beta)r - \alpha g(-1 + \beta)$$
$$(\alpha\beta + \alpha g + \beta)\} + O(|\mu_1|),$$

$$\tilde{G}_2(\mu_2) = \frac{r(\alpha(\alpha - 1) - \beta f)}{\alpha^2(\alpha - 1)^2} + O(|\mu_2|),$$

$$\tilde{G}_3(\mu_2) = \frac{1}{\alpha^4(\alpha - 1)^4}\{r^2(-\beta f + \alpha^2(\alpha - 1))(\alpha(\alpha-1)-\beta f) - rf\alpha^3(\alpha-1)\} + O(|\mu_2|).$$

When $g \neq -\beta(\alpha + 1)/\alpha$, for $\mu_1 = 0$ it shows that $G_2(0) \neq 0$ in (3.5) and the origin O is the unique equilibrium and that the other equilibrium arises from O as $\mu_1 \neq 0$. Moreover, the stabilities of the two equilibria exchange as μ_1 varies from negative to positive. Therefore, E_1 is a saddle-node at $\mu_1 = 0$ and system (2.1) undergoes a transcritical bifurcation at E_1 for $g \neq -\beta(\alpha + 1)/\alpha$. When $g = -\beta(\alpha + 1)/\alpha$, for $\mu_1 = 0$ it shows that $G_3(0) = -\alpha^2/r\beta^3(1 + \alpha)^2 < 0$ in (3.5). Then, the origin O is the unique equilibrium and that two equilibria arise from 0 as μ_1 varies from 0 to negative. Therefore, E_1 is a saddle at $\mu_1 = 0$ and system (2.1) undergoes a pitchfork bifurcation at E_1 for $g = -\beta(\alpha + 1)/\alpha$.

When $f \neq (\alpha - 1)\alpha/\beta$, for $\mu_2 = 0$ it shows that $\tilde{G}_2(0) \neq 0$ in (3.6) and the origin O is the unique equilibrium and that the other equilibrium arises from O as $\mu_2 \neq 0$. Moreover, the stabilities of the two equilibria exchange as μ_2 varies from negative to positive. Therefore, E_2 is a saddle-node at $\mu_2 = 0$ and system (2.1) undergoes a transcritical bifurcation at E_2 for $f \neq (\alpha - 1)\alpha/\beta$. When $f = (\alpha - 1)\alpha/\beta$, for $\mu_2 = 0$ it shows that $\tilde{G}_3(0) = -r/\beta(\alpha - 1)^2 < 0$ in (3.6). Then, the origin O is the unique equilibrium and that two equilibria arise from 0 as μ_2 varies from 0 to positive. Therefore, E_2 is a saddle at $\mu_2 = 0$ and system (2.1) undergoes a pitchfork bifurcation at E_2 for $f = (\alpha - 1)\alpha/\beta$. $\square$

Remark 3.1. Since we focus on U, V in $\mathcal{Q}$, there exist some different phenomena from classical transcritical and pitchfork bifurcations. Even we did not see two more equilibria lie in $\mathcal{Q}$ arise from the pitchfork bifurcation near E_1 and E_2, the bifurcation occurs when either E_1 or E_2 loses the stability.

3.2. Bifurcation of interior equilibrium

Next, we discuss the local bifurcation near E_+. Theorem 2.1 shows that system (2.1) has a center-focus equilibrium E_+ if $r = (fg - \sqrt{\Delta})U_+/D_+$. Consider parameters in the set

$$\mathcal{H} := \mathcal{H}_1 \cup \mathcal{H}_2, \tag{3.7}$$

where

$$\mathcal{H}_1 := \{(\alpha, \beta, f, g) : 0 < \alpha, \beta < 1, g > 0, 0 < \Delta < f^2 g^2, \max\{m_1, m_2, 0\} < f < (1+\alpha)/g,\},$$
$$\mathcal{H}_2 := \{(\alpha, \beta, f, g) : 0 < \beta < 1, g < -\beta, -\beta/g < \alpha < 1, 0 < \Delta < f^2 g^2,$$
$$\max\{-\alpha/\beta, (\alpha + 1)/g < f < \min\{0, m_3\}\}$$
$$\cup\{(\alpha, \beta, f, g) : f < 0, 0 < \beta < 1, g < -\beta, -\beta/g < \alpha < \min\{1, -\beta f\},\}.$$

Let

$$\mu_0 := r - \frac{(fg - \sqrt{\Delta})U_+}{D_+} \tag{3.8}$$

for r near $(fg - \sqrt{\Delta})U_+/D_+$ and regard μ_0 as the perturbation parameter. When $\mu_0 = 0$, i.e. $r = (fg - \sqrt{\Delta})U_+/D_+$, the linearization of system (2.1) at E_+ has eigenvalues

$$\pm U_+(1 - U_+ - D_+)\sqrt{fg\sqrt{\Delta} - \Delta}\ \mathbf{i}.$$

For sufficiently small $|\mu_0|$, the linearization of (2.1) at E_+ has a pair of conjugate complex eigenvalues $\lambda_{1,2} = \sigma(\mu_0) \pm \mathbf{i}\,\omega(\mu_0)$ such that

$$\sigma(0) = 0,\ \omega(0) = U_+(1 - U_+ - D_+)\sqrt{fg\sqrt{\Delta} - \Delta},$$

$$\frac{d\sigma}{d\mu_0}\Big|_{\mu_0=0} = \sqrt{\Delta}U_+D_+(1 - U_+ - D_+) \neq 0.$$

Then, for sufficiently small $|\mu_0|$, translating E_+ the origin and applying the linear transformation

$$U = \{2fg\big(D_+\mu_0 - (\sqrt{\Delta} - fg)U_+\big)\}^{-1}(K_1(\mu_0)x_1 + K_2(\mu_0 x_2),\quad D = x_2,$$

and the time rescaling $d\tau := -U_+(1 - U_+ - D_+)K_1 dt$, we can transform system (2.1) into the form

$$\begin{cases} x_1' = \epsilon(\mu_0)x_1 - x_2 + F_1(x_1, x_2, \mu_0), \\ x_2' = \ \ x_1 + \epsilon(\mu_0)x_2 + F_2(x_1, x_2, \mu_0), \end{cases} \tag{3.9}$$

where the linear part is standardized,

$$\epsilon(\mu_0) := \{2U_+K_1(\mu_0)\}^{-1}fD_+\mu_0,$$
$$K_1(\mu_0) := \{-f^2\big(D_+^2\mu_0^2 - 4\sqrt{\Delta}U_+D_+\mu_0 + 4\sqrt{\Delta}U_+^2(\sqrt{\Delta} - fg)\big)\}^{1/2},$$
$$K_2(\mu_0) := D_+\mu_0 - 2(\sqrt{\Delta} - fg)U_+$$

and $F_i(x_1, x_2, \mu_0)(i{=}1{,}2)$ are given in the Appendix.

Our second task is to compute a normal form for system (3.9). Let $z = x_1 + \mathbf{i}x_2$. Then (3.9) can be represented as the complex form

$$\dot{z} = (\epsilon(\mu_0) + \mathbf{i})z + F_1(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2\mathbf{i}}, \mu_0) + \mathbf{i}F_2(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2\mathbf{i}}, \mu_0). \tag{3.10}$$

Applying the near-identity transformation

$$z = w + \sum_{r+j=n} p_{rj}(\mu_0)\, w^r \overline{w}^j,$$

for $n = 2$ and $n = 3$ separately, where

$$p_{rj} = \begin{cases} 0, & r = j + 1, \\ ((r + j - 1)\tau + \mathbf{i}(r - j - 1))^{-1}g_{rj}, & r \neq j + 1, \end{cases}$$

and

$$g_{rj} = (\partial^{r+j}/\partial z^r \partial \bar{z}^j)\varphi(0,0,\epsilon)/(r!j!), \ \ r,j = 1,2,3\cdots$$

as done in [6], we normalize the second degree terms and the third degree terms separately and reduce system (3.10) to the normal form

$$\dot{w} = (\epsilon(\mu_0) + \mathbf{i})w + C_1(\mu_0)w^2\bar{w} + o(\|(w,\bar{w})\|^3). \tag{3.11}$$

Thus, the first Lyapunov quantity is given by

$$L_1|_{E_+} = \text{Re } C_1(0) = -\frac{f(\alpha\beta + 2\alpha g + \beta)}{8gU_+D_+(fg\sqrt{\Delta} - \Delta)^{3/2}}. \tag{3.12}$$

By the classical Hopf bifurcation theorem [5], we obtain the following results.

Theorem 3.2. *If* $r = (fg - \sqrt{\Delta})U_+/D_+$ *and* $(\alpha,\beta,f,g) \in \mathcal{H}_1(\mathcal{H}_2)$ *then the equilibrium* E_+ *of system* (2.1) *is a locally unstable(stable) weak focus of multiplicity 1 respectively, where* $\mathcal{H}_i(i=1,2)$ *are defined below* (3.7). *For sufficiently small* $|\mu_0|$*(defined in* (3.8)*) system* (2.1) *undergoes a Hopf bifurcation at* E_+ *as* μ_0 *passes* 0. *Moreover, there is a unique unstable(stable) limit cycle when* $\mu_0 > 0 (\mu_0 < 0)$ *but no limit cycle when* $\mu_0 \leq 0 (\mu_0 \geq 0)$ *as* $(\alpha,\beta,f,g) \in \mathcal{H}_1(\mathcal{H}_2)$ *respectively.*

Proof. When $(\alpha,\beta,f,g) \in \mathcal{H}_1$, it is easy to check that $L_1|_{E_+} < 0$, which implies that O is a locally stable weak focus of multiplicity 1 of system (3.11). When $(\alpha,\beta,f,g) \in \mathcal{H}_2$, we have $-\beta/g < \alpha$. Then, $\alpha\beta + 2\alpha g + \beta < \beta(\alpha - 1) < 0$. Thus, $L_1|_{E_+} > 0$ which implies that O is a locally unstable weak focus of multiplicity 1 of system (3.11). On the basis of the above results we can discuss the Hopf bifurcation for (2.1) at E_+. Notice that the time rescaling

$$d\tau := -U_+(1 - U_+ - D_+)K_1 dt, \quad -U_+(1 - U_+ - D_+) < 0$$

in system (3.9) changes the direction of the orbits. The results of Theorem 3.2 can be obtained by the classical Hopf bifurcation theorem and the proof is completed. $\square$

3.3. Bifurcations on singular line segment

Consider the bifurcations on singular line segment $\mathcal{L}$. The characteristic polynomial at equilibrium $E_{x_0} : (x_0, 1 - x_0)$ on singular line segment $\mathcal{L}$ is

$$\Phi(\lambda)|_{E_{x_0}} := \lambda\{\lambda + (1 - x_0)(x_0^2 + (gr - \alpha - f + r)x_0 + (\beta - 1)r)\}.$$

Then we consider the following two cases: **(S1)** $\alpha > \alpha_0 := \beta r + gr - f + 1$ and **(S2)** $\alpha \leq \alpha_0$. In case **(S1)**, the equilibria on singular line segment $\mathcal{L}$ except the point $(1,0)$ possess one dimensional center manifold and one dimensional unstable manifold and the equilibrium $(1,0)$ possesses two dimensional center manifold. In case **(S2)**, the equilibria on singular line segment

$$\mathcal{L}_+ := \{(U,D) : U + D = 1, 0 \leq U < x_+, \ \ 0 \leq D \leq 1\}$$

possess one dimensional center manifold and one dimensional unstable manifold, where $x_+ := \{\alpha - rg + f - r + (\alpha^2 + 2(f - gr - r)\alpha + g^2r^2 - 2fgr + 2gr^2 - 4\beta r + f^2 -$
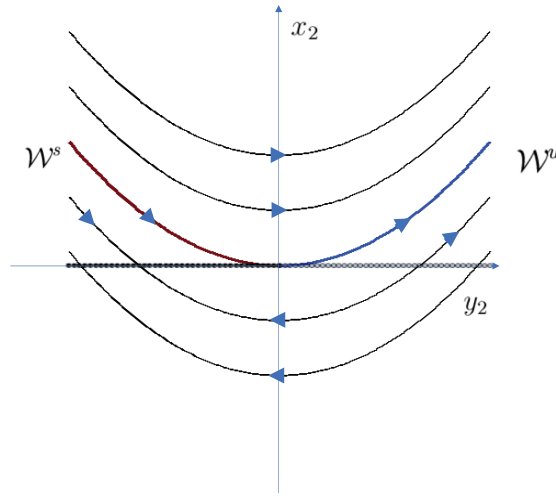
Figure 1. Bifurcation diagram of system (3.14). Stable manifold $\mathcal{W}^s$ of the origin in red, unstable manifold $\mathcal{W}^u$ in blue.

$2fr + r^2 + 4r\big)^{1/2}\}/2$. The equilibria $(1,0)$ and $(x_+, 1 - x_+)$ possess two dimensional center manifold. The equilibria on singular line segment

$$\mathcal{L}_- := \{(U, D) : U + D = 1, x_+ < U < 1, \ \ 0 \leq D \leq 1\}$$

possess one dimensional center manifold and one dimensional stable manifold.

Theorem 3.3. *The singular line segment $\mathcal{L}$ is a center submanifold of system* (2.1). *If $\alpha \neq \alpha_0$, system* (2.1) *undergoes the transcritical bifurcation without parameters. Then, system* (2.1) *is locally orbitally $\mathcal{C}^1$-equivalent to the normal form*

$$\dot{x} = xy, \quad \dot{y} = x,$$

near the singular line segment $\mathcal{L}$.

Proof.   It is easy to see that singular line segment $\mathcal{L}$ is an invariant curve. Computing the gradient of the function $U + D = 1$ at E_{x_0}, we obtain $(1, 1)$, which is a normal vector of singular line segment $\mathcal{L}$. On the other hand, the tangent space of center manifolds at E_{x_0} is spanned by the eigenvector $(-1, 1)$ of the linear part of system (2.1) at E_{x_0} corresponding to zero eigenvalue. Therefore, the line segment $\mathcal{L}$ is a center submanifold of system (2.1).

With the change of variables $x_2 := 1 - (U + D)$, $y_2 = D + 1 - x_0$, system (2.1) is changed into

$$\begin{cases} \dot{x}_2 = x_2\big(a_{10} + a_{11}x_2 + a_{12}y_2 + f_2(x_2, y_2)\big), \\ \dot{y}_2 = x_2\big(a_{20} + a_{21}x_2 + a_{22}y_2 + f_2(x_2, y_2)\big), \end{cases} \tag{3.13}$$

where

$$a_{10} = (1 - x_0)\{x_0^2 + (\alpha - gr + f - r - 4)x_0 + r\beta + 2rg - 2\alpha - 2f + r + 4\},$$
$$a_{11} = -3x_0^2 + (gr - 2\alpha - f + 10)x_0 - rg + 3\alpha + f - 8,$$

$$a_{12} = -3x_0^2 + (2gr - 2\alpha - 2f + 2r + 10)x_0 - r\beta - 3rg + 3\alpha + 3f - 2r - 8,$$
$$a_{20} = -r(1 - x_0)\{\beta + 2g + 1 - (1 + g)x_0\},$$
$$a_{21} = rg(1 - x_0),$$
$$a_{22} = r\{\beta + 3g + 2 - 2(1 + g)x_0\},$$
$$f_1(x_2, y_2) = (5 - 3x_0 - \alpha)x_2^2 + (gr - \alpha - f + r - 3x_0 + 5)y_2^2 - x_2^3 - 3y_2x_2^2 - 3x_2y_2^2 - y_2^3,$$
$$f_2(x_2, y_2) = -rgx_2y_2 - r(1 + g)y_2^2.$$

Then, system (3.13) has the same orbits as the rescaled system

$$\begin{cases} x_2' = a_{10} + a_{11}x_2 + a_{12}y_2 + f_2(x_2, y_2), \\ y_2' = a_{20} + a_{21}x_2 + a_{22}y_2 + f_2(x_2, y_2) \end{cases}$$

for $x_2 > 0$ but reverses their direction for $x_2 < 0$. If either $a_{12}a_{10} \neq 0$, $(a_{21}a_{12} - a_{11}a_{22}) \neq 0$ or $a_{12}a_{10} \neq 0$, $(a_{21}a_{12} - a_{11}a_{22}) > 0$ and $a_{11} + a_{22} \neq 0$, the equilibrium $(0,0)$ of system (3.13) is hyperbolic, i.e. has no purely imaginary eigenvalues. We choose

$$x_0 = \frac{1}{2}(rg - \alpha - f + r + 4) + \frac{1}{2}\{g^2r^2 - 2\alpha gr - 2fgr + 2gr^2$$
$$+ \alpha^2 + 2\alpha f - 2\alpha r - 4r\beta + f^2 - 2fr + r^2 + 4r\}^{1/2},$$

then $a_{10} = 0$, $a_{20} \neq 0$. From $a_{12} \neq 0$ we get $\alpha \neq \alpha_0$, which is the transversal condition. By the the rectification lemma or flow-box theorem (e.g. in Lemma 1.120 of [4]), we can locally transform system (3.13) into $x_2' = 0$, $y_2' = 1$. Then, on the center manifold there exists a local $\mathcal{C}^1$-diffeomorphism which (locally) maps orbits of the vector field (3.13) to orbits of the normal form

$$x_2' = x_2y_2, \qquad y_2' = x_2. \tag{3.14}$$

Thus, for $\alpha \neq \alpha_0$ system (3.13) undergoes the transcritical bifurcation without parameters by Theorem 4.2 in [7]. □

Note that $x_2 = 0$ is a line of equilibria for (3.13) and also for the resulting normal form (3.14). This zero eigenvalue becomes a double eigenvalue at $(0,0)$ if we choose $x_0 = 1$ or x_+, in a way such that the second system eigenvalue changes sign along the line of equilibria; specifically, this second eigenvalue is positive (resp., negative) if $y_2 > 0$ (resp., $y_2 < 0$). This means that the line of equilibria is normally hyperbolic for $y_2 \neq 0$, and a stability change occurs along the line of equilibria as a result of the loss of normal hyperbolicity at the origin, see the bifurcation diagram in Figure 1. This is the transcritical bifurcation without parameters.

4. Numerical simulations

In this section, we make numerically simulations to illustrate transcritical bifurcation, pitchfork bifurcation (given in Theorem 3.1) , one stable limit cycle arising from Hopf bifurcation (given in Theorem 3.2), which are not indicated in [11].

In order to display the transcritical bifurcation, we choose $\alpha = 0.4$, $\beta = 0.1$ and $g = -0.05$ in system (2.1). Theorem 3.1 shows that the parameter value of
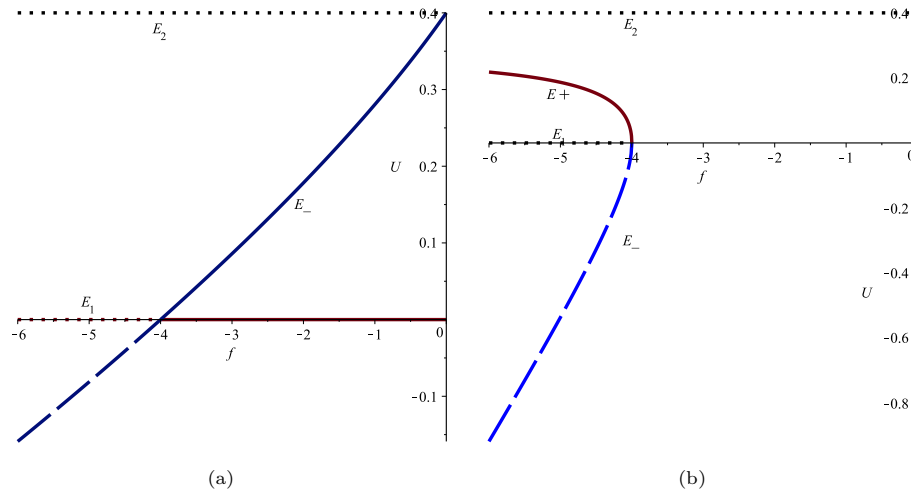
Figure 2. (a) Bifurcation diagram of system (2.1) for $\alpha = 0.4$, $\beta = 0.1$ and $g = -0.05$. (b) Bifurcation diagram of system (2.1) for $\alpha = 0.4$, $\beta = 0.1$ and $g = -0.35$.

transcritical bifurcation is $f = -\alpha/\beta = -4$. We use Maple software to plot in the (f, U)-coordinates the following four curves

$$U = 0, \quad U = \alpha, \quad U = U_-, \quad U = U_+$$

the abscissas of equilibria E_1, E_2, E_- and E_+ respectively depending on f. Figure 2 (a) shows that the $U = 0$ and $U = U_-$ intersect at the point $(-4, 0)$, i.e. E_- arises from the one E_1 as the the parameter f crosses -4 from $f < -4$ to $f > -4$, which demonstrate the result of Theorem 3.1. In addition, the saddle E_1 changes into a stable node when the parameter f crosses -4. Note that dashed line in Figure 2 (a) means the equilibrium E_- does not exist for $f \leq -4$. Choosing $\alpha = 0.4$, $\beta = 0.1$, then we compute $g = -\beta(\alpha + 1)/\alpha = -0.35$. Theorem 3.1 shows that system 2.1 undergoes pitchfork bifurcation near E_1 when f crosses -4. Figure 2 (b) shows that the three equilibria E_1, E_+ and E_- all arise from the one E_1 as the parameter f crosses $f = -4$ from $f > -4$ to $f < -4$, which demonstrate the result of Theorem 3.1. Note that dashed line in Figure 2 (b) means the equilibrium E_- does not exist for $f \leq -4$. Figure 2 (b) displays that for $F < -4$ there exist E_1, E_2 and E_+ in first quadrant but E_- is outside the first quadrant.

Consider system (2.1) with $\alpha = 0.4$, $\beta = 0.111$, $f = 0.5$, $g = 0.1$ and $r = 0.12$. Clearly, $(\alpha, \beta, f, g, r) \in \mathcal{H}_1$ in Theorem 3.2. As done in subsection 3.2 for weak focus E_+, from (3.8) and (3.12) we compute bifurcation parameter values $\mu_0 = 1.441173 \times 10^{-2}$, $\epsilon(\mu_0) = 9.930218 \times 10^{-2}$ and $L_1 = -7.769261 \times 10^4$. Simulation with the program "pplane8" in MATLAB shows that a unique unstable limit cycle(see Figure 3(a)) appears. When r increases, the unstable limit cycle will grow. When $r = 0.1261$, using "pplane8" in MATLAB software we see a homoclinic loop appears(see Figure 3(b)).
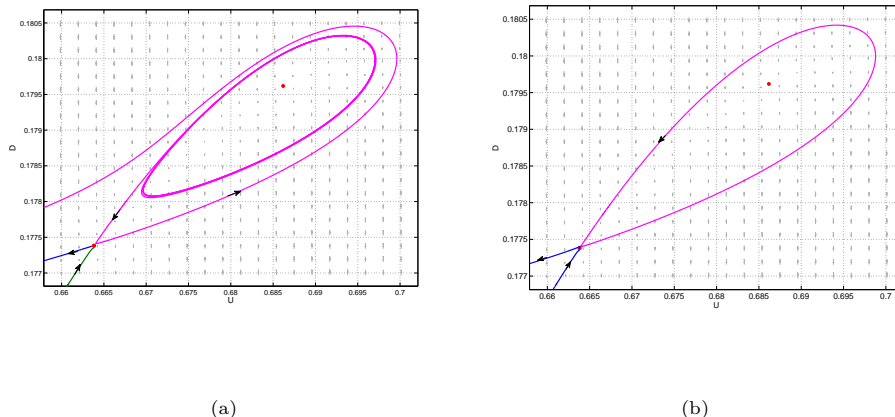
(a)   (b)

Figure 3. (a) A limit cycle of system (2.1) for $\alpha = 0.4$, $\beta = 0.111$, $f = 0.5$, $g = 0.1$ and $r = 0.12$. (b) A homoclinic loop of system (2.1) for $\alpha = 0.4$, $\beta = 0.111$, $f = 0.5$, $g = 0.1$ and $r = 0.1261$.

5. Conclusions

In this paper we analyzed the dynamics of system (1.1) near the boundary equilibria, interior equilibrium and singular line segment. We investigated the local bifurcations (transcritical bifurcation, pitchfork bifurcation, Hopf bifurcation) near the isolated equilibria(including the boundary and interior equilibria). We proved that the number of small amplitude limit cycle bifurcated from the Hopf bifurcation is at most one. Moreover, we discussed the bifurcations on the singular line segment. Versal unfolding was given in Theorem 3.3 for the normal form of the transcritical bifurcation without parameters.

For $f, g < 0$, the boundary equilibrium E_1, which means that nothing understood, will change the stability when the transcritical bifurcation or pitchfork bifurcation happens. Figure 2 (a) shows that if the relationship of doubt and comprehension becomes lower(from $f \leq -4$ to $f > -4$), different students will become either nothing understood E_1 or the stable state $\mathcal{L}_-$ which depends on different starting points. Figure 2 (b) shows that if the relationship of doubt and comprehension becomes higher(from $f \geq -4$ to $f < -4$), students may become the stable state E_+ by controlling the speed of the learning process $r > (fg - \sqrt{\Delta})U_+/(gU_+ + \beta)$ (see Theorem 3.2). Moreover, even E_+ is unstable, we can achieve a successful learning process if $(\alpha, \beta, f, g) \in \mathcal{H}_1$ by taking the speed of the learning process $r < (fg - \sqrt{\Delta})U_+/(gU_+ + \beta)$(see Theorem 3.2), in which a stable limit cycle exists. For $f, g > 0$(the case doubts destroy understanding), Figure 3(a) shows that students will become the stable state E_+ if the initial point is inside the limit cycle; either nothing understood E_1 or stable state $\mathcal{L}_-$ if the initial point is outside the limit cycle. Therefore, a good learning process depends on the initial point, the speed of the learning process and the relationship of doubt and comprehension. Teachers and professors should adopt their instructional strategies(such as background knowledge, formulation of interesting and puzzling questions in their classrooms, understanding).

When the number of interior equilibria is zero or one, the corresponding phase portraits can be given easily since system (1.1) has no limit cycle. However, when

the number of interior equilibria is two, the global dynamics is very complex since the number of limit cycles is not given. In the simulation, from Figure 3 we observed that the amplitude of the unstable limit cycle increases when r increases. When $\alpha = 0.4$, $\beta = 0.111$, $f = 0.5$, $g = 0.1$ and $r \to 0.1261$, a homoclinic loop exists. Thus, we can obtain that homoclinic bifurcations occurs. Numerical simulations of (1.1) seem to suggest that system (1.1) has at most one limit cycle. However, we have not been able to prove it analytically. Therefore, more complicated dynamical behaviors such as double limit cycle bifurcation may happen for the model (1.1) of learning process as an interplay between understanding and doubt and the global bifurcations will be our next work.

Acknowledgements

Appendix

Functions $g_i(x, y)$s, $\tilde{g}_i(x, y)$s$(i = 1, 2)$ in systems (3.1) and (3.2) are

$$g_1(x, y) = \frac{(2\beta - 1)x^2}{\beta(1 - \beta)} + \frac{\left(2gr\beta^2 + \alpha\beta g - \beta^2\mu_1 + \beta^2 r - \beta gr - \alpha g\right)xy}{gr\beta^2(1 - \beta)}$$

$$+ \frac{(\alpha\beta\mu_1 - \alpha\beta r - \alpha gr + \beta\mu_1 - \beta r)y^2}{r^2 g\beta^2} + \frac{x^3}{\beta(1 - \beta)} + \frac{(2\beta gr + \alpha g - \beta\mu_1 + \beta r)x^2 y}{gr\beta^2(1 - \beta)}$$

$$+ \frac{\left(-\beta gr^2 + \alpha\beta\mu_1 - \alpha\beta r - 2\alpha gr + \beta\mu_1 r - \beta r^2 + \alpha\mu_1 - \alpha r + \beta\mu_1 - \beta r\right)xy^2}{r^2 g\beta^2(-1 + \beta)}$$

$$+ \frac{y^3}{r^3 g^2 \beta^2(-1 + \beta)}\left\{-\beta(\alpha - \beta + 2)\mu_1{}^2 + r\left(\alpha\beta g + 2\alpha\beta + \alpha g - 2\beta^2 + \beta g + 4\beta\right)\mu_1\right.$$

$$\left. - r^2\left(\alpha\beta g + \alpha g^2 + \alpha\beta + \alpha g - \beta^2 + \beta g + 2\beta\right)\right\} + \frac{(\mu_1 - r)^2 xy^3}{r^3 g^2 \beta(-1 + \beta)}$$

$$+ \frac{(\mu_1 - r)^2(gr - \mu_1 + r)y^4}{r^4 g^3 \beta(-1 + \beta)},$$

$$g_2(x, y) = \frac{(-2\beta^2\mu_1 + \alpha\beta + \beta\mu_1 - \alpha)xy}{\beta^2 r(-1 + \beta)} + \frac{y^2}{r^2 g\beta^2(1 - \beta)}\left\{-\beta^2\mu_1{}^2 + \beta(2\beta gr + \alpha\beta + \beta r - gr\right.$$

$$\left. - \alpha + \beta - 1)\mu_1 - r(-1 + \beta)(\alpha\beta + \alpha g + \beta)\right\} + \frac{(-\beta\mu_1 + \alpha)x^2 y}{\beta^2 r(-1 + \beta)} + \frac{xy^2}{r^2 g\beta^2(-1 + \beta)}$$

$$\left\{\left(2\beta gr\mu_1 - \alpha\beta r + \alpha\beta\mu_1 - 2\alpha gr + \beta r\mu_1 - \beta\mu_1{}^2 - \alpha r + \alpha\mu_1 - \beta r + \beta\mu_1\right)\right\}$$

$$+ \frac{y^3}{r^3 g^2 \beta^2(1 - \beta)}\left\{-\beta(gr + \alpha - \beta + 2)\mu_1{}^2 + r(\beta g^2 r + \alpha\beta g + \beta gr + 2\alpha\beta + \alpha g\right.$$

$$\left. - 2\beta^2 + \beta g + 4\beta)\mu_1 - r^2\left(\alpha\beta g + \alpha g^2 + \alpha\beta + \alpha g - \beta^2 + \beta g + 2\beta\right)\right\}$$

$$- \frac{(\mu_1 - r)^2 xy^3}{r^3 g^2 \beta(-1 + \beta)} - \frac{(\mu_1 - r)^2(gr + r - \mu_1)y^4}{r^4 g^3 \beta(-1 + \beta)},$$

$$\tilde{g}_1(x, y) = \frac{(3\alpha - 1)x^2}{\alpha(\alpha - 1)} + \frac{xy}{\alpha^2(\alpha - 1)^3}\left\{\alpha fr(1 - 2\alpha)\mu_2 + (\alpha - 1)(\alpha^3 - 4\alpha^2 f + \beta fr - \alpha^2 + \alpha f)\right\}$$

$$+\frac{fy^2}{\alpha^2(\alpha-1)^4}\{-\alpha r\left(\alpha^2-2\alpha f-\alpha+f\right)\mu_2+(\alpha-1)\left(\alpha^2 f+\alpha^2 r-\beta fr-\alpha r\right)\}$$

$$+\frac{(3\alpha-2)x^3}{\alpha(\alpha-1)^2}+\frac{\left(-\alpha fr\mu_2+2\alpha^3-8\alpha^2 f+\beta fr-3\alpha^2+5\alpha f+\alpha\right)x^2 y}{\alpha^2(\alpha-1)^3}$$

$$-\frac{fxy^2}{\alpha^2(\alpha-1)^4}\{\alpha r(\alpha-2f-1)\mu_2+3\alpha^3-7\alpha^2 f-\alpha^2 r-\alpha\beta r+2\beta fr-4\alpha^2+4\alpha f$$

$$+\alpha r+\beta r+\alpha\}+\frac{fy^3}{\alpha^2\left(\alpha-1\right)^5}\{\alpha fr(\alpha-f-1)\mu_2+\alpha^3 f+\alpha^3 r-2\alpha^2 f^2-\alpha^2 fr$$

$$-\alpha\beta fr+\beta f^2 r-\alpha^2 f-2\alpha^2 r+\alpha f^2+\alpha fr+\beta fr+\alpha r\}+\frac{x^4}{\alpha\left(\alpha-1\right)^2}$$

$$+\frac{(\alpha-4f-1)x^3 y}{(\alpha-1)^3\alpha}-3\frac{(\alpha-2f-1)fx^2 y^2}{(\alpha-1)^4\alpha}+\frac{(3\alpha-4f-3)f^2 xy^3}{\alpha(\alpha-1)^5}$$

$$-\frac{f^3(\alpha-f-1)y^4}{(\alpha-1)^6\alpha},$$

$$\tilde{g}_2(x,y)=-\frac{\left(2\alpha^2\mu_2-\alpha\beta-\alpha\mu_2+\beta\right)rxy}{(\alpha-1)^2\alpha^2}-\frac{ry^2}{\alpha^2\left(\alpha-1\right)^3}\{\alpha\left(\alpha^2-2\alpha f-\alpha+f\right)\mu_2$$

$$-(\alpha-1)\left(\alpha^2-\beta f-\alpha\right)\}-\frac{(\alpha\mu_2-\beta)rx^2 y}{(\alpha-1)^2\alpha^2}$$

$$-\frac{\left(\alpha\left(\alpha-2f-1\right)\mu_2-\alpha^2-\alpha\beta+2\beta f+\alpha+\beta\right)rxy^2}{\alpha^2\left(\alpha-1\right)^3}$$

$$+\frac{r\left(\alpha-f-1\right)\left(\alpha f\mu_2+\alpha^2-\beta f-\alpha\right)y^3}{\alpha^2\left(\alpha-1\right)^4}.$$

Functions $F_i(x_1,x_2)$s $(i=1,2)$ in system (3.9) are

$$F_1(x_1,x_2)=\frac{\sqrt{\Delta}}{g\sqrt{fg\sqrt{\Delta}-\Delta}(1-U_+-D_+)}x_2^2-\frac{2\{(2g+1)(\alpha-U_+)+gf\}}{g(1-U_+-D_+)(\sqrt{\Delta}-fg-\alpha+1)}x_1 x_2$$

$$+\frac{g+1}{gD_+(1-U_+-D_+)}x_1^2 x_2+\frac{\sqrt{\Delta}}{gD_+(1-U_+-D_+)\sqrt{fg\sqrt{\Delta}-\Delta}}x_1 x_2^2,$$

$$F_2(x_1,x_2)=\frac{1}{8\sqrt{\Delta}gfU_+(1-U_+-D_+)}\{2\Delta^{3/2+}(-10fg-10f+2U_+)\Delta$$

$$+(9f^2 g^2-6\alpha fg+12f^2 g-3\alpha^2-12\alpha f-4fg+6\alpha-3)\Delta^{1/2}$$

$$+(fg-\alpha+1)(-f^2 g^2-2f^2 g+\alpha^2+2\alpha f+fg-\alpha+2f)-4D_+\}x_1^2$$

$$+\frac{1}{2fgU_+(1-U_+-D_+)\sqrt{fg\sqrt{\Delta}-\Delta}}\{(-3fg-2\alpha-8f+4)\Delta^{3/2}$$

$$+(6f^2 g^2+5\alpha fg+16f^2 g-2\alpha f-7fg-6f+1)\Delta$$

$$+(-3f^3 g^3-4\alpha f^2 g^2-8f^3 g^2+3\alpha^2 fg+2f^2 g^2+2\alpha^3+6\alpha^2 f$$

$$-10\alpha fg+4f^2 g-8\alpha^2-12\alpha f+5fg+10\alpha+4f-4)\Delta^{1/2}$$

$$-(fg+\alpha+2f-1)(-\alpha f^2 g^2+\alpha^2 fg-f^2 g^2-4\alpha fg+2fg+\alpha-1)$$

$$-4\alpha^2 f(1 - U_+ - D_+) - 12M^2 f D_+\}x_1 x_2 - \frac{-3U_+ + 2 + 2\alpha}{g(\sqrt{\Delta} - f*g)U_+}x_2^2$$

$$-\frac{(\sqrt{\Delta} + U_+)(g + 1) + D_+ - 1}{\sqrt{\Delta}g^2 U_+(1 - U_+ - D_+)}x_1^3$$

$$-\frac{1}{16f^2 g^2 U_+ D_+(1 - U_+ - D_+)\sqrt{fg\sqrt{\Delta} - \Delta}}\{3\Delta^2 - 24f(g+1)\Delta^{3/2}$$

$$+(42f^2 g^2 - 4\alpha fg + 56f^2 g - 6\alpha^2 - 12\alpha f - 4fg + 12\alpha - 6)\Delta$$

$$+8f(-3f^2 g^3 - 5f^2 g^2 + 3\alpha^2 g + 2\alpha fg + 3\alpha^2 - 6\alpha g - fg - 6\alpha + 3g + 3)\Delta^{1/2}$$

$$-4\alpha fg + 4fg - 16f^2 g + 12\alpha f - 14f^2 g^2 - 24\alpha^2 f + 8f^3 g^2 + 12\alpha^3 f + 3f^4 g^4$$

$$+8f^4 g^3 + 4f^3 g^3 + 32\alpha f^2 g - 4\alpha^2 fg - 16\alpha^2 f^2 g + 28\alpha f^2 g^2 + 4\alpha^3 fg$$

$$-4\alpha f^3 g^2 + 4\alpha f^3 g^3 - 14\alpha^2 f^2 g^2 + 3\alpha^4 - 12\alpha^3 + 18\alpha^2 - 12\alpha + 3\}x_1^2 x_2$$

$$-\frac{1}{16f^2 g^2 U_+ D_+(1 - U_+ - D_+)(fg - \sqrt{\Delta})}\{3\Delta^2 - 6f(3g + 4)\Delta^{3/2}$$

$$+(36f^2 g^2 - 2\alpha fg + 60f^2 g - 6\alpha^2 - 12\alpha f - 2fg + 12\alpha - 6)\Delta$$

$$+2f(-15f^2 g^3 + 2\alpha fg^2 - 24f^2 g^2 + 9\alpha^2 g + 8\alpha fg + 2fg^2 + 12\alpha^2 - 18\alpha g$$

$$-4fg - 24\alpha + 9g + 12)\Delta^{1/2} + 2fg - 2\alpha fg - 2\alpha^2 fg + 40\alpha f^2 g + 2\alpha^3 fg$$

$$-20\alpha^2 f^2 g + 24\alpha f^2 g^2 + 12\alpha f - 20f^2 g - 12f^2 g^2 - 24\alpha^2 f + 12\alpha^3 f$$

$$+8f^3 g^2 - 2f^3 g^3 + 9f^4 g^4 + 12f^4 g^3 + 3\alpha^4 - 12\alpha^3 + 18\alpha^2$$

$$-12\alpha + 3 - 2\alpha f^3 g^3 - 4\alpha f^3 g^2 - 12\alpha^2 f^2 g^2\}x_1 x_2^2$$

$$-\frac{\sqrt{\Delta}(-2f^2 g + 2f\sqrt{\Delta} + \alpha f + D_+)}{fg^2(1 - U_+ - D_+)U_+(\sqrt{\Delta} - fg)\sqrt{fg\sqrt{\Delta} - \Delta}}x_2^3$$

$$-\frac{2(g+1)}{2\sqrt{\Delta}g^3(1 - U_+ - D_+)U_+}x_1^4$$

$$-\frac{3g + 4}{g^3 U_+(1 - U_+ - D_+)\sqrt{fg\sqrt{\Delta} - \Delta}}x_1^3 x_2$$

$$+\frac{3(g + 2)}{g^3 U_+(1 - U_+ - D_+)(\Delta - fg)}x_1^2 x_2^2$$

$$+\frac{(g + 4)\sqrt{\Delta}}{g^3 U_+(1 - U_+ - D_+)(\Delta - fg)\sqrt{fg\sqrt{\Delta} - \Delta}}x_1 x_2^3$$

$$-\frac{\sqrt{\Delta}}{g^3 U_+(1 - U_+ - D_+)(\sqrt{\Delta} - fg)^2}x_2^4.$$

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