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PERSISTENCE OF TRAVELLING WAVEFRONTS IN A GENERALIZED BURGERS-HUXLEY EQUATION WITH LONG-RANGE DIFFUSION*

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Abstract In this paper, we study the persistence of travelling wavefronts in a generalized Burgers-Huxley equation with long-range diffusion. When the influence of long-range diffusion effect is sufficiently small, we prove the persistence of these waves by using geometric singular perturbation theory. When the influence becomes large, the behavior of these waves can only be investigate numerically. In this case, we find that the solutions lose monotonicity by using Matlab program bvp4c. Some previous results are extended.

Keywords Generalized Burgers-Huxley equation, travelling wavefronts, Fenichel's theory, persistence.

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1. Introduction

Travelling wave solutions are solutions of special type, and can be usually characterized as solutions invariant with respect to translation in space. The existence of traveling waves appears to be very common in nonlinear equations. From the physical point of view, travelling waves usually describe transition processes [23]. Transition from one equilibrium to another is a typical case, and the corresponding wave is called as travelling wavefronts. Since travelling wave solutions may provide more information for understanding the physical phenomena, its investigation plays an important role in the study of nonlinear physical phenomena. This is the reason why there are so many methods for exact travelling wave solutions, such as bifurcation method [9, 18], Lie symmetry method [19], tanh-function method [17], trigonometric function expansion [6] and so on.

In [20], the following generalized Burgers-Huxley (gBH) equation

$$u_t + \alpha u^n u_x - u_{xx} = \beta u(1 - u^n)(u^n - \gamma), \tag{1.1}$$

where α, β, $n > 0$ and $0 < \gamma < 1$, was used to model the interaction between reaction mechanisms, convection effects and diffusion transports. The solutions of Eq. (1.1) have been extensively studied, including numerical solutions [1,4,11,13,15] and exact travelling wave solutions [5, 10, 22].

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When $n = 1$, Eq. (1.1) becomes the Burgers-Huxley (BH) equation

$$u_t + \alpha u u_x - u_{xx} = \beta u(1-u)(u-\gamma). \tag{1.2}$$

When $\beta = 0$, it reduces to the Burgers equation. When $\alpha = 0$, it reduces to the Huxley equation, sometimes known as the FitzHugh-Nagumo [8]. In [16], Kyrychko et al. proved the persistence of travelling wavefronts of the BH equation with a small fourth-order derivative term

$$u_t + \alpha u u_x - u_{xx} + \delta u_{xxxx} = \beta u(1-u)(u-\gamma), \tag{1.3}$$

where $0 < \delta \ll 1$. It is worthwhile to note that, Fredholm theory in L^2 was used to prove the heteroclinic connection in the slow manifold. However, the travelling fronts are continuous and thus aren't appropriately studied in L^2. Moreover, they didn't investigate what would happen to the travelling wavefronts when δ becomes large.

In this paper, we study the gBH equation with long-range diffusion

$$u_t + \alpha u^n u_x - u_{xx} + D u_{xxxx} = \beta u(1-u^n)(u^n-\gamma), \tag{1.4}$$

where D is a positive parameter characterizing long-range diffusion effect [3]. When D is sufficiently small, we prove the persistence of the travelling wavefronts by using geometric singular perturbation theory [14]. In order to prove the heteroclinic connection in the slow manifold, we use the implicit function theorem. When D becomes large, we numerically investigate the behavior of the travelling wavefronts by using Matlab program bvp4c, and find that the solutions lose monotonicity.

2. Dynamical systems reformulation

The travelling wave solutions of Eq. (1.4) are of the form

$$u(x,t) = U(\xi) \quad \text{with} \quad \xi = x - ct, \tag{2.1}$$

where c is the wave speed. Substituting (2.1) into (1.4), we get

$$-cU' + \alpha U^n U' - U'' + DU'''' = \beta U(1-U^n)(U^n-\gamma). \tag{2.2}$$

Defining new variables

$$U' = v, \quad v' = w, \quad w' = z, \tag{2.3}$$

we rewrite Eq. (2.2) as

$$Y = \begin{pmatrix} U \\ v \\ w \\ z \end{pmatrix}, Y' = \begin{pmatrix} U' \\ v' \\ w' \\ z' \end{pmatrix} = \begin{pmatrix} v \\ w \\ z \\ \frac{1}{D}\left[\beta U(1-U^n)(U^n-\gamma) + cv - \alpha U^n v + w\right] \end{pmatrix} = F(Y). \tag{2.4}$$

Obviously, $Y^0 = (0,0,0,0)^T$ and $Y^1 = (1,0,0,0)^T$ are two equilibria of system (2.4). The linearization matrix at Y^0 is

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{\beta\gamma}{D} & \frac{c}{D} & \frac{1}{D} & 0 \end{pmatrix}$$

with the corresponding characteristic equation

$$\lambda^4 - \frac{1}{D}\lambda^2 - \frac{c}{D}\lambda + \frac{\beta\gamma}{D} = 0. \tag{2.5}$$

Similarly the linearization matrix at Y^1 is

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\beta n(\gamma-1)}{D} & \frac{c-\alpha}{D} & \frac{1}{D} & 0 \end{pmatrix}$$

with the corresponding characteristic equation

$$\lambda^4 - \frac{1}{D}\lambda^2 - \frac{c-\alpha}{D}\lambda + \frac{\beta n(1-\gamma)}{D} = 0. \tag{2.6}$$

We have the following result regarding the linearization of system (2.4).

Theorem 2.1. *In system* (2.4)*, the unstable manifold of* Y^0 *and the stable manifold of* Y^1 *both have dimension two.*

Proof. Our proof is based on Argument Principle. Spectrum of linearization at Y^0 is determined by the roots of Eq. (2.5), which can be written as $m_0(\lambda) = 0$ with

$$m_0(\lambda) = \lambda^4 - \frac{1}{D}\lambda^2 - \frac{c}{D}\lambda + \frac{\beta\gamma}{D}. \tag{2.7}$$

We want to show that $m_0(\lambda)$ has only two roots in the right half complex plane. Since $m_0(\lambda)$ is analytic, the number of roots in the right half complex plane is

$$\frac{1}{2\pi}\lim_{R\to\infty} \triangle_{C_0}\arg m_0(\lambda), \tag{2.8}$$

where the contour C_0 is the boundary, traversed anticlockwise, of the semicircle of radius R, centered at the origin, contained in $\mathrm{Re}\lambda \geq 0$, and $\triangle_{C_0}\arg m_0(\lambda)$ denotes the total change quantity in the argument of $m_0(\lambda)$ along C_0. The formula (2.8) equals

$$2 + \frac{1}{2\pi}[\triangle \arg m_0(iR)]_{R=\infty}^{R=-\infty}.$$

The quantity in the bracket denotes the change in the argument of $m_0(iR)$ as R goes from ∞ to $-\infty$, and thus we compute the number of times the image $m_0(iR)$ winds around the origin. Note that

$$m_0(iR) = \left(R^4 + \frac{1}{D}R^2 + \frac{\beta\gamma}{D}\right) + i\left(-\frac{c}{D}R\right).$$

Since $R^4 + \frac{1}{D}R^2 + \frac{\beta\gamma}{D} > 0$, the image $m_0(iR)$ only lies on the right half complex plane. For $|R|$ sufficiently large, $m_0(iR)$ has the asymptotic behavior

$$\mathrm{Re}m_0(iR) \sim R^4,\ \mathrm{Im}m_0(iR) \sim -\frac{c}{D}R \text{ as } R \to \pm\infty.$$

So $[\triangle \arg m_0(iR)]_{R=\infty}^{R=-\infty} = 0$, and thus the number of roots of Eq. (2.5) in the right half complex plane is two.

Similarly, for Y^1 we rewrite Eq. (2.6) as $m_1(\lambda) = 0$ with

$$m_1(\lambda) = \lambda^4 - \frac{1}{D}\lambda^2 - \frac{c-\alpha}{D}\lambda + \frac{\beta n(1-\gamma)}{D}. \tag{2.9}$$

Now let C_1 be the boundary of left half complex plane defined same as C_0, then the number of roots of $m_1(\lambda)$ in the left half complex plane is

$$2 + \frac{1}{2\pi}[\triangle \arg m_1(iR)]_{R=-\infty}^{R=\infty}.$$

Note that

$$m_1(iR) = \left(R^4 + \frac{1}{D}R^2 + \frac{\beta n(1-\gamma)}{D}\right) + i\left(\frac{\alpha - c}{D}R\right).$$

Since $R^4 + \frac{1}{D}R^2 + \frac{\beta n(1-\gamma)}{D} > 0$, the image $m_1(iR)$ only lies on the right half complex plane. For $|R|$ sufficiently large, $m_1(iR)$ has the asymptotic behavior

$$\mathrm{Re}m_1(iR) \sim R^4,\ \mathrm{Im}m_1(iR) \sim \frac{\alpha - c}{D}R \text{ as } R \to \pm\infty.$$

So $[\triangle \arg m_1(iR)]_{R=-\infty}^{R=\infty} = 0$, and thus the number of roots of Eq. (2.6) in the left half complex plane is two. This completes the proof of Theorem 2.1. □

From Theorem 2.1, we know that the stable manifold $W^s(Y^0)$ and the unstable manifold $W^u(Y^1)$ also both have dimension two. This is important for numerical investigation of the behavior of the travelling wavefronts, which will be discussed in Section 4. However, Theorem 1 isn't sufficient to show the intersection of $W^u(Y^0)$ and $W^s(Y^1)$, which is a heteroclinic orbit of system (2.4) amounting to a travelling wavefront of Eq. (1.4). In order to prove rigorously the existence of this intersection, we resort to geometric singular perturbation theory.

3. Persistence of travelling wavefronts for sufficiently small long-range diffusion

In this section, we prove the persistence of travelling wavefronts of Eq. (1.4) for sufficiently small long-range diffusion.

Let $D = \varepsilon^2 \ll 1$. Redefining (2.3) as

$$U' = v, \quad v' = w, \quad \varepsilon w' = z, \tag{3.1}$$

we rewrite system (2.4) as

$$\begin{cases} U' = v, \\ v' = w, \\ \varepsilon w' = z, \\ \varepsilon z' = \beta U(1-U^n)(U^n - \gamma) + cv - \alpha U^n v + w, \end{cases} \tag{3.2}$$

which is called the slow system. With $\eta = \xi/\varepsilon$, the dual fast system associated with system (3.2) is

$$\begin{cases} U_\eta = \varepsilon v, \\ v_\eta = \varepsilon w, \\ w_\eta = z, \\ z_\eta = \beta U(1-U^n)(U^n-\gamma) + cv - \alpha U^n v + w. \end{cases} \tag{3.3}$$

If ε is set to zero in system (3.2), then U and v are governed by

$$\begin{cases} U' = v, \\ v' = -\beta U(1-U^n)(U^n-\gamma) - cv + \alpha U^n v, \end{cases} \tag{3.4}$$

while w and z lie on the set

$$M_0 = \{(U,v,w,z) : z = 0,\ w = -\beta U(1-U^n)(U^n-\gamma) - cv + \alpha U^n v\},$$

which is a two-dimensional submanifold of R^4. Note that system (3.4) is the dynamical systems reformulation of Eq. (1.1).

By the definition in [7], the manifold M_0 is said to be normally hyperbolic if the linearization of the fast system, restricted to M_0, has exactly $\dim M_0$ eigenvalues on the imaginary axis, with the remainder of the spectrums hyperbolic. The linearization of the fast system (3.3) restricted to M_0 is

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ s & c-\alpha U^n & 1 & 0 \end{pmatrix}$$

with $s = \gamma - (1+\gamma)(1+n)U^n + (2n+1)U^{2n}$. So the matrix A has the eigenvalues $0, 0, 1, -1$, and thus M_0 is normally hyperbolic. Therefore by Fenichel's invariant manifold theory [7], for sufficiently small $\varepsilon > 0$ there exists a two-dimensional submanifold M_ε of R^4 which lies within $O(\varepsilon)$ of M_0 and is diffeomorphic to M_0. Moreover, M_ε is invariant under the flow (3.2) and C^r smooth for any $r < \infty$.

To determine the dynamics on M_ε, we write

$$M_\varepsilon = \{(U,v,w,z) : z = g(U,v,\varepsilon),\ w = h(U,v,\varepsilon) - H(U,v)\}, \tag{3.5}$$

where g and h depend smoothly on ε and satisfy $g(U,v,0) = h(U,v,0) = 0$, and $H(U,v) = \beta U(1-U^n)(U^n-\gamma) + cv - \alpha U^n v$. Now, we expand g and h in Taylor series in ε

$$g(U,v,\varepsilon) = g(U,v,0) + \varepsilon g_\varepsilon(U,v,0) + \frac{1}{2}\varepsilon^2 g_{\varepsilon\varepsilon}(U,v,0) + \cdots,$$
$$h(U,v,\varepsilon) = h(U,v,0) + \varepsilon h_\varepsilon(U,v,0) + \frac{1}{2}\varepsilon^2 h_{\varepsilon\varepsilon}(U,v,0) + \cdots.$$

Substituting the representations of z and w in M_ε from (3.5) into the third and fourth equations of system (3.2), equating the same order in ε (up to 2 order), we have

$$g(U,v,0)=h(U,v,0)=h_\varepsilon(U,v,0)=g_{\varepsilon\varepsilon}(U,v,0)=0,$$
$$g_\varepsilon(U,v,0)=[(2n+1)U^{2n}-(1+\gamma)(1+n)U^n+\gamma]v-(\alpha U^n-c)H(U,v)+\alpha nU^{n-1}v,$$
$$\frac{1}{2}h_{\varepsilon\varepsilon}(U,v,0)=vg_\varepsilon(U,v,0)-\frac{\partial g_\varepsilon(U,v,0)}{\partial v}H(U,v).$$

This allows us to rewrite system (3.2) as

$$\begin{cases} U'=v, \\ v'=-H(U,v)+\dfrac{1}{2}\varepsilon^2 h_{\varepsilon\varepsilon}(U,v,0)+O(\varepsilon^3), \end{cases} \tag{3.6}$$

which determines the dynamics on M_ε.

When $\varepsilon=0$, system (3.6) reduces to system (3.4). Now we are in the position to state and prove the following persistence theorem.

Theorem 3.1. *If Eq.*(1.1) *admits a strictly increasing travelling wavefront* $u_0(x,t)=U_0(\xi)$ *satisfying* $\lim_{\xi\to-\infty}U_0(\xi)=0$ *and* $\lim_{\xi\to\infty}U_0(\xi)=1$*, then for sufficiently small* $\varepsilon>0$ *this travelling wavefront persists in Eq.* (1.4). *In other words, Eq.* (1.4) *also admits a strictly increasing travelling wavefront* $u(x,t)=U(\xi)$ *satisfying* $\lim_{\xi\to-\infty}U(\xi)=0$ *and* $\lim_{\xi\to\infty}U(\xi)=1$.

Proof. Obviously, the trvelling wavefront $u_0(x,t)$ corresponds to a heteroclinic orbit of system (3.4) in (U,v) phase plane. This heteroclinic orbit connects the two equilibria E_- and E_+, where $E_-=(0,0)$ and $E_+=(1,0)$. Let c_0 be the wave speed of the travelling wavefront. For sufficiently small ε, E_- and E_+ are still two equilibria of system (3.6). Now we prove that system (3.6) admits a heteroclinic orbit connecting E_- and E_+. We rewrite system (3.6) as

$$\begin{cases} U'=v, \\ v'=\Phi(U,v,c,\varepsilon). \end{cases} \tag{3.7}$$

Note that

$$\Phi(U,v,c,0)=-H(U,v).$$

Since $U_0(\xi)$ is strictly increasing, it can be characterized as the graph of some function

$$v=f(U,c_0).$$

By the stable manifold theorem, for sufficiently small ε we can also characterize the unstable manifold of E_- as the graph of some function

$$v=f_1(U,c,\varepsilon),$$

where $f_1(0,c,\varepsilon)=0$. Furthermore, by continuous dependence of solutions on parameters, this manifold must cross the line $U=1/2$ somewhere.

Similarly, let $v=f_2(U,c,\varepsilon)$ be the function for the stable manifold of E_+. Then $f_2(1,c,\varepsilon)=0$, and for sufficiently small ε it must also cross the line $U=1/2$ somewhere. Thus

$$f_1(U,c_0,0)=f_2(U,c_0,0)=f(U,c_0). \tag{3.8}$$

To show that Eq. (3.7) admits a heteroclinic orbit, we prove that there exists a unique value $c = c(\varepsilon)$, near c_0, such that the manifolds f_1 and f_2 cross the line $U = 1/2$ at a same point. Define

$$G(c,\varepsilon) = f_1\left(\frac{1}{2}, c, \varepsilon\right) - f_2\left(\frac{1}{2}, c, \varepsilon\right). \tag{3.9}$$

Noticing that $v = f_1(U,c,\varepsilon)$ and $v = f_2(U,c,\varepsilon)$ both satisfy the equation

$$\frac{\mathrm{d}v}{\mathrm{d}U} = \frac{\Phi(U,v,c,\varepsilon)}{v}, \tag{3.10}$$

we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}U}\left(\frac{\partial f_1(U,c_0,0)}{\partial c}\right) &= \frac{\partial}{\partial c}\left(\frac{\mathrm{d}f_1(U,c,0)}{\mathrm{d}U}\right)\Big|_{c=c_0} \\
&= \frac{\partial}{\partial c}\left(\frac{\Phi(U,f_1(U,c,0),c,0)}{f_1(U,c,0)}\right)\Big|_{c=c_0} \\
&= \frac{\partial}{\partial c}\left(\frac{-\beta U(1-U^n)(U^n-\gamma) - cf_1(U,c,0) + \alpha U^n f_1(U,c,0)}{f_1(U,c,0)}\right)\Big|_{c=c_0} \\
&= \frac{\partial}{\partial c}\left(-c + \alpha U^n + \frac{-\beta U(1-U^n)(U^n-\gamma)}{f_1(U,c,0)}\right)\Big|_{c=c_0} \\
&= -1 + \frac{\beta U(1-U^n)(U^n-\gamma)}{f^2(U,c_0)} \cdot \frac{\partial f_1(U,c_0,0)}{\partial c}.
\end{aligned} \tag{3.11}$$

Let

$$P(U) = \frac{\beta U(1-U^n)(U^n-\gamma)}{f^2(U,c_0)}.$$

Since

$$\frac{\partial f_1(0,c,\varepsilon)}{\partial c} = 0,$$

we solved Eq. (3.11) and get

$$\frac{\partial f_1(U,c_0,0)}{\partial c} = -e^{\int_{\frac{1}{2}}^{U} P(\xi)\mathrm{d}\xi} \int_0^U e^{-\int_{\frac{1}{2}}^{s} P(\xi)\mathrm{d}\xi} \mathrm{d}s. \tag{3.12}$$

It follows that

$$\frac{\partial f_1(\frac{1}{2},c_0,0)}{\partial c} = -\int_0^{\frac{1}{2}} e^{-\int_{\frac{1}{2}}^{s} P(\xi)\mathrm{d}\xi} \mathrm{d}s. \tag{3.13}$$

Similarly, we have

$$\frac{\partial f_2(\frac{1}{2},c_0,0)}{\partial c} = -\int_1^{\frac{1}{2}} e^{-\int_{\frac{1}{2}}^{s} P(\xi)\mathrm{d}\xi} \mathrm{d}s. \tag{3.14}$$

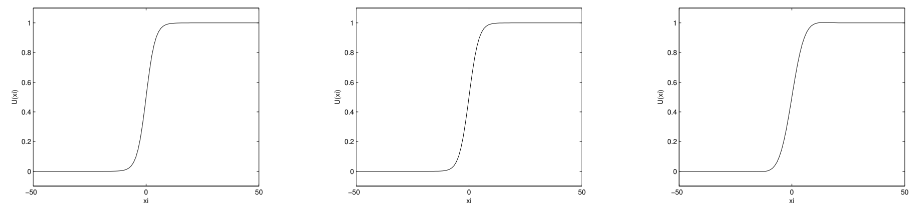
Therefore

$$\frac{\partial G(c_0,0)}{\partial c} = \frac{\partial f_1(\frac{1}{2},c_0,0)}{\partial c} - \frac{\partial f_2(\frac{1}{2},c_0,0)}{\partial c} = -\int_0^1 e^{-\int_{\frac{1}{2}}^{s} P(\xi)\mathrm{d}\xi} \mathrm{d}s < 0.$$

By the implicit function theorem, for sufficiently small ε, $G(c,\varepsilon) = 0$ has a unique root $c = c(\varepsilon)$ near c_0. This implies that the manifolds f_1 and f_2 cross the line $U = 1/2$ at a same point, that is, system (3.6) admits a heteroclinic orbit connecting E_- and E_+. So Eq. (1.4) also admits a travelling wavefront $u(x,t) = U(\xi)$ satisfying $\lim_{\xi\to-\infty} U(\xi) = 0$ and $\lim_{\xi\to\infty} U(\xi) = 1$. Moreover for sufficiently small ε, the strict monotonicity of $U_0(\xi)$ guarantees the strict monotonicity of $U(\xi)$. This completes the proof of Theorem 2. □

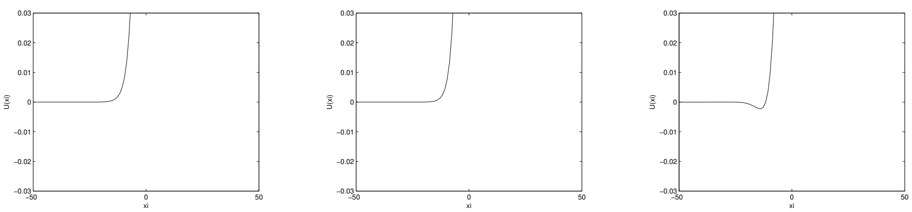
4. Numerical investigation of travelling wavefronts for large long-range diffusion

In this section, we numerically investigate the behavior of the travelling wavefronts for large long-range diffusion.



D=0.1 D=1 D=10

Figure 1. Heteroclinic orbits for (2.4) shown in $\xi - U$ plane, where $n = \alpha = \beta = 1$ and $\gamma = c = 0.5$.



D=0.1 D=1 D=10

Figure 2. Truncations of the three charts in Figure 1 on the region $[-50, 50] \times [-0.03, 0.03]$.

For our purpose, we look for a solution $U(\xi)$ of system (2.4) satisfying the boundary conditions

$$U(-\infty) = 0, \quad U(\infty) = 1. \tag{4.1}$$

We consider the boundary value problem (BVP) consisting of (2.4) and (4.1) on a finite interval $[L_1, L_2]$, with the approximate solution converging to a correct solution as $L_1 \to -\infty$ and $L_2 \to \infty$ [2, 12]. For this reason, we require the solution to have no projection on the stable manifold of Y^0 at $\xi = L_1$ and no projection on the unstable manifold of Y^1 at $\xi = L_2$. From Theorem 1, $W^s(Y^0)$ and $W^u(Y^1)$ both have dimension two, and thus constitute four boundary conditions for system (2.4). As regards numerical approximate solutions of BVP of ordinary differential equations, Matlab program bvp4c is an effective solver [21]. bvp4c implements a collocation method and requires users to supply a guess for the desired solution. In order to solve our problem, we use the solution [5]

$$u(x,t) = \left[\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{n(\rho-\alpha)}{4(n+1)}\left[x - \frac{\alpha-\rho+(\alpha+\rho)(n+1)\gamma}{2(n+1)}t + x_0\right]\right)\right]^{\frac{1}{n}}$$

as the guess solution. With the help of bvp4c, numerical simulations for some particular values of parameters are shown in Fig. 1 and Fig. 2, where the RelTol is 10^{-3} and AbsTol is 10^{-6}. As shown in Fig. 1 and Fig. 2, when D is small, the shape of the perturbed travelling wavefront is close to the unperturbed one; when D becomes large, the perturbed travelling wavefront loses monotonicity.

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References

[1] B. Batiha, M. Noorani, I. Hashim, *Application of variational iteration method to the generalized Burgers-Huxley equation*, Chaos Solitons Fractals, 2008, 36, 660–663.

[2] W. Beyn, *The numerical computation of connecting orbits in dynamical systems*, IMA J. Numer. Anal., 1990, 10, 379–405.

[3] D. Cohen, J. Murray, *A generalised diffusion model for growth and dispersal in a population*, J. Math. Biol., 1981, 12, 237–249.

[4] M. Darvishi, S. Kheybari, F. Khani, *Spectral collocation method and Darvishi's preconditionings to solve the generalized Burgers-Huxley equation*, Commun. Nonlinear Sci. Numer. Simulat. 2008, 13, 2091–2103.

[5] X. Deng, *Travelling wave solutions for the generalized Burgers-Huxley equation*, Appl. Math. Comput., 2008, 204, 733–737.

[6] E. Fan, *Extended tanh-function method and its applications to nonlinear equations*, Phys. Lett. A, 2000, 277, 212–218.

[7] N. Fenichel, *Geometric singular perturbation theory for ordinary differential equations*, J. Differential Equations, 1979, 31, 53–98.

[8] R. Fitzhugh, *Mathematical models of excitation and propagation in nerve, in: H.P. Schwan (Ed.), Biological Engineering*, McGraw-Hill, New York, 1969, 1–85.

[9] Y. Fu, J. Li, *Exact stationary-wave solutions in the standard model of the Kerr-Nonlinear optical fiber with the Bragg grating*, J. Appl. Anal. Comput., 2017, 7 (3), 1177–1184.

[10] H. Gao, R. Zhao, *New exact solutions to the generalized Burgers-Huxley equation*, Appl. Math. Comput., 2010, 217, 1598–1603.

[11] I. Hashim, M. Noorani, M. Al-Hadidi, *Solving the generalized Burgers-Huxley equation using the Adomian decomposition method*, Math. Comput. Model. 2006, 43, 1404–1411.

[12] F. Hoog, R. Weiss, *An approximation theory for boundary value problems on infinite intervals*, Computing, 1980, 24, 227–239.

[13] M. Javidi, *A numerical solution of the generalized Burger's-Huxley equation by pseudospectral method and Darvishi's preconditioning*, Appl. Math. Comput. 2006, 175, 1619–1628.

[14] C. Jones, *Geometric Singular Perturbaton Theory, in: L. Arnold, R. Johnson (Eds), CIME Lectures on Dynamical Systems, Lecture Notes in Mathematics, vol. 1*, Springer-Verlag, New York, 1995.

[15] A. Khattak, *A computational meshless method for the generalized Burger's-Huxley equation*, Appl. Math. Model. 2009, 33, 3718–3729.

[16] Y. Kyrychko, M. Bartuccelli, K. Blyuss, *Persistence of travelling wave solutions of a fourth order diffusion system*, J. Comput. Appl. Math., 2005, 176, 433–443.

[17] H. Lan, K. Wang, *Exact solutions for two nonlinear equations: I*, J. Phys. A: Math. Gen., 1990, 23, 3923–3928.

[18] J. Li, *Notes on exact travelling wave solutions for a long wave-short wave model*, J. Appl. Anal. Comput. 2015, 5(1), 138–140.

[19] H. Liu, J. Li, Q. Zhang, *Lie symmetry analysis and exact explicit solutions for general Burgers¡¯ equation*, J. Comput. Appl. Math., 2009, 228, 1–9.

[20] J. Satsuma, *Topics in soliton theory and exactly solvable nonlinear equations*, World Scientific, Singapore, 1987.

[21] L. Shampine, I. Gladwell, S. Thompson, *Solving ODEs with MATLAB*, Cambridge University Press, 2003.

[22] X. Wang, Z. Zhu, Y. Lu, *Solitary wave solutions of the generalized Burgers-Huxley equation*, J. Phys. A, 1990, 23, 271–274.

[23] A.I. Volpert, V.A. Volpert, *Traveling wave solutions of parabolic systems, in : Translations of Mathematical Monographs,* Amer. Math. Soc.£¬ Providence, RhodeIsland, 1994.